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Approximation in composition Real Functions Spaces

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ABSTRACT

An approximation in space of continuous functions, as we well known had studied in many books and researches, but approximation in composition of real continuous functions space had not studied previously. In these papers we studied in detail an approximation in this space, where we studied this approximation in terms of existence, uniqueness, and its degree of an element (function) of best approximation of real function that consisting from composition of two real continuous functions. We also noticed that the extent of the impact that make submission two real functions to laws of approximation in this space on the approximation of the composite function from them and degree of that approximation. After that, we studied the relationship between the module of smoothness for the original functions and the resulting function from their composition in different forms of this study and deducing their important relations for them, in the end we studied the possibility of the compact set to providing an element of best approximates of the composite function of two real functions depending on its provision these elements for the original functions which belong in it, then we concluded that this set provides a best approximation element of the function in question also vice versa for original functions.

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1. Introduction:

It is known that an approximation of continuous functions was studied in several books and researches such as [1],[2],[3] ...... etc. Those researchers which Their research mentioned, studied an approximation in space of continuous functions, where they studied for example the degree of that approximation by means of modulus of smoothness or other means without any mention of the subject of the composition of those functions.

After that, some researchers tried to introduce the idea of composition of functions into their studies, such as [4].[5],.....etc. but without any study of the subject of approximation in space of composition functions in their researches, for example the first researcher which studied in his research the approximation in compose special case of functions, that is composite analytic function with an operator in Hilbert space, knowing that the researcher here did not deal in his research to the approximation in the sense known to us and did not use the modulus of smoothness or generally known approximation theorems, but rather limited to theorems that dealt with very special cases of approximation and far from the considered concepts of approximation. In the second research, as in the previous research, the researcher here attempted to approximate a continuous function using a single operator composed from two well-known operators, they are Chlodowsky operator and Sz´azs–Durrmeyer operator. This

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research only refers to approximation in space of compost functions in name only. Thus, there are no studies in subject of approximation of composition real functions. Hence the idea of my current research.

In addition to the above, the idea of the modulus of smoothness of composition real continuous functions has never been studied, neither in the aforementioned references nor in the modern, that have specialized to some extent in studying this specific topic in approximation theory, such as [6],[7].... etc. Those researchers referred to in their research studied the modulus of smoothness in special and general cases without addressing this modulus in the case of compost functions, and this gives an additional reason to study and write this research.

From the above an approximation of the composition real functions was not studied previously. First we recall that the norm of the function $f \in A \subseteq R$ is define as $\|f\| = \underbrace{\sup}_{a\in A} |f(A)|$.also if we assume that $g: A \longrightarrow B$ and $f: B \longrightarrow C$ be two real continuous functions such that an image set of g is a subset of domain of function f then composition of the function g and f which symbolled by the symbol is define as:

$f \circ \mathrm{g}: \mathrm{A} \longrightarrow \mathrm{C}$ where $(f \circ g)(a) = f(g(a)$.It is clear that the function $f \circ \mathrm{g}$ is continuous [8], also since every continuous function is a bounded [1], so $f \circ \mathrm{g}$ is bounded and then it has a norm as:

$\|f \circ \mathrm{g}\| = \underbrace{\sup}_{a\in A} |(f \circ \mathrm{g})(\mathrm{a})| = \underbrace{\sup}_{a\in A} |f(\mathrm{g}(\mathrm{a}))|$.

In these papers we are supposed to compare between norm of original functions f , g and the compost function $f \circ \mathrm{g}$, follow this our conclusion for its differences and modulus of smoothness, same case when we composition more than two functions, not that and as we mention above: $\|(f \circ \mathrm{g})(a)\| = \underbrace{\sup}_{a\in A} |f(\mathrm{g}(\mathrm{a}))|$.

Now, since by definition of composition of functions range set of $g \subseteq$ domain set of f and by definition of function we get:

$n$umber of elements of range of $g \leq$ numer of elements domain of f

So, with above and if we assume that $\mathrm{g}(\mathrm{a}) = b, f\big(\mathrm{g}(\mathrm{a})\big) = f(b), a \in A, b \in B$

then:

$\underbrace{sup}_{a\in A} |f(\mathrm{g}(\mathrm{a}))| \leq \underbrace{sup}_{b\in B} |f(b)|$ and this mean

$\|(f \circ \mathrm{g})(a)\| \leq \|f(b)\|, a \in A , b \in B$............1.1

For the comparison between $\|\mathrm{g}(a)\|, a \in A$ and $\|f(b)\|, b \in B$ Nothing can be deduced.

