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On the Relationship Between Smoothness and Best Approximation in Weighted Function Space

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ABSTRACT

This work investigates the relationship between the smoothness measure and the function norm in the space $Z_{\alpha,\mathrm{w}}^{\mathfrak{p}}(I)$, where modified symmetric difference based on the Ditzian-Totik function $\varphi(\mathrm{x}) = \sqrt{1-\mathrm{x}^2}$ are employed to assess the behavior of functions within centered subintervals. The focus is on the weighted smoothness measure of order m, used to derive both local and global estimates of the function. The weighted smoothness measure can be bound in terms of a series involving $E_n(\mathrm{x})$, which represents the optimal approximation of the function by polynomials. The following estimate incorporates the effects of the weight, partial smoothness, and derivative behavior into a precise quantitative expression linking approximation properties with smoothness analysis. The results contribute to a deeper understanding of the interplay between function smoothness and behavior in approximation function spaces, and they open pathways to accurate numerical applications.

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1. Introduction

In this paper, a weighted normed space $Z_{\alpha,\mathrm{w}}^{\mathfrak{p}}(I)$ where as $1 \le \mathfrak{p} < \infty$, $\alpha \in (0,1]$ $W: I \to R$, $\mathrm{w} \in W$, $|\mathrm{w}| \le k$, k positive then the norm distance on $Z_{\alpha,\mathrm{w}}^{\mathfrak{p}}(I)$ is $\|\mathrm{F}\|_{Z_{\alpha,\mathrm{w}}^{\mathfrak{p}}(I)}: Z_{\alpha,\mathrm{w}}^{\mathfrak{p}}(I) \to R$. Defined the function $\mathrm{F} \in Z_{\alpha,\mathrm{w}}^{\mathfrak{p}}(I)$ as follows:

$$\|\mathrm{F}\|_{Z_{\alpha,\mathrm{w}}^{\mathfrak{p}}(I)} = \|\mathrm{F}\|_{L_{\mathrm{w},\mathfrak{p}}(I)} + \left|\Delta_{\frac{\varphi h}{(m)}}^{(m)}(\mathrm{F},\mathrm{x})_{\mathrm{w}}\right|, \forall \mathrm{x} \in I, m \in N, 1 \le \mathfrak{p} < \infty \tag{1.1}$$

The modulus of smoothness of order $m \in N$ take the form of :

$$\widetilde{\omega}_m^{\varphi}(\mathrm{F},\delta,I)_{\mathrm{w},\mathfrak{p}} = \widetilde{\omega}_m^{\varphi}(\mathrm{F},\delta)_{\mathrm{w},\mathfrak{p}} = \sup_{|h|\le\delta} \left\|\Delta_{\frac{\varphi h}{(m)}}^{(m)}(\mathrm{F},.)\right\|_{Z_{\alpha,\mathrm{w}}^{\mathfrak{p}}(I)}, I = [a,b] \tag{1.2}$$

Let $\mathrm{F} \in L_{\mathrm{w},\mathfrak{p}}(I)$, $\mathrm{F}: I \to R$, $I = [a,b]$, then the symmetric difference (see [1, 2]) of order m defined by:

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$$\Delta^{(m)}_{\frac{\varphi h}{(m)}}(\text{F},\text{x})_{\text{w}}=\begin{cases}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\text{F}\left(\text{x}+\frac{(\varphi(\text{x})hi)}{m}\right)\text{w}\left(\text{x}+\frac{(\varphi(\text{x})h)}{m}\right), if\ \text{x}+\frac{(\varphi(\text{x})h)}{m}, \text{x}\in I\\ 0 \qquad , o.w\end{cases} \tag{1.3}$$

Where $\binom{m}{i}=\frac{m!}{i!\,(m-i)!}$ is the binomial coefficient.

The error of optimal approximation of a function F with polynomials of degree $\leq n$ in norm $Z^{\text{þ}}_{\alpha,\text{w}}(I)$, and expressed by the symbol $\tilde{E}_n(\text{F})_{\text{w},\text{þ}}$, it can be estimated in terms of the $m-$order smoothness scale as:

$$\tilde{E}_n(\text{F})_{\text{w},\text{þ}}\leq c(m)\,\tilde{\omega}^{\varphi}_m\left(\text{F},\frac{1}{n}\right)_{\text{w},\text{þ}}$$

Where $\tilde{E}_n(\text{F})_{\text{w},\text{þ}}$ it is the error of the optimal approximation of the degree $\leq n$ in the space $Z^{\text{þ}}_{\alpha,\text{w}}(I)$.

