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On some type of H-functions

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ABSTRACT

In this paper ,we introduce the definitions of some types of H-functions namely ,H-continuous ,H-irresolute ,H-closed function ,H-open function, H-homomorphism function and investigate some properties of composition and study relation between functions.

MSC..

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Introduction

One of the very important concepts in a topology is the concept of function. there are several types of function related to types of open set in topological space. Introduce in [1] , the definition of H-open set and they introduce several concepts related to H-open set. in this paper , we study an important classes of function namely ,H-continuous ,H-irresolute,

H-closed function ,H-open function ,H-homomorphism function and give the concept of H-closed function. in this work ,we use the concepts of H-open set to construct some types of H-function namely.

The work consists of two section. section one introduces the definition

of H-functions and some proposition and example and results ,we will be needed in the next section. In section two we review some ,example and proposition and results.

1. Basic Concepts

1.1 Definition [2]: A subset A of the topological space (X,T) is called H-open set if for every non-empty set U in X ,$U \neq X$,U and $U \in T$, $A \subseteq \text{int}(A \cup U)$.

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The complement of the H-open set is called H-closed. we denoted the

Family of all H-open set of a topological space (X,T)by T^H.

1.2 Example[2]: Let $X = \{a, b, c, d\}$ and $T = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. Then $T^H = \{\phi, x, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

1.3 Proposition [1]: Every open set in a topological space (X ,T) is H-open.

1.4 Proposition[2]: Let(X,T) be a space, A and B two H-open sets. Then

(i) $A \cup B$ is an H-open set.

(ii) $A \cap B$ is an H-open set.

1.5 Definition [1]: Let X be a space and $A \subseteq X$, The union of all H-open sets of X contained in A is called the H-interior of A and denoted by A^{oH}.

1.6 Proposition [1]: Let (X,T) be a space and $A \subseteq B \subseteq X$. Then:

(i) A is an H-open set if and only if $A = A^{oH}$

(ii) If $A \subseteq B$ then $A^{oH} \subseteq B^{oH}$

(iv) A^{oH} is an H-opened set.

1.7 Definition [2]: Let X be a space and the $A \subseteq X$, The intersection of all H-closed sets of X containing A is called the H-closure of A and is denoted by $\overline{A}^{H}$.

1.8 Proposition [1]: Let (X, T) be a space and $A, B \subseteq X$ then:

(i) A is H-closed set if and only if $A = \overline{A}^{H}$.

(ii) If $A \subseteq B$ then $\overline{A}^{H} \subseteq \overline{B}^{H}$

(iii) $A \subseteq \overline{A}^{H}$

1.9 Definition [4]: Let $f : X \to Y$ be a function of a spaces. Then:

(i) f is called **continuous function** if $f^{-1}(A)$ is an open set in X for every open set A in Y.

(ii) f is called an **open function** if $f(A)$ is a open set in Y for every open set A in X.

(iii) f is called **closed function** if $f(A)$ is a closed set in Y for every closed setA in X.

1.10 Theorem [7]:

Let $f : X \to Y$ be a function of a spaces . then:

(i) f H is a **continuous function** if and only if $f^{-1}(A)$ is a closed set in X for every closed set A in A.

(ii) f is a **continuous function** if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset of X.

(iii) f is a **continuous function** if and only if $\overline{f^{-1}(A)} \subseteq f^{-1}(\overline{A})$ for every subset A of Y .

(iv) f is a **continuous function** if and only if $f^{-1}(A^0) \subseteq (f^{-1}(A))^o$for every subset A of Y.

1.11 Theorem [3]: Let f : X→Y be a function of a spaces . Then the following statements are equivalent:

(i) f is an open function.

(ii) $f(A^0) \subseteq (f(A))^o$, for every subset A of X.

(iii) $(f^{-1}(A))^o \subseteq f^{-1}(A^0)$,for every subset A of Y.

(iv) $f^{-1}(\overline{A}) \subseteq \overline{f^{-1}(A)}$, for every subset A of Y.

1.12 Definition [6]: A bijective function f : X→Y from a space X onto a space Y is called Homeomorphism if H is continuous and open (or closed).

Also , X is called homeomorphic to the space Y (written X≅Y) .

1.13 Proposition [7]: Let X and Y be spaces and f : X→Y be a homeomorphism function . Let A⊆ X and B⊆ X such that $f(A) = B$.Thenf_A: A→B is an homeomorphism Function.

1.14 Definition [1,2]: Let X and Y be spaces and f: X→Y be a function .then :

(i) f is called **H-continuous function** if $f^{-1}(A)$ is an H-open set in X For every open set A in Y .