If we want to study effect the of differences Δ_h (t represents the difference between two values in domain of the function f) is known that:

$\Delta_t(f, a) = f(a + t) - f(a)$ If we continue to find module of continuity of f i.e. $\omega_h(f. a) = \underbrace{\max}_{a\in A} (\underbrace{sup}_{|t|\leq h} \{(|\Delta_h(f, a)|)\}$ as the following, and if we take into consideration composition of two difference on f,then by definition of difference of a function and properties of functions ,the composition of two difference becomes :

$$\begin{aligned} \Delta_t(\Delta_t f(a)) &= \Delta_t\big(f(a + t) - f(a)\big) \\ &= \Delta_t(f(a + t)) - \Delta_t(f(a)) \\ &= f(a + t + t) - f(a + t) - f(a + t) + f(a) \\ &= f(a + t + t) - 2f(a + t) + f(a) \\ &= \Delta_t^2\big(f(a)\big) \qquad \text{.........................1.2} \end{aligned}$$

As we known this is the second difference, if we continue to composition of n difference, then we get $\Delta_t^n\big(f(a)\big)$ say n-difference.

Now, and as above we recall definition of modulus of smoothness as:

$\omega_h(f, a) = \underbrace{max}_{a\in A} \|\Delta_h^r(f, a)\| = \underbrace{max}_{a\in A} (\underbrace{sup}_{|t|\leq h} (|\Delta_h^r(f, a)|))$

As we previously comparisons if we compose two modulus of continuity (or smoothness) for the function f,then by definition of supremum [9], it is clear that this equal to one value only (i.e. effect of differences on this single value) and let's assume that this value is in point $x_\circ$ this mean:

$\underbrace{sup}_{a\in A} \{(|\Delta_h(f, a)|) = |f(a_\circ + t) - f(a_\circ)|, a_\circ \in A$

Then, $\omega_h(\omega_h(f. a)) = \omega_h(\underbrace{max}_{a\in A} \{(\|\Delta_h(f, a)\|)\big)$

$$= \omega_h\left(\underbrace{max}_{a\in A}\left(\underbrace{sup}_{|t|\le h}|f(a+t)-f(a)|\right)\right)$$
$$= \omega_h(\underbrace{max}_{a\in A}(|(|f(a_\circ+t)-f(a_\circ)|)|))$$
$$= \omega_h(|f(a_\circ+t)-f(a_\circ)|)$$
$$= \underbrace{max}_{a\in A}(\underbrace{sup}_{|t|\le h}|f(a_\circ+t)-f(a_\circ)|$$
$$= |f(a_\circ+t)-f(a_\circ)|$$
$$= \underbrace{max}_{a\in A}(\underbrace{sup}_{|t|\le h}|f(a+t)-f(a)| = \omega_h$$

And we refer to this result by ………...1.3

This mean composition of two modulus of continuity is equal to one modulus only, also for modulus of smoothness of our function.

finally for the compact set, and as we known that, it is provide an element of best approximation for a continuous function belong to it (or is located within) [10], So we should ask here, is this property still remains present for a composition for two real continuous functions say f, g belong in it? The answer here is yes, and this by convergent of sequences in our compact set and composition of these sequences and its convergent as we will see later.

2. Auxiliary results:

In this part, firstly we will study relationship between composition of subtraction (or addition) of two functions with a third function, and we will prove that the process of subtraction (or addition) is distributed over the composition of functions. After that, study the relationship between the modules of smoothness of composition of two functions and modulus of smoothness of the original functions, and in the process, we will study an image of modulus of smoothness for the function f and notice the relationship between it and the modulus of the composition function. Also, we will notice during this study existence of many important relationships

2.1 Theorem:

Suppose that $f: B \longrightarrow C$, $g: B \longrightarrow C$,and $P: A \longrightarrow B$ such that rang(P) ⊆domin of (f) and (g) then:
$[(f-g)\circ P](a) = (f\circ P)(a) - (g\circ P)(a)$.