2. Main Result

In this part , we show several dependencies between the modulus of smoothness and the norm of the function F in the space $Z^{\text{þ}}_{\alpha,\text{w}}(I)$. And provide estimates for the approximation error based on the order of smoothness, in the first figure, the relationship between the modulus of smoothness at different orders is shown, where the relation remains valid for different values of h, in the second figure, the validity of the relation is shown with the existence of a constant satisfying Jacksons inequality (see [3]).

Lemma 2.1 [4]. Let $\text{F}\in L_{\text{w},p}(I)\cap\Delta^1(A_s), 0<\text{þ}<1$ then

$$\left\|\Delta^{(m)}_{\frac{\varphi h}{(m)}}(\text{F},.)\right\|_{L_{\text{w},\text{þ}}(I)}\leq c(m)\,\|\text{F}\|_{L_{\text{w},\text{þ}}(I)}$$

Corollary 2.2. Let $\text{F}\in Z^{\text{þ}}_{\alpha,\text{w}}(I)$ then for $1\leq\text{þ}<\infty$ we have

$$\tilde{\omega}^{\varphi}_m(\text{F},\delta)_{\text{w},\text{þ}}\leq B\,\omega^{\varphi}_m(\text{F},\delta)_{\text{w},\text{þ}}$$

Where $B=1+c$, $c=c(m,M,k,\alpha,A_s)$, $\delta>0$

Proof. From the inequality

$$\left\|\Delta^{(m)}_{\frac{\varphi h}{(m)}}(\text{F},.)\right\|_{Z^{\text{þ}}_{\alpha,\text{w}}(I)}\leq(1+c)\left\|\Delta^{(m)}_{\frac{\varphi h}{(m)}}(\text{F},.)\right\|_{L_{\text{w},\text{þ}}(I)}$$

When $0<h\leq\delta$ then

$$\sup_{|h|\leq\delta}\left\|\Delta^{(m)}_{\frac{\varphi h}{(m)}}(\text{F},.)\right\|_{Z^{\text{þ}}_{\alpha,\text{w}}(I)}\leq(1+c)\,\sup_{|h|\leq\delta}\left\|\Delta^{(m)}_{\frac{\varphi h}{(m)}}(\text{F},.)\right\|_{L_{\text{w},\text{þ}}(I)}$$

By using definition the modulus of smoothness (1.2) and (1.3)

We get

$$\tilde{\omega}^{\varphi}_m(\text{F},\delta)_{\text{w},\text{þ}}\leq B\,\omega^{\varphi}_m(\text{F},\delta)_{\text{w},\text{þ}}$$

Where $B=(1+c)$ and $c=c(\alpha,m,M,k,A_s)$ and $\delta>0$

Theorem 2.3. Let $m\geq 1$ and $J_r\subset I$,then for $\text{F}\in Z^{\text{þ}}_{\alpha,\text{w}}(I)$, $1\leq\text{þ}<\infty$ there exist a fixed $c>0$, so that :

$$\|\mathrm{F}\|_{Z^{\mathfrak{p}}_{\alpha,\mathrm{w}}(I)} \leq m^{\frac{1}{\mathfrak{p}}}\, c\, \|\mathrm{F}\|_{Z^{\mathfrak{p}}_{\alpha,\mathrm{w}}(L_r)} + \widetilde{\omega}^{\varphi}_{m}(\mathrm{F},|J_r|,J_r)_{\mathrm{w},\mathfrak{p}}$$

where $J_r = \left[\mathrm{v} - (m+1)\frac{(h_r\varphi(\mathrm{x}))}{2}, \mathrm{v} + m\frac{(h_r\varphi(\mathrm{x}))}{2}\right]$, $r = 1,\dots,s$, where $c = c(\mathfrak{p},m)$

Proof. We will first find a relation between the subintervals of the interval $l_k = [a_{t-1}, a_t]$ and $\tilde{l}_{\tilde{k}} = [a_t, a_{t+1}]$, $t = 1,3,5,\dots,s$, we will work on the interval l_k.