(ii) f is called **H-irresolute function** if $f^{-1}(A)$ is an H-open set in X for every H-open set A in Y.

1.15 Remark [1,2]:

(i) every H-irresolute function is an H-continuous function .

(ii) every continuous function is H-continuous function .

The converse of Remark (1.15,i,ii) , is not true in general as the
Following example shows.

1.16 Eexample[1]:

Let X=Y = {a,b,c},τ= {Ø,X, {a}}, $τ^h$ ={Ø,X,{a},{b,c}}, σ={Ø,Y,{b,c}} and $σ^h$ = {Ø,{a},{b},{c}{a,b},{a,c},{b,c}}. Clearly , the identity Function : $(X,\tau)\to(Y,\sigma)$ is H- continuous but not H-irresolute .

1.17 Eexample [2] : Let X= {a,b,c} and Y = {1,2,3},τ= {Ø,X, {b}}, $τ^h$={Ø,X,{b},{a,c}}, σ={Ø,Y,{1}, {1,3}}. A Function f: (X,τ)→(Y,σ) is defined by f({a})={2}, f({b})={1}, f({c})={3}.Clearly ,f is an H-continuous but not continuous.

1.18 Eexample[1]: Let X=Y = {a,b,c},τ= {Ø,X, {a},{a,c}, $τ^h$={Ø,X,{a},{c},{a,c},{b,c}}, σ={Ø,Y,{a},{c},{a, c}} and $σ^h$ = {Ø,Y,{a},{c},{a,c}}. Clearly , the identity Function f: (X,τ)→(Y,σ) is H- irresolute but not continuous .

1.19 Proposition [1,2]: Let X,Y and Z be a spaces , and f :X→Y ,g:Y→Z be function. Then:

(i)if f is H-continuous ,g is H-continuous ,then gof is H-continuous.

(ii) if is H-irresolute ,g is H-irresolute , then gof is H-irresolute.

(iii) if f is H-irresolute ,g is H-continuous , then gof is H-irresolute .

1.20 Definition [2]:

(i) f is called H-open function if f(A) is an H-open set in Y For every open set A in X.

1.21 Example[2]**:** Let X=Y = {a,b,c },τ= {Ø,X,{b,c}}, σ={Ø,Y,{a}} and $\sigma^{h=}$ {Ø,X,{a},{b,c}}. Clearly ,the identity Function : (X,τ)→(Y,σ) is an H-open.

1.22 Remark [1]: Every open function is H-open function.

The convers of the Remark (1.22) need not be true as shown in the following in example(1.21),the identity function, the identity Function f : (X,τ)→(Y,σ) is an H-open but not open.

1.23 Definition [2]: A bijective function f: X→Y is said to be H-homeomorphism if f is an H-continuous and H-open function.

1.24 Remark [2]: Every homeomorphism function is an H-homeomorphism function.

The converse of the remark (1.24) ,need not be true as shown in the following example .

1.25 Example [2]: Let X=Y = {a,b,c},τ={Ø,X, {a,c}}, τ^h = {Ø,X,{a},{b},{c},{a,b},{a,c},{b,c}}, σ={Ø,Y,{b, c}} and σ^h = {Ø,{a},{b},{c}{a,b},{a,c},{b,c}}. Clearly, the identity Function f: (X,τ)→(Y,σ) is H-homeomorphism but not homeomorphism.

2-The Main Results

2.1 Proposition: Let f :X→Y be a function of a space . Then the following

Statements are equivalent

(i) f is H-irresolute .

(ii) for x ∈X and any H-open subset V of Y containing f(x),there exists

an H-open set U in X such that x∈U and f(U) ⊆V.

(iii) The inverse image of every H-closed subset of Y is an H-closed

Subset of X.

Proof:

(i→ii) Let x∈ X and V be is an H-open set in Y such that f(x)∈V . since f is

H-irresolute ,we have x∈f^{-1} (V) ⊆ X .let U=f^{-1} (V). then x∈U and f(U)⊆V .

(ii→i) Let V be an H-open set in Y and Let x∈f^{-1} (V) .then, f(x)∈V and thus

there exists U is be an H-open set such that x∈U and f(U)⊆ V.

Now, x∈U ⊆ f^{-1} [f(V)]⊆ f^{-1} (V).

then , f^{-1}(V) is an H-nhd of all its points. thus f^{-1} (V) is H-open set.

Hence f is an H-irresolute.