Proof:

Since $[(f-g)\circ P](a) = (f-g)(P(a))$

Suppose that $P(a) = b \in B$ which an element in domain of functions f and g.

So, by properties of functions we have:

$(f-g)(b) = f(b) - g(b)$

And then $f(b) - g(b) = f\big(P(a)\big) - g\big(P(a)\big)$

$$= (f\circ \mathrm{P})(a) - (g\circ \mathrm{P})(a)$$

Thus $[(f-g)\circ P](a) = (f\circ \mathrm{P})(a) - (g\circ \mathrm{P})(a)$

As above clear that proof is complete■

We will prove the following theorem with respect to the modulus of continuity for our chosen functions in order to simplify form of the proof, after that the proof can be generalized:

2.2 Theorem:

Suppose that $f: B \longrightarrow C$, $g: A \longrightarrow B$ such that rang(g) ⊆domin of (f)then:
$\omega_h(f\circ g, a) \le \omega_h(f, b)$ where $a \in A$ and $b \in B$.

Proof:

$$\omega_h(f\circ g, a) = \underbrace{max}_{a\in A}\|\Delta_h(f\circ g, a)\| = \underbrace{max}_{a\in A}(\underbrace{sup}_{|t|\le h}(|\Delta_h(f\circ g, a)|))$$
$$= \underbrace{max}_{a\in A}(\underbrace{sup}_{|t|\le h}(|(f\circ g)(a+h) - (f\circ g)(a)|))$$
$$= \underbrace{max}_{a\in A}(\underbrace{sup}_{|t|\le h}(|f(g(a+h)) - f(g(a)|))$$

Since rang(g) ⊆domin of (f) and by definition of supremum and the maximum of $|f(g(a+h)) - f(g(a)|$ we get:

$\underbrace{max}_{a\in A}(\underbrace{sup}_{|t|\le h}(f\circ g)(a) = \underbrace{max}_{a\in A}(\underbrace{sup}_{|t|\le h}(f(g(a)))$ So, and like we make in the introduction, it must be in one point say $a_\circ \in A$ then the last term is become:

$$= |f(g(a_\circ + h)) - f(g(a_\circ)|$$

First suppose that $g(a_\circ) = b_\circ \in B$ and $h = \Delta a$,We note $b_\circ$ is one point in B.
Second by use the hypothesis in calculus that
$f(a+\Delta a) = b + \Delta b$ where $f(a) = b$
So, the last term becomes:

$$= |f(b_\circ + \Delta b) - f(b_\circ)|$$

Since $b_\circ \in B$, So there exist $b \neq b_\circ, b \in B$ (and by existence of absolute value) the last term becomes:

$$\le |f(b+\Delta b) - f(b)|$$

If we compare between Δb and $h = \Delta a$ we get the following cases, if $\Delta b > h$ (in general) then result become greater than, or if $\Delta b < h$ by existence of absolute value the result become greater than, also if $\Delta y = h$ it is clear. of course, note that all the previous cases are true whether the function f is increasing or decreasing, so for the last term we have:

$$\le |f(b+h) - f(b)|$$
$$= |\Delta_h(f,b)| \le \underbrace{sup}_{|t|\le h}(|\Delta_h(f,b)|)$$
$$\le \underbrace{max}_{a\in A}\left(\underbrace{sup}_{|t|\le h}(|\Delta_h(f,b)|)\right)$$
$$= \underbrace{max}_{a\in A}\|\Delta_h(f,b)\| = \omega_h(f,b).$$

The proof can be generalized to the modulus of smoothness for our functions,
So, $\underbrace{max}_{a\in A}\|\Delta_h^r(f\circ g,a)\| = \omega_h(f\circ g,a)$

$$\le \omega_h(f,b) = \underbrace{max}_{a\in A}\|\Delta_h^r(f,b)\|$$

Clear that, we completed the proof ■
In the following theorem we compare between $f(\omega_r(g,a))$ which belong in C and $\omega_h(fog,a)$ which also belong in C.