We have $L_r = \left[\mathrm{v} - m\frac{(\varphi(\mathrm{x})h_r)}{2}, \mathrm{v} + m\frac{(\varphi(\mathrm{x})h_r)}{2}\right]$, $h_r = h_0 + (r-1)d$, $r = 1,\dots,s$ and $L_r \subset l_k$. Now by symmetric difference of order $m \in N$, and for $\mathrm{F} \in Z^{\mathfrak{p}}_{\alpha,\mathrm{w}}(I)$, we obtain $\Delta^{(m)}_{\frac{\varphi h}{(m)}}(\mathrm{F},\mathrm{x})_{\mathrm{w}} = \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\mathrm{F}\left(x + \frac{(\varphi(\mathrm{x})hj)}{m}\right)\mathrm{w}\left(x + \frac{(\varphi(\mathrm{x})h)}{m}\right)$

$$= \mathrm{F}(\mathrm{x} + \varphi(\mathrm{x})h)\ \mathrm{w}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})h)}{m}\right) + \sum_{j=0}^{m-1}(-1)^{m-j}\binom{m}{j}\mathrm{F}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})hj)}{m}\right)\mathrm{w}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})h)}{m}\right)$$

$$\mathrm{F}\big(\mathrm{x} + (\varphi(\mathrm{x})h)\big)\ \mathrm{w}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})h)}{m}\right) = \Delta^{(m)}_{\frac{\varphi h}{(m)}}(\mathrm{F},\mathrm{x})_{\mathrm{w}} - \sum_{j=0}^{m-1}(-1)^{m-j}\binom{m}{j}\mathrm{F}\left(x + \frac{(\varphi(\mathrm{x})hj)}{m}\right)\mathrm{w}\left(x + \frac{(\varphi(\mathrm{x})h)}{m}\right).$$

Since for all $\mathrm{x} \in l_k$ and $0 < h \leq \delta$, we have $\mathrm{x} + \frac{\varphi(\mathrm{x})h}{m} \leq \mathrm{x} + \varphi(\mathrm{x})h$ and since F is increasing in an interval l_k, then for $1 \leq \mathfrak{p} < \infty$, we have

$$\left|\mathrm{F}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})h)}{m}\right)\ \mathrm{w}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})h)}{m}\right)\right| \leq \left|\mathrm{F}(\mathrm{x} + \varphi(\mathrm{x})h)\ \mathrm{w}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})h)}{m}\right)\right|.$$

Then

$$\left|\mathrm{F}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})h)}{m}\right)\mathrm{w}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})h)}{m}\right)\right|^{\mathfrak{p}} \leq \left|\Delta^{(m)}_{\frac{\varphi h}{(m)}}(\mathrm{F},\mathrm{x})_{\mathrm{w}}\right|^{\mathfrak{p}} + \left|\sum_{j=0}^{m-1}(-1)^{m-j}\binom{m}{j}\mathrm{F}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})hj)}{m}\right)\mathrm{w}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})h)}{m}\right)\right|^{\mathfrak{p}}$$

$$\leq \left|\Delta^{(m)}_{\frac{\varphi h}{(m)}}(\mathrm{F},\mathrm{x})_{\mathrm{w}}\right|^{\mathfrak{p}} + \sum_{j=0}^{m-1}\left|(-1)^{m-j}\binom{m}{j}\right|^{\mathfrak{p}}\left|\mathrm{F}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})hj)}{m}\right)\mathrm{w}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})h)}{m}\right)\right|^{\mathfrak{p}}.$$

By take the integration of both sides with respect to J_r, where

$$J_r = \left[\mathrm{v} - (m+1)\frac{(\varphi(\mathrm{x})h_r)}{2}, \mathrm{v} + m\frac{(\varphi(\mathrm{x})h_r)}{2}\right] \ni L_r \subset J_r,\ r = 1,\dots,s.$$

Then

$$\int_{J_r}\left|\mathrm{F}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})h)}{m}\right)\mathrm{w}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})h)}{m}\right)\right|^{\mathfrak{p}} dx \leq c(\mathfrak{p},m)\int_{J_r}\sum_{j=0}^{m-1}\left|\mathrm{F}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})hj)}{m}\right)\mathrm{w}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})h)}{m}\right)\right|^{\mathfrak{p}} dx + \int_{J_r}\left|\Delta^{(m)}_{\frac{\varphi h}{(m)}}(\mathrm{F},\mathrm{x})_{\mathrm{w}}\right|^{\mathfrak{p}} dx.$$

Now, consider the interval

$$T_r = \left[\mathrm{v} - \frac{(m+1)}{2}\varphi(\mathrm{x})h_r, \mathrm{v} - \frac{m}{2}\varphi(\mathrm{x})h_r\right]$$

such that $T_r \subset J_r$.

