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Large Small compressible and Large Small retractable Modules

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ABSTRACT

We consider R is any ring with identity and M is a non-zero unitary left R-module. In this article we introduce the notion of large small Compressible modules and large small Retractable module. Also, we will discuss and study many basic properties about these concepts with its generalization to the ring. Where a submodule X of M is said to be large small of M denoted by $X \ll_L M$ if for $F \leq M$ such that $X + F = M$ then F is essential in M.

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1. Introduction

Suppose M is a unitary left R-module and R is an associative ring with identity. A proper submodule X of M is named small in M $(X \ll M)$, if for any submodule S of M such that $X + S = M$ implies that $M = S$, see [1] and [2]. A non-zero submodule X of M considered as essential in $M(X \leq_e M)$ if for every $0 \neq S \leq M$ then $X \cap S \neq 0$ [3], [4]. A non-zero module M called uniform if all its non-zero submodule are essential in M see [5], [6] and [7]. The annihilator of a module M is the set $ann(M) = \{r \in R: rM = 0\}$, as well as $M$is said to be faithful if $ann(M) = 0$, see these [8], [9], and [10]. A module M is called multiplication module if for all submodule B of M, $B = JM$ for some ideal J in R. Equivalently if for all $L \leq M, L = [L: M]. M$, where $[L: M] = \{r \in R: rM \subseteq L\}$ see [11], [12] and [13]. Many authors present generalizations of a small submodule, to see more of these generalizations, note the following sources [14-20].

A. Abduljaleel in [21] and [22] introduced the definition of large small submodule (LS-submodule) as a submodule X of M called large small (LS) of M denoted by $(X \ll_L M)$ if for $F \leq M$ such that $X + F = M$, then F is essential in M. The ideal A is called LS-ideal if for $A \subseteq R$ such that $A + S = R$, then S is essential ideal in R. And M is called L-hollow (every proper submodule of M is large small in M). Let us recall the most important definitions which are the

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essential concepts for our study such as Compressible modules and retractable modules as A module M is said to be compressible if it could be immersed in any of its non-zero submodule X of M. That is for all $0 \neq X \leq M$, there is a monomorphism $\Psi: M \to X$. A ring R is compressible if R is compressible module. see [23], [24], and [25]. M is said to be small compressible if M could be immersed in any its non-zero small submodule. Equivalently if there is a monomorphism from M into X whenever $0 \neq X \ll M$, then M is called small compressible [26] A module M is called retractable if $Hom(M, X) \neq 0$ for each submodule $0 \neq X$ of M. and called small retractable if $Hom_R(M, X) \neq 0$ for any $0 \neq X \ll M$ [27-32]. Our work is to study the large small Compressible (LS- Compressible) modules and large small retractable (LS- retractable) modules as a generalization of Compressible and retractable modules. Also, we generalize it on the rings. And discuss some of these results on finitely generated faithful multiplication modules (FGFM) and give some Characterizations of LS- Compressible and LS- retractable modules

2. Large Small Compressible Modules.

In this section we have provided a definition of LS-Compressible modules and study its basic properties. Also provides many of remarks to help understand this topic. Also, we give some characterizations of this concept and discuss the relationships with some other modules.

Definition 2.1: A module M is called LS-Compressible if M could be immersed in any of its non-zero LS-submodule.

Equivalently, M is LS-Compressible if there is a monomorphism from M into X whenever $0 \neq X \ll_L M$. A ring R is called LS-Compressible if R is LS-Compressible module. That is R could be immersed in any of its non-zero LS-ideal.

Examples and Remarks 2.2:

1. Each compressible module is LS-Compressible, but the converse is not generally true, and as an example: Z_6 as Z-module is not compressible by [26], Remark and Example (2.1.4)] but Z_6 is LS-Compressible since $\langle 0 \rangle$ is the only LS-submodule of Z_6.

