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Study a New Subclasses of Analytic Bi-Univalent Functions Defined by New Integral Operator

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ABSTRACT

This paper aims to introduce and examine specific subclass $\mathcal{K}_{\Sigma}^{\mathfrak{s},\mathfrak{t}}(\alpha,\mu,\mathfrak{p},\mathfrak{q})$ and $\mathcal{H}_{\Sigma}^{\mathfrak{s},\mathfrak{t}}(\beta,\mu,\mathfrak{p},\mathfrak{q})$ of analytic and bi-univalent functions defined in the open unit disk U, linked to a new integral operator $\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}$. We examine the upper bounds for the initial Taylor-Maclaurin coefficients $|a_2|$ also $|a_3|$ for functions these subclasses.

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1. Introduction

We refer to Δ to be the class of functions that are analytic within the open unit disk $U=\{z\in\mathbb{C} : |z|<1\}$ and have the subsequent normalization from:

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n. \tag{1.1}$$

We refer via $\varXi$ the subclass of Δ including functions that are univalent in U. As per the Koebe One-Quarter theorem [22], all function $f\in\varXi$ possesses an inverse f^{-1} characterized via

$$f^{-1}\big(f(z)\big)=z,\ (z\in U)$$

also

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$$f(f^{-1}(w)) = w, \qquad \left(|w| < r_0(f), r_0(f) \geq \tfrac{1}{4}\right),$$

location

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \quad (1.2)$$

A function $f \in \Delta$ is said to be bi-univalent in U if both of them f also f^{-1} are univalent in U. We refer to Σ the class of bi-univalent functions in U characterized as given in (1.1).

Furthermore, in discussing geometric function theory, it is imperative to address the examination of operators that are fundamental to mathematics in general and particularly within geometric function theory. Dziok and Srivastava [15], along with Libera and Zlatkiewicz [28], introduced an integral operator and investigated particular properties of starlike functions under this operator. Sălăgean, in 1983 [10], studied the class of analytic functions defined by differential also linear operators, which are significant in geometric function theory. The estimation of coefficients for bi-univalent functions is a significant and intriguing aspect of the geometric function theory of analytic functions, which plays a crucial role in this domain.

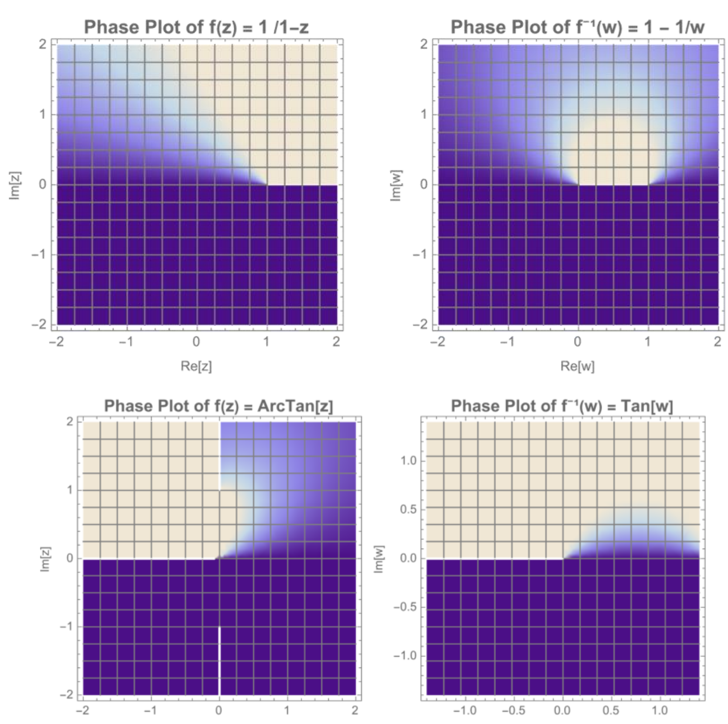
In 1967, Lewin [18] established the concept of the family Σ of bi-univalent analytic functions. He demonstrated that $|a_2| \leq 1.51$ for each function $f \in \Sigma$. Brannan and Clunie [6] subsequently postulate that $|a_2| \leq \sqrt{2}$. The most recognized estimate for functions in Σ was obtained by Tan in 1984 [9], $|a_2| \leq 1.487$. For a concise history and notable examples within the family Σ, refer to the seminal do tasks by Srivastava et al. [14], which has revitalized the examination of bi-univalent functions in recent years.