What we do is split the interval J_r into two intervals, which are:

$T_r = \left[\mathrm{v} - \frac{(m+1)}{2}\varphi(\mathrm{x})h_r\, , \mathrm{v} - \frac{m}{2}\varphi(\mathrm{x})h_r\right]$ and $L_r = \left[\mathrm{v} - \frac{m}{2}\varphi(\mathrm{x})h_r\, , \mathrm{v} + \frac{m}{2}\varphi(\mathrm{x})h_r\right]$, see that $|J_r| = |T_r| + |L_r| \quad , r = 1, \dots, s$.

This retail achieves the following:

$$\int_{J_r}\left|\mathrm{F}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})h)}{m}\right)\mathrm{w}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})h)}{m}\right)\right|^{\mathrm{b}} dx$$

$$\le \int_{J_r}\left|\Delta^{(m)}_{\frac{\varphi h}{(m)}}(\mathrm{F},\mathrm{x})_{\mathrm{w}}\right|^{\mathrm{b}} dx + c(\mathrm{b},m)\int_{T_r}\sum_{j=0}^{m-1}\left|\mathrm{F}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})hj)}{m}\right)\mathrm{w}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})h)}{m}\right)\right|^{\mathrm{b}} dx$$

$$+c(\mathrm{b},m)\int_{L_r}\sum_{j=0}^{m-1}\left|\mathrm{F}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})hj)}{m}\right)\mathrm{w}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})h)}{m}\right)\right|^{\mathrm{b}} dx \quad .$$

Since the interval L_r is to the right of interval T_r and the length of interval L_r is greater then the length of interval T_r , such that

$|L_r| = m\varphi(\mathrm{x})h_r$ and $|T_r| = \frac{1}{2}m\varphi(\mathrm{x})h_r \quad , \forall m \ge 1$, then

$$\int_{J_r}\left|\mathrm{F}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})h)}{m}\right)\mathrm{w}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})h)}{m}\right)\right|^{\mathrm{b}} dx \le \int_{J_r}\left|\Delta^{(m)}_{\frac{\varphi h}{(m)}}(\mathrm{F},\mathrm{x})_{\mathrm{w}}\right|^{\mathrm{b}} dx + 2c(\mathrm{b},m)\int_{L_r}\sum_{j=0}^{m-1}\left|\mathrm{F}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})hj)}{m}\right)\mathrm{w}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})h)}{m}\right)\right|^{\mathrm{b}} dx$$

$$\int_{L_r}\sum_{j=0}^{m-1}\left|\mathrm{F}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})hj)}{m}\right)\,\mathrm{w}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})h)}{m}\right)\right|^{\mathrm{b}} dx = \int_{L_r}\left|\mathrm{F}(\mathrm{x})\,\mathrm{w}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})h)}{m}\right)\right|^{\mathrm{b}} dx$$

$$+\int_{L_r}\left|\mathrm{F}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})h)}{m}\right)\,\mathrm{w}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})h)}{m}\right)\right|^{\mathrm{b}} dx + \int_{L_r}\left|\mathrm{F}\left(\mathrm{x}+\frac{(2\varphi(\mathrm{x})h)}{m}\right)\,\mathrm{w}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})h)}{m}\right)\right|^{\mathrm{b}} dx$$

$$+\dots+\int_{L_r}\left|\mathrm{F}\left(\mathrm{x}+\frac{((m-1)\varphi(\mathrm{x})h)}{m}\right)\,\mathrm{w}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})h)}{m}\right)\right|^{\mathrm{b}} dx$$

$$= I_0^{\mathrm{b}} + I_1^{\mathrm{b}} + I_2^{\mathrm{b}} + \dots + I_{m-1}^{\mathrm{b}}$$

Since $\mathrm{x} < \mathrm{x} + \frac{\varphi(\mathrm{x})h}{m}$ and $\mathrm{w}(\mathrm{x})$ decreasing,

hence $\mathrm{w}\left(\mathrm{x}+\frac{\varphi(\mathrm{x})h}{m}\right) \le \mathrm{w}(\mathrm{x})$, then

$$I_0^{\mathrm{b}} = \int_{L_r}\left|\mathrm{F}(\mathrm{x})\,\mathrm{w}\left(\mathrm{x}+\frac{(\varphi(x)h)}{m}\right)\right|^{\mathrm{b}} dx \le \int_{L_r}|\mathrm{F}(\mathrm{x})\,\mathrm{w}(\mathrm{x})|^{\mathrm{b}} dx = \|\mathrm{F}\|^{\mathrm{b}}_{Z^{\mathrm{b}}_{\alpha,\mathrm{w}}(L_r)} \quad .$$