2. Each small compressible module is LS-Compressible, but the converse is not generally true. And as an example: $Z \oplus Z$ as Zmodule is LS-Compressible but not small Compressible since for every submodule $nZ \oplus mZ$, $g.c.d(n,m) \neq 1$, is LS- submodule of $Z \oplus Z$. For all $\varphi \neq 0$, φ: $Z \oplus Z \longrightarrow nZ \oplus mZ$ define by $\varphi(x,y) = (nx, my)$, clear that φ is homomorphism and $Ker\varphi = \{(x,y) \in Z \oplus Z, \varphi(x,y) = (nx, my) = (0,0)\} = \{0\} + \{0\}$. So φ is monomorphism. And clear that $nZ \oplus mZ$ is proper in $Z \oplus Z$, which is LS- submodule in $Z \oplus Z$, but not small.

3. Q as Z-module isn't LS-Compressible because $Z \ll_L Q$ and $Hom\ (Q, Z) = 0$. Also Z_4 as Z-module isn't LS-Compressible, because $\langle \bar{2} \rangle \ll_L Z_4$ and Z_4 can not be immersed in $\langle \bar{2} \rangle$.

4. The opposite of the point (1) appears if that each submodule of M has non-zero LS- sub of M, so M is compressible. the same applies if the module M is L-hollow.

Proof: Suppose $0 \neq X \leq M$. By default there is LS- submodule $0 \neq F \ll_L X$, then $F \ll_L M$ [22], proposition (2.1.3)] and because M is LS-Compressible there exists a monomorphism $\varphi: M \to F$, and by the inclusion homomorphism $i: F \to X$ we have a monomorphism $i\varphi: M \to X$, hence M is compressible.

5. R is LS-Compressible ring if and only if R is an integral domain.

Proof: ⇒) Let $0 \neq x \in J$, and $J = \langle x \rangle$, where $J$is LS-ideal in R, and suppose that $x, y \in R$, such that $xy = 0$. but R is LS-Compressible so there is a monomorphism $\emptyset: R \to J$. Let $\emptyset(1) = rx$ for some $0 \neq r \in R$. Then $\emptyset(y) = y\emptyset(1) = y(rx) = r(xy) = 0$, implies $y(rx) = 0$ therefore $y = 0$. So, R is integral domain.

⇐) Suppose R is an integral domain and J be a non-zero ideal of R. Then there is an element $0 \neq x \in J$. Define$\emptyset: R \to J$ by $\emptyset(r) = rx\ \forall\ r \in R$. Since R is an integral domain so that $\emptyset$ is a homomorphism and monomorphism. Hence R is LS-Compressible. By the way Z as Z-module is LS-Compressible ring.

6. Each simple module is LS-Compressible but the converse isn't true as an example Z as a Z-module is LS-Compressible by (5) and clear it isn't simple.

Now we can give a characterization of LS-Compressible module.

Proposition 2.3: M is LS-Compressible module if and only if it could be immersed in Rx for all $0 \neq x \in M$ and $Rx \ll_L M$.

Proof: ⇒) By the definition of LS-Compressible.

⇐) Let $0 \neq X \ll_L M$ and let $0 \neq x \in X$. Then, $Rx \ll_L M$ ([22], proposition (2.1.3)). By default there exists a monomorphism $\vartheta: M \to Rx$ so, the composition $M \overset{\vartheta}{\to} Rx \overset{i}{\to} X$ is a monomorphism with $i: Rx \to X$ is the inclusion homomorphism. Hence M is LS-Compressible.

Corollary 2.4: If every cyclic submodule of M is LS- sub in M then the LS-Compressible module M is compressible.

Proof: Let $0 \neq X \leq M$ and $0 \neq x \in X$. By default, $Rx \ll_L M$ so there is $\vartheta: M \to Rx$ which is a monomorphism and hence the composition $M \overset{\vartheta}{\to} Rx \overset{i}{\to} X$ is a monomorphism where i is the inclusion map, that is M is compressible.

Corollary 2.5: Let M be a module in which every cyclic submodule of M is LS-submodule in M. Then M is compressible if and only if M is LS-Compressible.

Proposition 2.6: The LS- submodule of LS-Compressible module is LS-Compressible.