In this paper, we introduced a novel subclasses $\mathcal{K}_{\Sigma}^{\mathfrak{s},\mathfrak{t}}(\alpha, \mu, \mathfrak{p}, \mathfrak{q})$ and $\mathcal{H}_{\Sigma}^{\mathfrak{s},\mathfrak{t}}(\beta, \mu, \mathfrak{p}, \mathfrak{q})$ of bi-univalent functions in the open unit disk U and also we estimated the coefficients $|a_2|$ and $|a_3|$ by using a new integral operator $\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}$.

Example 1. The well-known instances regarding this class are as detailed below:

1. The inverse of function $\frac{1}{1-z}$ is $1 - \frac{1}{w}$.
2. The inverse of function $\tan^{-1} z$ is $\tan w$.
3. The inverse of function $\log(z^2 + 1)$ is $\sqrt{e^w - 1}$.

The following images (**Figure 1**) clarify the notion of the analytic function and its equivalent inverse. This also signifies that structure (1.2) is fulfilled.



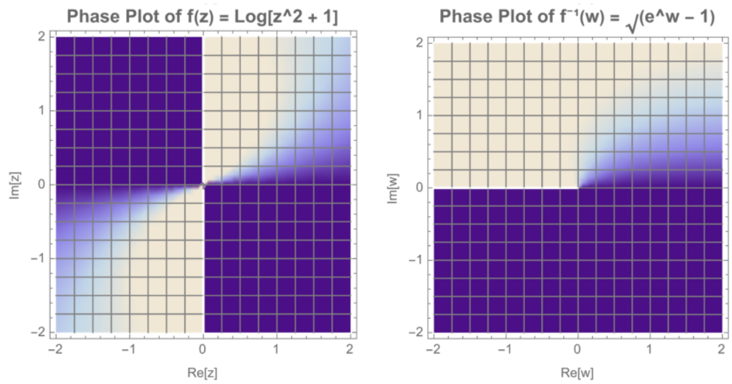


Figure 1. Graphs of the analytic function and its inverse.

A substantial number of sequels to the work of Srivastava et al. [12] have introduced also examined various subclasses of the bi-univalent function class Σ by numerous authors (for instance, [5,8,15,16,19,20,21,34,35,336,37,38]); however, many of these recent publications have solely yielded non-sharp estimates for the preliminary coefficients $|a_2|$ also $|a_3|$ in the Taylor-Maclaurin expansion (1.1). The issue of determining the universal boundaries for the Taylor-Maclaurin coefficients.

$$|a_n|\ (\, n \in \mathbb{N} \setminus \{1,2\}; \mathbb{N} \coloneqq \{1,2,3,\ldots\}),$$

the issue concerning functions $f \in \Sigma$ remains inadequately resolved for numerous subclasses of the bi-univalent function class Σ (see, for instance, [4,11,13,23,33]).

In [3], Frasin introduced the subclass $\mathcal{G}(\alpha, \mathfrak{s}, \mathrm{t})$ of analytic functions f fulfilling the subsequent criterion:

$$\mathcal{R}e\left\{\frac{(\mathfrak{s}-t)zf'(\mathfrak{s}z)}{f(\mathfrak{s}z)-f(\mathrm{t}z)}\right\} > \alpha,$$

for certain individuals $0 \le \alpha < 1$ $\mathfrak{s}, t \in \mathbb{C}$ accompanied by $|\mathfrak{s}| \le 1$; $|t| \le 1$; $\mathfrak{s} \ne \mathrm{t}$ also for everyone $z \in U$.

Lemma 1.1 . The operator of $f \in \Delta$, for $0 < \mathfrak{p} < 1$, $\mathfrak{q} \ge 1$ is denoted by $\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}$ and we defined it as following :

$$\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}} : \Delta \to \Delta,$$

$$\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}} f(z) = \frac{1}{\Gamma(\mathfrak{p})\left(\frac{1-\mathfrak{p}}{1+\mathfrak{q}}\right)^{\mathfrak{p}+1}} \int_0^\infty t^{\mathfrak{p}-1} e^{-\left(\frac{t(1+\mathfrak{q})}{1-\mathfrak{p}}\right)} f(zt)dt.$$