And $I_1^{\mathrm{b}} = \int_{L_r}\left|\mathrm{F}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})h)}{m}\right)\,\mathrm{w}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})h)}{m}\right)\right|^{\mathrm{b}} dx = \|\mathrm{F}\|^{\mathrm{b}}_{Z^{\mathrm{b}}_{\alpha,\mathrm{w}}(L_r)}$.

We will discuss the remaining cases for when $j = 2, \dots, m-1$

$$\sum_{j=2}^{m-1}\int_{L_r}\left|\mathrm{F}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})hj)}{m}\right)\,\mathrm{w}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})h)}{m}\right)\right|^{\mathrm{b}} dx \le \sum_{j=2}^{m-1}\int_{L_r}\left|\mathrm{F}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})hj)}{m}\right)\,\mathrm{w}(\mathrm{x})\right|^{\mathrm{b}} dx.$$

Let $u = \mathrm{x} + \frac{\varphi(\mathrm{x})hj}{m}$, then $\mathrm{x} = u - \frac{\varphi(\mathrm{x})hj}{m}$ and $dx = du$, then

$$\sum_{j=2}^{m-1} \int_{L_r} \left| \mathrm{F}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})hj)}{m}\right) \mathrm{w}(\mathrm{x}) \right|^{\mathrm{þ}} dx = \sum_{j=2}^{m-1} \int_{L_r + \frac{\varphi(\mathrm{x})hj}{m}} \left| \mathrm{F}(u)\, \mathrm{w}\left(u - \frac{(\varphi(\mathrm{x})hj)}{m}\right) \right|^{\mathrm{þ}} du \,.$$

We have $-u - \frac{(\varphi(\mathrm{x})hj)}{m} \le u - \frac{(\varphi(\mathrm{x})hj)}{m}$ $\forall j = 2, \dots, m-1$,since $\mathrm{w}(\mathrm{x})$ decreasing function then $\left(u - \frac{(\varphi(\mathrm{x})hj)}{m}\right) \le w\left(-u - \frac{(\varphi(\mathrm{x})hj)}{m}\right)$, hence

$$\sum_{j=2}^{m-1} \int_{L_r} \left| \mathrm{F}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})hj)}{m}\right) \mathrm{w}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})h)}{m}\right) \right|^{\mathrm{þ}} dx \le \sum_{j=2}^{m-1} \int_{L_r + \frac{\varphi(\mathrm{x})hj}{m}} \left| \mathrm{F}(u)\, \mathrm{w}\left(u - \frac{(\varphi(\mathrm{x})hj)}{m}\right) \right|^{\mathrm{þ}} du \,.$$

$$\le \sum_{j=2}^{m-1} \int_{L_r + \frac{\varphi(\mathrm{x})hj}{m}} \left| \mathrm{F}(u)\; \mathrm{w}\left(-u - \frac{(\varphi(\mathrm{x})hj)}{m}\right) \right|^{\mathrm{þ}} du \,.$$

Since $\mathrm{w}(\mathrm{x})$ is even function then

$\mathrm{w}\left(-u - \frac{(\varphi(\mathrm{x})hj)}{m}\right) = \mathrm{w}\left(-\left(u + \frac{(\varphi(\mathrm{x})hj)}{m}\right)\right) = \mathrm{w}\left(u + \frac{(\varphi(\mathrm{x})hj)}{m}\right)$ and since $u \le u + \frac{(\varphi(\mathrm{x})hj)}{m}$ and $\mathrm{w}(\mathrm{x})$ decreasing function then $\mathrm{w}\left(u + \frac{(\varphi(\mathrm{x})hj)}{m}\right) \le \mathrm{w}(u)$, hence

$$\sum_{j=2}^{m-1} \int_{L_r} \left| \mathrm{F}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})hj)}{m}\right) \mathrm{w}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})hj)}{m}\right) \right|^{\mathrm{þ}} dx \le \sum_{j=2}^{m-1} \int_{L_r + \frac{\varphi(\mathrm{x})hj}{m}} |\mathrm{F}(u)\, \mathrm{w}(u)|^{\mathrm{þ}} du \,,$$

that is

$$\sum_{i=2}^{m-1} I_j^{\mathrm{þ}} \le \sum_{j=2}^{m-1} \int_{L_r + \frac{\varphi(\mathrm{x})hj}{m}} |\mathrm{F}(u)\, \mathrm{w}(u)|^{\mathrm{þ}} du \,.$$