Proof: Let M be LS-Compressible module and $0 \neq X \ll_L M$. Let $0 \neq K \ll_L X$. Then $K \ll_L M$ [22], proposition (2.1.3)]. As M is LS-Compressible implies there exists a monomorphism, say $\varphi: M \to K$ and therefore $\varphi i: X \to K$ is a monomorphism where $i: X \to M$ is the inclusion homomorphism. Hence X is LS-Compressible.

Proposition 2.7: A direct summand of LS-Compressible module is LS-Compressible.

Proof: Let $M = C \oplus D$ be a LS-Compressible module and let $0 \neq K \ll_L C$. Then $K \oplus 0 \ll_L M$ [22], proposition (2.1.17)] and hence there is a monomorphism, $\varphi: M \to K \oplus 0$ clearly $K \oplus 0 \simeq K$, so $\varphi: M \to K$ is a monomorphism and the composition $C \overset{jC}{\to} M \overset{\varphi}{\to} K$ is a monomorphism where jC is the injection of C into M. Therefore C is LS-Compressible.

Proposition 2.8: Let M_1 and M_2 be two isomorphic modules. Then M_1 is LS-Compressible if and only if M_2 is LS-Compressible.

Proof: Assume that M_1 is LS-Compressible and let $\varphi: M_1 \to M_2$ be an isomorphism. Let $0 \neq X \ll_L M_2$. Then $0 \neq \varphi^{-1}(X) \ll_L M_1$. [22], proposition. (2.1.12)]. Put $A = \varphi^{-1}(X)$. Let $\vartheta: M_1 \to A$ be a monomorphism and let $g = \varphi|_A$ then $g: A \to M_2$ is a monomorphism and $g(A) = \varphi(\varphi^{-1}(X)) = X$, so $g: A \to X$ is a monomorphism. Now, we have the composition $M_2 \overset{\varphi^{-1}}{\longrightarrow} M_1 \overset{\vartheta}{\to} A \overset{g}{\to} X$. Let $h = g\vartheta\varphi^{-1}$ is a monomorphism. Implies that M_2 is LS-Compressible.

Remark 2.9: A homomorphic image of LS-Compressible module need not be LS-Compressible in general. For example, Z as a Z-module is LS-Compressible and $Z/4Z \simeq Z_4$ is not LS-Compressible.

Proposition 2.10: Let $M = M_1 \oplus M_2$ be an R-module such that $annM_1 + annM_2 = R$. Then M is LS-Compressible if and only if M_1 and M_2 are LS-Compressible.

Proof: ⇒) Follows from proposition (2.7).

⇐) Let $0 \neq X \ll_L M$. Then by [33, proposition (4.2)], $X = K_1 \oplus K_2$ for some $0 \neq K_1 \leq M_1 \leq M$ and $0 \neq K_2 \leq M_2 \leq M$. And as $X \ll_L M$, then $K_1 \ll_L M_1$ and $K_2 \ll_L M_2$ by [22], proposition (2.1.19)]. But M_1 and M_2 are LS-Compressible, so there are monomorphisms $\varphi: M_1 \to K_1$ and $\vartheta: M_2 \to K_2$. Define $h: M \to N$ by $g(a, b) = (\varphi(a), \vartheta(b))$. It can be easily checked that g is a monomorphism and hence M is LS-Compressible.

3. large Small Retractable Modules.

In this section we will introduce the definition of LS- retractable modules as a generalization of retractable module and study the form of the relationship between it and LS-Compressible.

Definition 3.1: An R-module M is called LS-retractable if $Hom_R(M,X) \neq 0$ for each nonzero LS-submodule $X$of M. A ring R is called LS-retractable if the R-module R is LS- retractable. That is $Hom_R(R, I) \neq 0$ for each non-zero LS- ideal I of R.