Proof:

$$\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}} f(z) = \frac{1}{\Gamma(\mathfrak{p})\left(\frac{1-\mathfrak{p}}{1+\mathfrak{q}}\right)^{\mathfrak{p}+1}} \int_0^\infty t^{\mathfrak{p}-1} e^{-\left(\frac{t(1+\mathfrak{q})}{1-\mathfrak{p}}\right)} f(zt)dt$$

$$= \frac{1}{\Gamma(\mathfrak{p})\left(\frac{1-\mathfrak{p}}{1+\mathfrak{q}}\right)^{\mathfrak{p}+1}} \int_0^\infty t^{\mathfrak{p}-1} e^{-\left(\frac{t(1+\mathfrak{q})}{1-\mathfrak{p}}\right)} \left[zt + \sum_{n=2}^{\infty} a_n z^n\, t^n\right] dt$$

$$=\frac{1}{\Gamma(\mathfrak{p})\left(\frac{1-\mathfrak{p}}{1+\mathfrak{q}}\right)^{\mathfrak{p}+1}}\left[z\int_0^\infty t^{\mathfrak{p}}e^{-\left(\frac{t(1+\mathfrak{q})}{1-\mathfrak{p}}\right)}dt+\sum_{n=2}^{\infty}a_n z^n\int_0^\infty t^{\mathfrak{p}+n-1}e^{-\left(\frac{t(1+\mathfrak{q})}{1-\mathfrak{p}}\right)}dt\right]$$

Let $x=\frac{t(1+\mathfrak{q})}{1-\mathfrak{p}}\to t(1+\mathfrak{q})=x(1-\mathfrak{p})$, then if $t=0$, we get $x=0$, $t=\infty$, we get $x=\infty$, then $dt(1+\mathfrak{q})=(1-\mathfrak{p})dx$. Thus

$$\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(z)=\frac{1}{\Gamma(\mathfrak{p})\left(\frac{1-\mathfrak{p}}{1+\mathfrak{q}}\right)^{\mathfrak{p}+1}}\left[z\int_0^\infty\left(\frac{1-\mathfrak{p}}{1+\mathfrak{q}}\right)^{\mathfrak{p}}x^{\mathfrak{p}}e^{-(x)}\left(\frac{1-\mathfrak{p}}{1+\mathfrak{q}}\right)dx\qquad+\sum_{n=2}^{\infty}a_n z^n\int_0^\infty\left(\frac{1-\mathfrak{p}}{1+\mathfrak{q}}\right)^{\mathfrak{p}+n-1}x^{\mathfrak{p}+n-1}\,e^{-(x)}\left(\frac{1-\mathfrak{p}}{1+\mathfrak{q}}\right)dx\right]$$

$$=\frac{1}{\Gamma(\mathfrak{p})\left(\frac{1-\mathfrak{p}}{1+\mathfrak{q}}\right)^{\mathfrak{p}+1}}\left[z\left(\frac{1-\mathfrak{p}}{1+\mathfrak{q}}\right)^{\mathfrak{p}+1}\Gamma(\mathfrak{p})+\sum_{n=2}^{\infty}a_n z^n\left(\frac{1-\mathfrak{p}}{1+\mathfrak{q}}\right)^{\mathfrak{p}+n}\Gamma(\mathfrak{p}+n)\right]$$

$$=z+\sum_{n=2}^{\infty}\mathfrak{D}(\mathfrak{p},\mathfrak{q},n)a_n z^n,$$

where

$$\mathfrak{D}(\mathfrak{p},\mathfrak{q},n)=\frac{\Gamma(\mathfrak{p}+n)}{\Gamma(\mathfrak{p})}\left(\frac{1-\mathfrak{p}}{1+\mathfrak{q}}\right)^{n-1}. \qquad \square$$

We now reiterate the subsequent lemma that shall be utilized to substantiate our principal results.

Lemma 1.2 [22]. If $j\in\beth$, subsequently $|c_\varkappa|\le 2$ for each $\varkappa\in\mathbb{N}$, in which denoted by $\beth$ is the class of all functions j analytic in U for which

$$\mathcal{R}e\big(j(z)\big)>0,(z\in U),$$

Location

$$j(z)=1+c_1z+c_2z^2+\cdots,(z\in U). \qquad \square$$

Unless stated otherwise, we assume across this document that

$$\mathfrak{s},\mathfrak{t}\in\mathbb{C}\text{ with }|\mathfrak{s}|\le 1;|\mathfrak{t}|\le 1;\mathfrak{s}\ne\mathfrak{t};0<\alpha\le 1;0\le\beta<1;0<\mu\le 1;0<\mathfrak{p}<1\,;\ \mathfrak{q}\ge 1.$$

2. Bounds on Coefficients for the Class $\mathcal{K}_{\Sigma}^{\mathfrak{s},\mathfrak{t}}(\alpha,\mu,\mathfrak{p},\mathfrak{q})$