Since $u = \mathrm{x} + \frac{(\varphi(\mathrm{x})hi)}{m}$, then

$$\sum_{i=2}^{m-1} I_j^{\mathrm{þ}} \le \sum_{j=2}^{m-1} \int_{L_r} \left| \mathrm{F}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})hj)}{m}\right) \mathrm{w}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})hj)}{m}\right) \right|^{\mathrm{þ}} dx$$
$$= (m-2)\, \|\mathrm{F}\|^{\mathrm{þ}}_{Z^{\mathrm{þ}}_{\alpha,\mathrm{w}}(L_r)} \,.$$

Then

$$\int_{L_r} \sum_{j=0}^{m-1} \left| \mathrm{F}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})hj)}{m}\right) \mathrm{w}\left(\mathrm{x} + \frac{(\varphi(\mathrm{x})h)}{m}\right) \right|^{p} dx \le I_0^{\mathrm{þ}} + I_1^{\mathrm{þ}} + \sum_{i=2}^{m-1} I_j^{\mathrm{þ}}$$
$$= \|\mathrm{F}\|^{\mathrm{þ}}_{Z^{\mathrm{þ}}_{\alpha,\mathrm{w}}(L_r)} + \|\mathrm{F}\|^{\mathrm{þ}}_{Z^{\mathrm{þ}}_{\alpha,\mathrm{w}}(L_r)} + (m-2)\|\mathrm{F}\|^{\mathrm{þ}}_{Z^{\mathrm{þ}}_{\alpha,\mathrm{w}}(L_r)}$$
$$= c(\mathrm{þ}, m)\, m\, \|\mathrm{F}\|^{\mathrm{þ}}_{Z^{\mathrm{þ}}_{\alpha,\mathrm{w}}(L_r)} \,.$$

Then

$$\int_{J_r}\left|\text{Ƒ}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})h)}{m}\right)\ \mathrm{w}\left(\mathrm{x}+\frac{(\varphi(\mathrm{x})h)}{m}\right)\right|^{\text{ þ}} dx \le \int_{J_r}\left|\Delta^{(m)}_{\frac{\varphi h}{(m)}}(\text{Ƒ},\mathrm{x})_{\mathrm{w}}\right|^{\text{þ}} dx + c(\text{þ},m)\ m\ \|\text{Ƒ}\|^{\text{þ}}_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(L_r)}$$

$$\|\text{Ƒ}\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(J_r)} \le \left\|\Delta^{(m)}_{\frac{\varphi h}{(m)}}(\text{Ƒ},.)\right\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(J_r)} + c(\text{þ},m)\ m^{\frac{1}{\text{þ}}}\ \|\text{Ƒ}\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(L_r)}\ .$$

By definition the modulus of smoothness (1.2), we get

$$\left\|\Delta^{(m)}_{\frac{\varphi h}{(m)}}(\text{Ƒ},.)\right\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(I)} \le \widetilde{\omega}^{\varphi}_m(\text{Ƒ},\delta)_{\mathrm{w},\text{þ}},$$

then

$$\|\text{Ƒ}\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(J_r)} \le c(\text{þ},m)\ m^{\frac{1}{\text{þ}}}\ \|\text{Ƒ}\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(L_r)} + \widetilde{\omega}^{\varphi}_m(\text{Ƒ},|J_r|\ ,J_r\ )_{\mathrm{w},\text{þ}} \tag{2.1}$$

where $m \ge 1\ ,L_r \subset J_r\ .$

By the same way in (2.1) , on an interval $\tilde{l}_{\tilde{k}}$ where f is decreasing on $\tilde{l}_{\tilde{k}}$, consider $\tilde{T}_r = \left[\mathrm{v} + m\frac{(\varphi(\mathrm{x})h)}{2}, \mathrm{v} + (m+1)\frac{(\varphi(\mathrm{x})h_r)}{2}\right]$ and split the interval $\tilde{J}_r$ into two interval , which are: $\tilde{T}_r$ and L_r see that $|\tilde{J}_r| = |L_r| + |\tilde{T}_r|\ \ ,r = 1, \ldots, s$.