Examples and Remarks 3.2:

1. Every retractable module is LS-retractable but the converse is not always hold. See example (3.3).
2. Every small retractable module is LS- retractable. But not converse in general see Examples and remarks (2.2(2)).
3. The L-hollow module M is retractable if and only if M is LS-retractable.
4. Q as Z-module isn't LS-retractable since $Z \ll_L Q$ but $Hom_R(Q,Z) = 0$.
5. Every integral domain is small retractable ring [26] and so is LS- retractable ring by (2) but not conversely, for instance Z_6 as Z_6-module is LS- retractable but Z_6 not an integral domain.
6. Every semisimple module is LS-retractable, however the converse is not true in general, for example Z is LS-retractable Z-module but it is not semisimple.
7. Every module over a semisimple ring is small retractable [26] and by (1) is LS-retractable.
8. Every LS-Compressible module is LS-retractable and the converse is not true in general, for example Z_{24} as Z-module is LS-retractable but not LS-Compressible since $\{\bar{0}, \bar{12}\}$ is the LS-submodule in Z_{24} and $\varphi: Z_{24} \to \{\bar{0}, \bar{12}\}$ such that $\varphi(\bar{x}) = 12\bar{x}$ for all $\bar{x} \in Z_{24}$ is a homomorphism but not monomorphism.
9. Z_4 as Z-module is LS- retractable since $Hom_Z(Z_4, (\bar{2})) \neq 0$. In fact, Z_n as Z-module is LS-retractable for all $n \in Z^+$ (since it is retractable)[31].
10. $Zp\infty$ as Z-module is not LS-retractable (Since every submodule of $Zp\infty$ is LS-submodule. And $Zp\infty$ is not retractable**)** [31].
11. M is LS- retractable R-module if and only if M is LS- retractable $R/annM$-module. **Proof:**⇒) Put $\bar{R}=R/annM$ We have $Hom_R(M,X) = Hom_{\bar{R}}(M,X)$, for all $X \leq M$, by [34]. Let $0 \neq X \ll_L M$ as R-module, then $X \ll_L M$ as $\bar{R}$-module. But M as R-module is LS- retractable R-module then $Hom_R(M,X) \neq 0$ for all $X \ll_L M$, so $Hom_{\bar{R}}(M,X) \neq 0$ for all $X \ll_L M$, thus M is LS- retractable $R/annM$-module. The converse is similarly.

Example 3.3: Let $S = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in R\}$ where R be a commutative ring with identity. S is a ring with identity with respect to addition and multiplication of matrices. The non-zero ideals of S are: $I_1 = S$, $I_2 = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}: a, b \in R\}$, $I3 = \{\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}: a, c \in R\}$, $I_4 = \{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}: a \in R\}$ *or* $I_5 = \{\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}: c \in R\}$.

In each of these cases one can easily define a non-zero homomorphism from S to I, which means that S is a retractable S-module. Now, let $I = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in R\}$. we claim that I is not a retractable submodule of S. Note that $I = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S$, and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is an idempotent element and hence I is an idempotent ideal. Let $J = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}: b \in R\}$. J is a subideal of I and $JI = 0$. Suppose that there is a homomorphism, $f: I \to J$. Then $f(I) = f(I^2) = f(I)I \subseteq JI = 0$ and hence $f(I) = 0$, that means $f = 0$, therefore $Hom\ (I, J) = 0$. Hence, I is not retractable. on the other hand, the only LS-submodule of I is the zero submodule, hence I is LS-retractable.

Recall that a ring R is called Boolean ring in case each of its element is an idempotent [35].

Proposition 3.4: Let M be an R-module such that $End_R(M)$ is a Boolean ring. If M is LS-retractable, then every non-zero LS-submodule of M is LS-retractable.

Proof: Let $0 \neq N \ll_L M$ and $0 \neq K \leq N.K \ll_L M$ [22], proposition (2.1.3)]. Then $Hom_R(M,K) \neq 0$. Let $f: M \to K$ be a non-zero homomorphism. Hence $fi: N \to K$ is a homomorphism where $i: N \to M$ is the inclusion homomorphism. We claim that $fi \neq 0$, Suppose that $fi = 0$, then $(fi)(N) = 0 = f(N)$, So $N \subseteq Kerf$ and hence $K \subseteq Kerf$, which implies that $f(M) \subseteq Kerf$ therefore $f(f(M)) = 0$. Let $j: K \to M$ be the inclusion homomorphism. Then $jf \in End_R(M)$ and $jf(M) = f(M)$ *but* $(jf)^2(M) = (jf)(jf)(M) = jf(f(M)) = j(f(f(M)) = j(0) = 0$, and $(jf)^2(M) = (jf)(M)$ since $End_R(M)$ is a Boolean ring. Hence $j(f(M)) = f(M) = 0$. Therefore $f = 0$ which is a contradiction, thus $fi \neq 0$, therefore N is LS-retractable submodule**.**

Proposition 3.5: Let M_1 and M_2 be two isomorphic R-modules. Then M_1 is LS- retractable if and only if M_2 is LS-retractable.