(iii→ii)

Let x∈X and V ∈Y such that f(x)∈V. then Y−V is H-open in Y. by given hypothesis ,f^{-1} (Y−V)= X−f^{-1}(V) is an H-closed set in X, then f^{-1}(V) is an H-open in X. Let U= f^{-1}(V) and clearly ,x∈U.

therefore, f(U)=f[f^{-1}(V)]⊆V.

(i→iii)

Let F is an H-closed set of Y. then Y−F is an H-open in Y.

than f^{-1} (Y−F) is an H-open in X.

since f^{-1}(Y−F) =f^{-1}(Y)−f^{-1}(F)= X−f^{-1}(F) is an H-open in X.

Hence, f^{-1}(F)Is an H-closed set in X.

(iii→i)

Let U is an H-open in Y. then Y−U is an H-closed in Y. by (iii), then f^{-1}(Y−U) is an H-closed. Since f^{-1}(Y−U) =X−f^{-1}(U). then, X−f^{-1}(U) is an H-closed in Y .we have f^{-1}(U) is H-open set in X.

Hence, f is H-irresolute.

2. 2 Proposition:

Let f :X→Y be a function of spaces. Then the following statements

are equivalent:

(i)f is an H-continuous function .

(ii) $f^{-1}(A^{o}) \subseteq f^{-1}(A^{o})^{oH}$ for every subset A of Y .

(iii) f^{-1}(A)is an H-closed set in X for every closed set in X for every

Closed set in X for every closed set A in Y.

(iv) $f(\overline{A}^{H}) \subseteq \overline{f(A)}$ for every subset A of X .

(v) $\overline{f^{-1}(A)}^{H} \subseteq f^{-1}(\overline{A})$ for every subset A of Y .

Proof:

(i→ii)

Let A⊆ X ,since A^{o} is open in Y, then $f^{-1}(A^{o})$is H-open in X ,since $A^{0} \subseteq A$, then $f^{-1}(A^{o}) \subseteq f^{-1}(A)$, thus $f^{-1}(A^{o})=((f^{-1}(A^{o}))^{oH} \subseteq ((f^{-1}(A))^{oH}$.

(ii→iii)

Let A be closed subset of Y, then Y−A is open set in Y ,thus by hypothesis $f^{-1}(Y-A)\subseteq(f^{-1}(Y-A))^{oH}$, then $f^{-1}(Y)-f^{-1}(A) \subseteq (f^{-1}(Y)-f^{-1}(A))^{oH}$ and therefore $(f^{-1}(A))^{c}\subseteq ((f^{-1}(A))^{c})^{oH}$.

Then, $(f^{-1}(A))^{c}$ is an H-open set in X .

Hence $f^{-1}(A)$ is an H-closed set in X.

(iii→iv)

let $A\subseteq X$ .then $f(A)$ is closed set in Y , then by(iii) $f^{-1}(f(A))$ is a

H-closed set in X containing A. thus $\overline{A}^{H}\subseteq f^{-1}(f(A))$. Hence $f(\overline{A}^{H})\subseteq \overline{f(A)}$.

(iv→v)

let $A\subseteq Y$. then by (iv) ,$f(\overline{f^{-1}(A)}^{H})\subseteq \overline{f(f^{-1}(A))}$.

Hence, $\overline{f^{-1}(A)}^{H}\subseteq f^{-1}(\overline{A})$.

(v→i)

suppose that $\overline{f^{-1}(A)}^{H}\subseteq f^{-1}(\overline{A})$ and A is an open set in Y.

then, A is an closed set then.

$A^{c}\subseteq \overline{A^{c}}\Rightarrow f^{-1}(A^{c})\subseteq f^{-1}(\overline{A^{c}})$

$\overline{f(A^{c})}^{H}\subseteq f^{-1}(\overline{A^{c}}) = f^{-1}(A^{c})$

$\Rightarrow \overline{f(A^{c})}^{H}\subseteq f^{-1}(A^{c})$ ... (1)

$f^{-1}(A^{c})\subseteq \overline{f(A^{c})}^{H}$ ... (2)

from (1) and (2) we have that $\overline{f(A^{c})}^{H}= f^{-1}(A^{c})$.

Then, $f^{-1}(A^{c})=X-f^{-1}(A)$ is an H-closed .

Hence, $f^{-1}(A)$ is an H-open then f is an H-continuous function.

2.4 Proposition: Let f: X→Y be an H-continuous (H-irresolute) function and A be an H-open set in X. then the restriction function$f_{/A}$:A→Y is an H-continuous (H-irresolute) function.