2.3 <u>Theorem:</u>

Suppose that $f: B \longrightarrow C$, $g: A \longrightarrow B$ such that rang(g) ⊆domin of (f)then: $f(\omega_h(g,a)) \le \omega_h(f\circ g,a)$ where $a \in A$ and $b \in B$.

<u>Proof:</u>
Since:

$$\omega_r(g,a) = \underbrace{max}_{a\in A}(\underbrace{sup}_{|t|\le h}|\Delta_h^r(f,a)| = \underbrace{max}_{a\in A}(\underbrace{sup}_{|t|\le h}|\textstyle\sum_{k=0}^{r}(-l)^{r-k}\binom{r}{k}g(a+kt)|).$$

As before, and by properties of supremum the last term is in one pointe say $a_\circ \in A$ and since $t \in R$ and by use properties of Absolut value, the last term becomes:

$$= |\textstyle\sum_{k=0}^{r}(-l)^{r-k}\binom{r}{k}g(a_\circ + kt)|$$

So:

$$f(\omega_r(g,a)) = f(|\textstyle\sum_{k=0}^{r}(-l)^{r-k}\binom{r}{k}g(a_\circ + kt)|)$$
$$\le |f(\textstyle\sum_{k=0}^{r}\left(-l)^{r-k}\binom{r}{k}g(a_\circ + kt)\right)|$$

The follow result can be gotten If f is convex (or concave) function, so by use properties of absolute value and properties of convexity (or concavity) the last term becomes:

$$\le |\textstyle\sum_{k=0}^{r}(-l)^{r-k}\binom{r}{k}f(g(a_\circ + kt)))|$$
$$= |\textstyle\sum_{k=0}^{r}(-l)^{r-k}\binom{r}{k}(f\circ g)(a_\circ + kt)))|$$
$$= \underbrace{\sup}_{|t|\le h}|\textstyle\sum_{k=0}^{r}(-l)^{r-k}\binom{r}{k}(f\circ g)(a + kt)))|$$
$$= \underbrace{\sup}_{|t|\le h}|\Delta_r^h(f\circ g,x)| \le \underbrace{max}_{x\in X}(\underbrace{\sup}_{|t|\le h}|\Delta_r^h(f\circ g,a)|)$$
$$= \underbrace{max}_{x\in X}\|\Delta_h^r(f\circ g,x)\| = \omega_h(f\circ g,a)$$

Clear that, the proof is complete■

2.4 Theorem:

Suppose that $f: B \longrightarrow C$, $g: A \longrightarrow B$ then $\omega_h(f \circ g, a) \le \omega_h(g, a) + \omega_h(f, b)$ where $a \in A$ and $b \in B$.

Proof:

By use theorem 2.2 $\omega_h(f \circ g, a) \le \omega_h(f, b)$ and since by definition of modulus of smoothness $\omega_h(g, a) \ge 0$,so we will get:

$\omega_h(f \circ g, a) \le \omega_h(f, b) \le \omega_h(g, a) + \omega_h(f, b)$

And the proof is complete■

3. *Main Results:*

In this part: First, we will study the relationship between degree of approximation of composition two continuous real functions and a degree of approximation of the original functions. After that, we will study a possibility of providing the compact set to a best approximation element to the composite function, since and as is known to us that, the compact set provides an element of best approximation for the continuous functions which belong in it, so we will study this hypothesis here in detail.

3.1. Theorem: [1]

for each continuous function $f \in C[-1,1]$ there exists a constant M and a sequence of polynomials $P_n(x)$ for which:$|f(x) - P_n(x)| \le M\omega_h(f, \Delta_n x)$.

The proof of this theorem can be generalized for any period $[a, b]$.

3.2 Theorem:

Suppose that $g: A \longrightarrow B$ and $f: B \longrightarrow C$ be two real continuous functions such that P_1 and P_2 (resp.) be a best approximation element of f and g (resp.) then:

$\|(f \circ g)(a) - (P_1 \circ P_2)(a)\| \le 3\omega_r(f, b)$, $a \in A, b \in B$.