Proof: Assume that M_1 is LS-retractable and let $\varphi : M_1 \to M_2$ be an isomorphism. Let $0 \neq N \ll_L M_2$. Then $0 \neq \varphi^{-1}(N) \ll_L M_1$ [22, proposition (2.1.7)]. Put $K = \varphi^{-1}(N)$. Let $f: M_1 \to K$ be a non-zero homomorphism and let $g = \varphi|_K$ then $g: K \to M_2$ is a homomorphism and $g(k) = \varphi(\varphi^{-1}(N)) = N$, hence $g: K \to N$ is a homomorphism. Now, we have the composition $M_2 \xrightarrow{\varphi^{-1}} M_1 \xrightarrow{f} K \xrightarrow{g} N$. Let $h = gf\varphi^{-1}$, then $h \in Hom(M_2, N)$. If $h = 0$, then $0 = g(f(\varphi^{-1}(M_2)) = g(f(M_1))$, implies that $f(M_1) \subseteq Kerg \subseteq Ker\varphi = 0$. Thus, $f(M_1) = 0$, which is a contradiction. Therefore $HomR\ (M_2, N) \neq 0$. M_2 is LS- retractable.

Remark 3.6: A direct summand (and a homomorphic image, or a quotient module) of a LS-retractable module may not be LS-retractable in general. For example, $M = Z \oplus Zp\infty$ as Z-module is retractable module by [26] and so it is LS-retractable by Example and remark (2.2.2 (1)**)**, however $Zp\infty$ is not LS-retractable, $M/Z \simeq Zp\infty$ is not LS-retractable and $Zp\infty$ is L-hollow Z- module.

In the following proposition we investigate the direct sum of LS-retractable modules.

Proposition 3.7: If M_1 and M_2 are LS-retractable modules such that $ann\ M_1 + ann\ M_2 = R$, then $M_1 \oplus M_2$ is LS-retractable.

Proof: Let $0 \neq K \ll_L M_1 \oplus M_2$. As $ann\ M_1 + ann\ M_2 = R$ by [33, proposition 4.2] gives $K = N_1 \oplus N_2$ with $N_1 \leq M_1$ and $N_2 \leq M_2$. But $N_1 \oplus N_2 \ll_L M_1 \oplus M_2$ implies $N_1 \ll_L M_1$ and $N_2 \ll_L M_2$[22**,** proposition (2.1.19)]. Therefore, $Hom(M_1, N_1) \neq 0$ and $Hom(M_2, N_2) \neq 0$. Let $0 \neq f: M_1 \to N_1$ and $0 \neq g: M_2 \to N_2$. Define $h: M_1 \oplus M_2 \to N_1 \oplus N_2$ by $h(m_1, m_2) = (f(m_1), g(m_2))$ clearly h is a homomorphism. If $h = 0$, then $h(m_1, m_2) = 0$ for all $m_1 \in M_1$, $m_2 \in M_2$, so $f(m_1) = 0$ and $g(m_2) = 0$ for all $m_1 \in M_1$, $m_2 \in M_2$, which is a contradiction since $f \neq 0$ and $g \neq 0$. Therefore $Hom(M_1 \oplus M_2, K) \neq 0$. So $M_1 \oplus M_2$ is LS- retractable**.**

In the following proposition we give a sufficient condition for LS- retractable module to be retractable.

Proposition 3.8: Let M be LS- retractable module. If every non-zero submodule of M contains a non-zero LS-submodule then M is retractable.