Definition 2.1. A function $f\in\Sigma$ defined accompanied by (1.1) is classified within the category $\mathcal{K}_{\Sigma}^{\mathfrak{s},\mathfrak{t}}(\alpha,\mu,\mathfrak{p},\mathfrak{q})$ if it fulfills the subsequent criteria:

$$\left|\arg\left(\frac{(\mathfrak{s}-\mathfrak{t})z\left(\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(z)\right)'}{\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(\mathfrak{s}z)-\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(\mathfrak{t}z)}+(1-\mu)z\left(\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(z)\right)''\right)\right|<\frac{\alpha\pi}{2}\,,(0<\alpha\le 1;\ z\in U),\quad(2.1)$$

also

$$\left|\arg\left(\frac{(\mathfrak{s}-\mathfrak{t})w\left(\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(w)\right)'}{\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(\mathfrak{s}w)-\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(\mathfrak{t}w)}+(1-\mu)w\left(\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(w)\right)''\right)\right|<\frac{\alpha\pi}{2},(0<\alpha\leq 1;\ w\in U). \quad (2.2)$$

Theorem 2.1 delineates our primary finding.

Theorem 2.1. Let $f\in\mathcal{K}_{\Sigma}^{\mathfrak{s},\mathfrak{t}}(\alpha,\beta,\mu,\mathfrak{p},\mathfrak{q})$ be given by (1.1). Then

$$|a_2|\leq\frac{2\mu\sqrt{\alpha}}{\sqrt{m|(6-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})-2(2\mathfrak{s}+2\mathfrak{t}-\mathfrak{s}^2-t^2-2\mathfrak{t}\mathfrak{s})\mu^2-\alpha(\alpha-1)(4-\mathfrak{s}-\mathfrak{t}-2\mu)^2|}}$$

and

$$|a_3|\leq\frac{2\alpha}{(3-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s}-6\mu)k}+\frac{2\mu^2(6-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})}{(3-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s}-6\mu)(4-\mathfrak{s}-\mathfrak{t}-2\mu)^2k},$$

where

$$\mathfrak{m}=\mathfrak{D}(\mathfrak{p},\mathfrak{q},2),\quad k=\mathfrak{D}(\mathfrak{p},\mathfrak{q},3).$$

Proof. It follows from conditions (2.1) and (2.2) that

$$\left[\frac{(\mathfrak{s}-\mathfrak{t})z\left(\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(z)\right)'}{\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(\mathfrak{s}z)-\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(\mathfrak{t}z)}+(1-\mu)z\left(\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(z)\right)''\right]=[\mathfrak{u}(z)]^{\alpha} \quad (2.3)$$

and

$$\left[\frac{(\mathfrak{s}-\mathfrak{t})w\left(\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(w)\right)'}{\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(\mathfrak{s}w)-\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(\mathfrak{t}w)}+(1-\mu)w\left(\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(w)\right)''\right]=[\mathfrak{v}(w)]^{\alpha} \quad (2.4)$$

Location $g=f^{-1}$ also $\mathfrak{u},\mathfrak{v}$ in $\mathcal{P}$ possess the subsequent series representations:

$$\mathfrak{u}(z)=1+\mathfrak{u}_1z+\mathfrak{u}_2z^2+\mathfrak{u}_3z^3+\cdots \quad (2.5)$$

and

$$\mathfrak{v}(w)=1+\mathfrak{v}_1w+\mathfrak{v}_2w^2+\mathfrak{v}_3w^3+\cdots. \quad (2.6)$$

By equating the corresponding coefficients of (2.3) and (2.4), we determine that

$$(4-\mathfrak{s}-\mathfrak{t}-2\mu)ma_2=\alpha\mathfrak{u}_1, \quad (2.7)$$

$$[(3-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s}-6\mu)ka_3-(2\mathfrak{s}+2\mathfrak{t}-\mathfrak{s}^2-t^2-2\mathfrak{t}\mathfrak{s})m^2a_2^2]=\alpha\mathfrak{u}_2+\frac{\alpha(\alpha-1)}{2}\mathfrak{u}_1^2, \quad (2.8)$$

$$-(4-\mathfrak{s}-\mathfrak{t}-2\mu)ma_2=\alpha\mathfrak{v}_1 \quad (2.9)$$

and

$$\left[\left((6-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})-2(2\mathfrak{s}+2\mathfrak{t}-\mathfrak{s}^2-t^2-2\mathfrak{t}\mathfrak{s})\right)m^2a_2^2-(3-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s}-6\mu)ka_3\right]$$

$$=\alpha\mathfrak{v}_2+\frac{\alpha(\alpha-1)}{2}\mathfrak{v}_1^2. \quad (2.10)$$