Proof:

$\|(f \circ g)(a) - (P_1 \circ P_2)(a)\|$

$= \|(f \circ g)(a) - (P_1 \circ \mathrm{g})(a) + (P_1 \circ \mathrm{g})(a) - (P_1 \circ P_2)(a)\|$

$\le \|(f \circ g)(a) - (P_1 \circ \mathrm{g})(a)\| + \|(P_1 \circ \mathrm{g})(a) - (P_1 \circ P_2)(a)\|$

By use theorem 2.1 we get:

$= \|(f - P_1) \circ \mathrm{g}\| + \|P_1 \circ (\mathrm{g} - P_2)\|$

$= \|(f - P_1)(\mathrm{g(a)})\| + \|P_1((\mathrm{g} - P_2)(a))\|$

Again, by use theorem 2.1 in first part only, the last term becomes:

$= \|(f(\mathrm{g(a)}) - P_1(\mathrm{g(a)}))\| + \|P_1((\mathrm{g} - P_2)(a))\|$

Suppose that $\mathrm{g(a)} = b$ so the last term becomes:

$= \|f(\mathrm{b}) - P_1(\mathrm{b})\| + \|P_1((\mathrm{g} - P_2)(a))\|$

$= \|f(\mathrm{b}) - P_1(\mathrm{b})\| + \|P_1(\mathrm{g(a)} - P_2(a))\|$

By use $\mathrm{g(a)} = b$ and assume that $P_2(a) = \grave{b} \in B$ the last term becomes:

$= \|f(\mathrm{b}) - P_1(\mathrm{b})\| + \left\|P_1(\mathrm{b}) - P_1(\grave{b}))\right\|$

$= \|f(\mathrm{b}) - P_1(\mathrm{b})\| + \left\|\left(f(b) - P_1(\grave{b}) - f(b) + P_1(b)\right\|$

$\le \|f(\mathrm{b}) - P_1(\mathrm{b})\| + \left\|\left(f(b) - P_1(\grave{b})\right\| + \|f(b) - P_1(b)\|$

Since by definition of norm we have:

$\left\|\left(f(b) - P_1(\grave{b})\right\| = \underbrace{\sup}_{b,\grave{b}\in B} \left|f(b) - P_1(\grave{b})\right|$

Since $\grave{b}$ is a single point *in* B

So, $\underbrace{\sup}_{b,\grave{b}\in B} |f(b) - P_1(\grave{b}| \le \underbrace{\sup}_{b,\grave{b}\in B} |f(b) - P_1(b)|$ for any two functions, So:

$\left\|\left(f(b) - P_1(\grave{b})\right\| \le \|(f(b) - P_1(b)\|$ and then the last term becomes:

$\le \|f(\mathrm{b}) - P_1(\mathrm{b})\| + \|(f(b) - P_1(b)\| + \|f(b) - P_1(b)\|$

By use theorem 3.1we get for the last term:

$\le 3M\omega_r(f, b)$

Clear that, we completed proof ■

3.3 Theorem:

Suppose that $g: A \longrightarrow B$ and $f: B \longrightarrow C$ be two real continuous functions such that $g(a) \in M \subseteq B$ where the set M is a compact set, then $f(M)$ provide an element of best approximation of the compost function $f \circ g$.

Proof:
Since M is a compact, then M *provide* an element of best approximation of the function g.
Then by properties of the compact set, there exists a sequence $\propto_i \in B$, $\forall i \in I$
such that $\propto_i \longrightarrow g \subseteq M \subseteq B$ As $i \longrightarrow \infty$.
Since f be a continuous function.
Then, $f(\propto_i) \longrightarrow f(g) \subseteq f(M) \subseteq f(B)$ As $i \longrightarrow \infty$.
Again, since f is a continuous, then $f(M)$ be a compact set.
So, $f(M)$ be a compact set and there exist a sequence $f(\propto_i) \in f(M) \forall i \in I$ such that
$f(\propto_i) \longrightarrow \big(f(g)\big)(a) = (f \circ g)(a)$ As $i \longrightarrow \infty$. This means the compact set $f(M)$ proved an element of best approximation of composition of two real continuous function f, g (i.e. $f \circ g$)
As above we complete proof■

4.Conclusions

Since the real composite function consists of composition of two real functions (or more), the important conclusion here is that an approximation of this function will be affected by the approximation of the functions that compose it and vice versa. This is what was proven in this research, where we studied this approximation in terms of existence, degree, and the polynomial (polynomials) of the best approximation.

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