Proof: Let $0 \neq N \leq M$. By hypothesis N contains a non-zero LS-submodule. Let $0 \neq K \ll_L N$. Then $K \ll_L M$ [22], proposition (2.1.3)]. Hence $Hom(M, K) \neq 0$ (since M is LS-retractable), and therefore $Hom\ (M, N) \neq 0$, so M is retractable.

Note that the converse of Examples and Remarks (3.2 (8)) can be hold under certain conditions:

Proposition 3.9: If M is LS- retractable quasi-Dedekind module, then every nonzero element of $Hom\ (M, N)$ is a monomorphism for any non-zero LS-submodule N of M.

Proof: Let $0 \neq N \ll_L M$ and let $f: M \to N$ be a non-zero homomorphism. Then $if \in End(M)$ and $if \neq 0$. Since if $if = 0$, then $if(M) = f(M) = 0$ implies $f = 0$, which is a contradiction. Hence $0 \neq if \in End(M)$ and by hypothesis if is a monomorphism which gives that f is a monomorphism.

Corollary 3.10: Let M be LS-retractable module. If M is quasi-Dedekind, then M is LS-Compressible.

Proof: Let $(0) \neq N \ll_L M$. Since M is LS-retractable, then $Hom(M, N) \neq (0)$. Let $f \in Hom(M, N)$ and $f \neq (0)$. Consider the diagram: $M \xrightarrow{f} N \xrightarrow{i} M$ Since M is quasi-Dedekind $Ker(if) = (0)$. But $Ker(f) = Ker(if)$, then $kerf = (0)$. Thus, M is LS-Compressible.

We shall introduce some characterizations of LS-retractable modules

Proposition 3.11: An R-module M is LS-retractable if and only if there exists $0 \neq f \in End_R(M)$ such that $Im\, f \subseteq N$ for each non-zero LS-submodule N of M.

Proof: ⇒) Suppose that M is LS-retractable. Let $0 \neq N \ll_L M$. Then $Hom_R(M, N) \neq 0$. Let $g: M \to N$ be a non-zero homomorphism and $f = ig$ where $i: N \to M$ be the inclusion homomorphism, then $f \in End_R(M)$ and $f \neq 0$ since $g \neq 0$ and i is a monomorphism. Clearly, $f(N) = g(N) \subseteq N$.

⇐) Let $0 \neq N \ll_L M$. By hypothesis, there exists a non-zero endomorphism $f: M \to M$ and $f(M) \subseteq N$.

Therefore $f: M \to N$ is a non-zero homomorphism that is M is LS-retractable.

The following is another characterization of LS-retractable modules

Proposition 3.12: An R-module M is LS-retractable if and only if for each $0 \neq x \in M$ with $Rx \ll_L M$, $Hom\, R(M, Rx) \neq 0$.

Proof: ⇒) Clear.

⇐) To prove M is LS-retractable. Let $0 \neq N \ll_L M$ and let $0 \neq x \in N$, then $Rx \ll_L N$, so by hypothesis, $Hom(M, Rx) \neq 0$ which implies that $Hom(M, N) \neq 0$ and therefore M is LS- retractable.

A sufficient condition for a faithful finitely generated multiplication R-module to be LS-retractable is that R is LS-retractable ring, as it is shown in the following proposition

Proposition 3.13: Let M be FGFM module. Then M is LS-retractable.

Proof: By [26, proposition 2.3.12] we have M is small-retractable. So M is LS-retractable by (Examples and Remarks 3.2 (2)).

Remark 3.14: The ring Z is LS-retractable but Q as Z-module is not LS-retractable, in fact Q is not finitely generated multiplication Z-module. This means that these two conditions cannot be dropped in the proposition (3.13).

Corollary 3.15: Every faithful cyclic R-module is LS- retractable.

Proof: By [26], Corollary (1.3.23)], every faithful cyclic R-module is retractable and hence is LS-retractable.

Conclusions

We defined large small Compressible (LS-Compressible) modules and large small retractable (LS- retractable) modules as a generalization on small compressible and small retractable modules, respectively. We also presented several key properties and illustrative examples, which will serve as a foundation for future research and establish a connection between our work and previous studies in our field of work.

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