Utilizing (2.7) and (2.9), we derive

$$\mathfrak{u}_1 = -\mathfrak{v}_1 \tag{2.11}$$

and

$$2(4-\mathfrak{s}-\mathfrak{t}-2\mu)^2 m^2 a_2^2 = \mu^2(\mathfrak{u}_1^2+\mathfrak{v}_1^2). \tag{2.12}$$

If we add (2.8) to (2.10), we obtain

$$\big((6-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})-2(2\mathfrak{s}+2\mathfrak{t}-\mathfrak{s}^2-t^2-2\mathfrak{t}\mathfrak{s})\big)m^2a_2^2 = \alpha(\mathfrak{u}_2+\mathfrak{v}_2)+\frac{\alpha(\alpha-1)}{2}(\mathfrak{u}_1^2+\mathfrak{v}_1^2\ ). \tag{2.13}$$

From (2.12), we conclude that

$$a_2^2 = \frac{\alpha\mu^2(\mathfrak{u}_2+\mathfrak{v}_2)}{[(6-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})-2(2\mathfrak{s}+2\mathfrak{t}-\mathfrak{s}^2-t^2-2\mathfrak{t}\mathfrak{s})\mu^2-\alpha(\alpha-1)(4-\mathfrak{s}-\mathfrak{t}-2\mu)^2]m^2}. \tag{2.14}$$

By calculating the modulus of (2.14) also utilizing Lemma 1.1 regarding the coefficients $\mathfrak{u}_2$ also $\mathfrak{v}_2$, we derive

$$|a_2| \le \frac{2\mu\sqrt{\alpha}}{\sqrt{m|(16-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})-2(2\mathfrak{s}+2\mathfrak{t}-\mathfrak{s}^2-t^2-2\mathfrak{t}\mathfrak{s})\mu^2-\alpha(\alpha-1)(4-\mathfrak{s}-\mathfrak{t}-2\mu)^2|}}$$

To determine the limit on $|a_3|$, by deducting (2.10) from (2.8), we derive

$$2(3-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s}-6\mu)ka_3-(6-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})m^2a_2^2 = \alpha(\mathfrak{u}_2-\mathfrak{v}_2)+\frac{\alpha(\alpha-1)}{2}(\mathfrak{u}_1^2-\mathfrak{v}_1^2). \tag{2.15}$$

It follows from (2.11), (2.12) and (2.15) that

$$a_3 = \frac{\mu^2(6-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})(\mathfrak{u}_1^2+\mathfrak{v}_1^2)}{4(3-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s}-6\mu)(4-\mathfrak{s}-\mathfrak{t}-2\mu)^2k}+\frac{\alpha(\mathfrak{u}_2-\mathfrak{v}_2)}{2(3-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s}-6\mu)k}. \tag{2.16}$$

Calculating the modulus of (2.16) also reapplying Lemma 1.1 for the coefficients $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{v}_1$ also $\mathfrak{v}_2$, we derive

$$|a_3| \le \frac{2\alpha}{(3-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s}-6\mu)k}+\frac{2\mu^2(6-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})}{(3-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s}-6\mu)(4-\mathfrak{s}-\mathfrak{t}-2\mu)^2k},$$

This concludes the proof of Theorem 2.1. □

3. Bounds on Coefficients for the Class $\mathcal{H}_{\Sigma}^{\mathfrak{s},\mathfrak{t}}(\beta,\mu,\mathfrak{p},\mathfrak{q})$

Definition 3.1. A function $f \in \Sigma$ As stated in (1.1), it is classified inside the category $\mathcal{H}_{\Sigma}^{\mathfrak{s},\mathfrak{t}}(\beta,\mu,\mathfrak{p},\mathfrak{q})$ if it fulfills the subsequent criteria:

$$\mathcal{R}e\left\{\frac{(\mathfrak{s}-\mathfrak{t})z\left(\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(z)\right)'}{\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(\mathfrak{s}z)-\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(\mathfrak{t}z)}+(1-\mu)z\left(\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(z)\right)''\right\}>\beta\ ,(0\leq\beta<1;\ z\in U)\,, \tag{3.1}$$

and

$$\mathcal{R}e\left\{\frac{(\mathfrak{s}-\mathfrak{t})w\left(\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(w)\right)'}{\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(\mathfrak{s}w)-\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(\mathfrak{t}w)}+(1-\mu)w\left(\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(w)\right)''\right\}>\beta\,,(0\leq\beta<1;\ w\in U). \tag{3.2}$$