Proof:

Let B is an open set in Y, since f is an H-continuous. then, $f^{-1}(B)$ is an H-open set in X. thus $f^{-1}(B) \cap A$ is an H-open set in A.

but, $(f_{/A}(B))^{-1}=f^{-1}(B)\cap A$. Hence, $(f_{/A}(B))^{-1}$is an H-open in A. thus, $f_{/A}$:A→Y is an H-continuous.

2.5 Definition: f is called H-closed function if $f(A)$ is an H-closed set in Y for every closed set A in X .

2.6 Proposition: Let f :X→Y be an H-closed function and g :Y→Z be an H-closed Then gof: X→Z is an H-closed function.

Proof:

Let F be a closed subset of X,then f(F) is an H-closed in Y and then g(f(F) is an H-closed in Z. Hence go$f$: X→Z is an H-closed function.

2.7 Proposition:

Let X and Y be a spaces and f : X→Y be a function .then :

(i) f is H-closed if and only if $\overline{f(A)}^{H} \subseteq f(\overline{A})$ for all A⊆ X .

(iii) if $f(\overline{A}) = \overline{f(A)}^{H}$ for each subset A of X .Then f is a continuous, H-closed ,function.

Proof:

(i) suppose that f is H-closed .Let A⊆ X ,since $\overline{A}$ is a closed, then $\overline{A}$

is H-closed set ,then $f(\overline{A})$ is H-closed in Y. and since A ⊆ $\overline{A}$

then $f(A) \subseteq f(\overline{A})$, by proposition (1.8,iii) thus $\overline{f(A)}^{H} \subseteq \overline{f(\overline{A})}^{H} = f(\overline{A})$,Hence $\overline{f(A)}^{H} \subseteq f(\overline{A})$.

conversely: Let A be any closed set in X .then A=$\overline{A}$,since $\overline{f(A)}^{H} \subseteq f(\overline{A}) = f(A)$, thus $f(A) = \overline{f(A)}^{H}$. Hence f(A) is an H-closed set in Y .therefore f:X→Y is an H-closed f unction.

(ii)

Let F be a closed subset of X, To prove that f is an H-closed function.

Since F=$\overline{F}$, then $f(F) = f(\overline{F}) = \overline{f(F)}^{H}$. Hence f(F) is an H-closed subset of Y. therefore f:X→Y is an H-closed function.

Now, To prove that f is a continuous function. Let B ⊆ X, then $f(\overline{B}) = \overline{f(B)}^{H} = f(\overline{B})$. Thus, f:X→Y is a continuous function.

2.8 Proposition: A bijective function f:X→Y is an H-closed function if and only if f is an H-open function.

Proof:

(→) Let f:X→Y be a bijective ,H-closed ,function and B be an open subset of X, thus $B^c$is closed set. since f is H-closed, then $f(B^c)$ is an

H-closed in Y, thus $(f(B^c))^c$is H-open. since f is a bijective function ,then $f(B^c))^c = f(B)$,hence f(B)is open in Y. therefore f is an H-open function.

(←) by same for (→).

2.9 Proposition: Let X and Y be any spaces and f: X→Y be an H-homeomorphism

Function .Let A⊆ X and B is a H-open subset of Y such that $f(A) = B$ then f_A:A→B is an H-homeomorphism function.

Proof:

Since f is one to one and f_A(A)=f(A)=B, then f_A is bijective function.

Now f_A :A→B is a H- continuous function .To prove f_A: A→B is H-open function.

Let W be any open subset of A ,then there exists open set K in X such that W=A∩K. since f is one to one function .

then . f(A∩K)=f(A)∩ f(K),thus f_A(W)=f(W)=f(A)∩ f(K)= B∩ f(K).

Since f(K)is an H-open set in Y and B is a H-open set in Y, then by proposition (1.4,ii),B∩ f(K) is an H-open set in B. Hence f_A: A→B is an H-open function. Therefore f_A :A→B is an H-homeomorphism function.

References

[1] Fahhil Abbas , "h-open set and H-continuous function. Johannes Kepler University ,Linz-Austria , (2021).

[2] Fahhil Abbas , "h-open set in Topological spaces. Johannes Kepler University ,Linz-Austria , (2021).

[3] AL-Badairy ,M,H " on Feebly proper Action " .M .Sc , Thesis ,University of Al-Mustansiriyah ,(2005).

[4] Bourbaki ,N., Elements of Mathematics , " General Topology" ,Chapter 1-4 ,springer-Verlog ,Berlin, Heidelbery , New-York, London ,Paris ,Tokyo,2^nd Edition(1989).