Theorem 3.1 below articulates our second fundamental result. □

Theorem 3.1. Consider the function $f\in\mathcal{H}_{\Sigma}^{\mathfrak{s},\mathfrak{t}}(\beta,\mu,\mathfrak{p},\mathfrak{q})$ be provided by (1.1). We have

$$|a_2|\leq\sqrt{\frac{(1-\beta)}{m|[(6-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})-2(2\mathfrak{s}+2\mathfrak{t}-\mathfrak{s}^2-t^2-2\mathfrak{t}\mathfrak{s})]|}}\,, \tag{3.3}$$

and

$$|a_3|\leq\frac{2(1-\beta)^2(6-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})}{(3-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s}-6\mu)(4-\mathfrak{s}-\mathfrak{t}-2\mu)^2n}+\frac{2(1-\beta)}{(3-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s}-6\mu)n}\,. \tag{3.4}$$

Proof. Conditions (3.1) and (3.2) imply the existence of $\mathfrak{u},\mathfrak{v}\in\mathcal{P}$ in such a manner that

$$\frac{(\mathfrak{s}-\mathfrak{t})z\left(\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(z)\right)'}{\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(\mathfrak{s}z)-\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(\mathfrak{t}z)}+(1-\mu)z\left(\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(z)\right)''=\beta+(\ 1-\beta)\mathfrak{u}(z) \tag{3.5}$$

and

$$\frac{(\mathfrak{s}-\mathfrak{t})w\left(\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(w)\right)'}{\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(\mathfrak{s}w)-\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(\mathfrak{t}w)}+(1-\mu)w\left(\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}f(w)\right)''\quad=\beta+(\ 1-\beta)\mathfrak{v}(w), \tag{3.6}$$

where $\mathfrak{u}(z)$ and $\mathfrak{v}(w)$ are defined by equations (2.5) and (2.8), respectively. Equating the coefficients of (3.5) and (3.6) we get

$$(4-\mathfrak{s}-\mathfrak{t}-2\mu)ma_2=(1-\beta)\mathfrak{u}_1, \tag{3.7}$$

$$[(3-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s}-6\mu)ka_3-(2\mathfrak{s}+2\mathfrak{t}-\mathfrak{s}^2-t^2-2\mathfrak{t}\mathfrak{s})m^2a_2^2]=(1-\beta)\mathfrak{u}_2, \tag{3.8}$$

$$-(4-\mathfrak{s}-\mathfrak{t}-2\mu)ma_2\ =(1-\beta)\mathfrak{v}_1 \tag{3.9}$$

and

$$\left[\left((6-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})-2(2\mathfrak{s}+2\mathfrak{t}-\mathfrak{s}^2-t^2-2\mathfrak{t}\mathfrak{s})\right)m^2a_2^2-(3-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s}-6\mu)ka_3\right]=(1-\beta)\mathfrak{v}_2. \tag{3.10}$$

From (3.7) and (3.9), we get

$$\mathfrak{u}_1=-\mathfrak{v}_1 \tag{3.11}$$

and

$$2(4-\mathfrak{s}-\mathfrak{t}-2\mu)^2 m^2 a_2^2=(1-\beta)^2(\mathfrak{u}_1^2+\mathfrak{v}_1^2). \tag{3.12}$$

Adding (3.8) and (3.10), we obtain

$$\big((6-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})-2(2\mathfrak{s}+2\mathfrak{t}-\mathfrak{s}^2-t^2-2\mathfrak{t}\mathfrak{s})\big)m^2 a_2^2=(1-\beta)(\mathfrak{u}_2+\mathfrak{v}_2). \tag{3.13}$$

Therefore, we have

$$\mathrm{a}_2^2=\frac{(1-\beta)(\mathfrak{u}_2+\mathfrak{v}_2)}{m^2[(6-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})-2(2\mathfrak{s}+2\mathfrak{t}-\mathfrak{s}^2-t^2-2\mathfrak{t}\mathfrak{s})]}.$$

Utilizing Lemma 1.1 regarding the coefficients $\mathfrak{u}_2$ also $\mathfrak{v}_2$, we derive

$$|a_2|\le\sqrt{\frac{(1-\beta)}{m|[(6-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})-2(2\mathfrak{s}+2\mathfrak{t}-\mathfrak{s}^2-t^2-2\mathfrak{t}\mathfrak{s})]|}}.$$

This provides the requisite estimate regarding $|a_2|$ as stated in (3.3).

To ascertain the limit on $|a_3|$, we deduct equation (3.10) from the equation (3.8), yielding

$$2(3-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s}-6\mu)k a_3-(6-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})m^2 a_2^2=(1-\beta)(\mathfrak{u}_2-\mathfrak{v}_2),$$

By replacing the value of a_2^2 from (3.12), it may be inferred that

$$a_3=\frac{2(1-\beta)^2(6-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})(\mathfrak{u}_1^2+\mathfrak{v}_1^2)}{(3-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s}-6\mu)(4-\mathfrak{s}-\mathfrak{t}-2\mu)^2 k}+\frac{2(1-\beta)}{(3-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s}-6\mu)k}. \tag{3.14}$$

Reapplying Lemma 1.1 for the coefficients $\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{v}_1$ and $\mathfrak{v}_2$, we obtain

$$|a_3|\le\frac{2(1-\beta)^2(6-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})}{(3-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s}-6\mu)(4-\mathfrak{s}-\mathfrak{t}-2\mu)^2 k}+\frac{2(1-\beta)}{(3-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s}-6\mu)k},$$

This concludes the proof of Theorem 3.1. □

4. Corollaries and Conclusions

By placing $\mu=1$ from Theorem (2.1), we derive the subsequent corollary:

Corollary 1. If $f(z)\in\mathcal{K}_\Sigma^{\mathfrak{s},\mathfrak{t}}(\alpha,\beta,1,\mathfrak{p},\mathfrak{q})$ defined in (1.1), then we have

$$|a_2|\le\frac{2\sqrt{\alpha}}{\sqrt{m|(6-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})-2(2\mathfrak{s}+2\mathfrak{t}-\mathfrak{s}^2-t^2-2\mathfrak{t}\mathfrak{s})-\alpha(\alpha-1)(2-\mathfrak{s}-\mathfrak{t})^2|}},$$

and

$$|a_3|\le\frac{2\alpha}{(-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s}-3)k}+\frac{2(6-2\mathfrak{s}^2-2\mathfrak{t}-2\mathfrak{t}\mathfrak{s})}{(-3-\mathfrak{s}^2-t^2-\mathfrak{t}\mathfrak{s})(2-\mathfrak{s}-\mathfrak{t})^2 k}.$$

By putting $\mathfrak{s}=1,\mu=1$ from Theorem (2.1), we derive the subsequent corollary:

Corollary 2. If $f(z) \in \mathcal{K}_{\Sigma}^{1,t}(\alpha, \beta, 1, \mathfrak{p}, \mathfrak{q})$ defined in (1.1), then we have

$$|a_2| \leq \frac{2\sqrt{\alpha}}{\sqrt{m|4(1-t) - 2(1 + 2t - t^2 - 2t) - \alpha(\alpha - 1)(1 - t)^2|}}$$

and

$$|a_3| \leq \frac{2\alpha}{(-t^2 - t - 4)k} + \frac{8(1 - t)}{(-4 - t^2 - t)(1 - t)^2 k}.$$

By putting $t = -1, \mathfrak{s} = 1$ from Theorem (2.1), we derive the subsequent corollary:

Corollary 3. If $f(z) \in \mathcal{K}_{\Sigma}^{1,-1}(\alpha, \beta, \mu, \mathfrak{p}, \mathfrak{q})$ defined in (1.1), then we have

$$|a_2| \leq \frac{2\mu\sqrt{\alpha}}{\sqrt{m|8 - 2\alpha(\alpha - 1)(2 - \mu)^2|}}$$

and

$$|a_3| \leq \frac{\alpha}{(1 - 3\mu)k} + \frac{8\mu^2}{(1 - 3\mu)(2 - \mu)^2 k},$$

By putting $t = 0, \mu = 1$, from Theorem (2.1), we derive the subsequent corollary:

Corollary 4. If $f(z) \in \mathcal{K}_{\Sigma}^{\mathfrak{s},0}(\alpha, \beta, 1, \mathfrak{p}, \mathfrak{q})$ defined in (1.1), then we have

$$|a_2| \leq \frac{2\sqrt{\alpha}}{\sqrt{m|2(3 - \mathfrak{s}^2) - 2\mathfrak{s}(2 - \mathfrak{s}) - \alpha(\alpha - 1)(2 - \mathfrak{s})^2|}},$$

and

$$|a_3| \leq \frac{2\alpha}{(-\mathfrak{s}^2 - 3)k} + \frac{4(3 - \mathfrak{s}^2)}{(-3 - \mathfrak{s}^2)(2 - \mathfrak{s})^2 k}.$$

By placing $\mu = 1$ from Theorem (3.1), we derive the subsequent corollary:

Corollary 5. If $f(z) \in \mathcal{H}_{\Sigma}^{\mathfrak{s},t}(\beta, 1, \mathfrak{p}, \mathfrak{q})$ defined in (1.1), then we have

$$|a_2| \leq 2\sqrt{\frac{(1 - \beta)}{m|[(6 - 2\mathfrak{s}^2 - 2t - 2t\mathfrak{s}) - 2(2\mathfrak{s} + 2t - \mathfrak{s}^2 - t^2 - 2t\mathfrak{s})]|}},$$

and

$$|a_3| \leq \frac{2(1 - \beta)^2(6 - 2\mathfrak{s}^2 - 2t - 2t\mathfrak{s})}{(3 - \mathfrak{s}^2 - t^2 - t\mathfrak{s})(2 - \mathfrak{s} - t)^2 k} + \frac{2(1 - \beta)}{(3 - \mathfrak{s}^2 - t^2 - t\mathfrak{s})k}.$$

By Placing $\mathfrak{s} = 1, \mu = 1$ from Theorem (3.1), we derive the subsequent corollary:

Corollary 6. If $f(z) \in \mathcal{H}_{\Sigma}^{1,t}(\beta, 1, \mathfrak{p}, \mathfrak{q})$ defined in (1.1), then we have

$$|a_2| \leq 2\sqrt{\frac{(1-\beta)}{m|[4(1-\mathrm{t})-2(1-t^2)]|}},$$

and

$$|a_3| \leq \frac{8(1-\beta)^2}{(2-t^2-\mathrm{t})(1-t)k}+\frac{2(1-\beta)}{(2-t^2-\mathrm{t})k}.$$

By putting $\mathrm{t}=-1, \mathfrak{s}=1$ from Theorem (3.1), we derive the subsequent corollary:

Corollary 7. If $f(z) \in \mathcal{H}_{\Sigma}^{1,-1}(\beta, \mu, \mathfrak{p}, \mathfrak{q})$ defined in (1.1), then we have

$$|a_2| \leq 2\sqrt{\frac{(1-\beta)}{8m}},$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(1-3\mu)(2-\mu)^2 k}+\frac{(1-\beta)}{(1-2\mu)k}.$$

By putting $\mathrm{t}=0$ in Theorem (3.1), we have the following corollary :

Corollary 8. If $f(z) \in \mathcal{H}_{\Sigma}^{\mathfrak{s},0}(\beta, \mu, \mathfrak{p}, \mathfrak{q})$ defined in (1.1), then we have

$$|a_2| \leq 2\sqrt{\frac{(1-\beta)}{2m|[(3-\mathfrak{s}^2)-\mathfrak{s}(2-\mathfrak{s})]|}},$$

and

$$|a_3| \leq \frac{4(1-\beta)^2(3-\mathfrak{s}^2)}{(3-\mathfrak{s}^2-6\mu)(4-\mathfrak{s}-2\mu)^2 k}+\frac{2(1-\beta)}{(3-\mathfrak{s}^2-6\mu)k}.$$

Conclusions

This study introduces and examines two novel subclasses of bi-univalent functions associated with a novel integral operator $\mathcal{M}_{\mathfrak{q}}^{\mathfrak{p}}$ of analytic functions within the open unit disk U. Additionally, we acquired the second and third Taylor–Maclaurin coefficients of functions inside these subclasses. Plays a significant function in geometric function theory to establish novel generalized subclasses of analytic univalent functions also thereafter examine their owners features [1,2]. The specific instances derived from the primary findings validate the aforementioned outcomes. We indicated that the aforementioned estimations for the coefficients $|a_2|$ also $|a_3|$ pertaining to the function classes $\mathcal{K}_{\Sigma}^{\mathfrak{s},\mathfrak{t}}(\alpha, \mu, \mathfrak{p}, \mathfrak{q})$ and $\mathcal{H}_{\Sigma}^{\mathfrak{s},\mathfrak{t}}(\beta, \mu, \mathfrak{p}, \mathfrak{q})$ are not precise. Determining the precise upper bounds for the aforementioned estimations remains an intriguing unresolved challenge, particularly for $|a_n|$, where $n \geq 4$.

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