We get

$$\|\text{Ƒ}\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(\tilde{J}_r)} \le m^{\frac{1}{\text{þ}}}\ c\ \|\text{Ƒ}\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(L_r)} + \widetilde{\omega}^{\varphi}_m(\text{Ƒ},|\tilde{J}_r|\ ,\tilde{J}_r\ )_{\mathrm{w},\text{þ}}\ .$$

where $L_r \subset \tilde{J}_r\ ,m \ge 1\ \ ,c = c(\text{þ},m) > 0$

Example 2.4. Let

1) $\text{Ƒ}(\mathrm{x}) = \frac{1}{1+\mathrm{x}^2}\ \ ,\mathrm{w}(\mathrm{x}) = \frac{1}{1+\mathrm{x}^2}\ \ ,\mathrm{x} \in I$.
2) $\text{þ} = 4\ ,m = 2\ ,h = 0.2\ ,\mathrm{x}_0 = 0$

Using the definition of the modulus of smoothness in formula (1.2) at $= [-2,2]$, and using the definition of the norm Ƒ on the space $Z^{\text{þ}}_{\alpha,\mathrm{w}}(I)$ in formula (1.1) once at interval $I = [-2,2]$ and once at subinterval $L_r = [-1,1]$, we obtain the following results:

$$\|\text{Ƒ}\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(I)} \approx 0.9192, \|\text{Ƒ}\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(L_r)}\ \ \approx 0.9188$$

$$\widetilde{\omega}^{\varphi}_m(\text{Ƒ},\delta)_{\mathrm{w},\text{þ}} \approx 0.0146,\ \ m^{\frac{1}{\text{þ}}} \approx 1.189$$

By Jackson inequality $\exists\ c = c(m,\text{þ})$ and $c \ge 0.8275$ such that

$$\|\text{Ƒ}\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(I)} \approx 0.9192 \le 1.0187829814$$

$$\approx c(\text{þ},m)\ m^{\frac{1}{\text{þ}}}\ \ \|\text{Ƒ}\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(L_r)} + \widetilde{\omega}^{\varphi}_m(\text{Ƒ},\delta)_{\mathrm{w},\text{þ}}$$

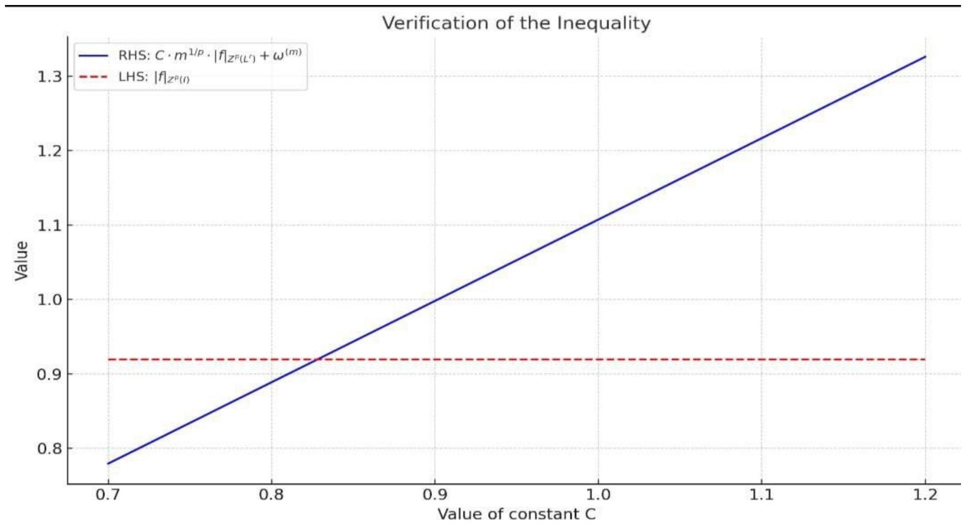

Fig. 1 - The figure shows that the inequality is satisfied when the constant value $c = 0.8275$.

Lemma 2.5. For $\text{Ƒ} \in Z^{\text{þ}}_{\alpha,\mathrm{w}}(I)$, $1 \le \text{þ} < \infty$, $\alpha \in (0,1]$ and $m, k \in N$ we have $\tilde{\omega}^{\varphi}_{m}(\text{Ƒ}, \delta)_{\mathrm{w},\text{þ}} \le c\, \tilde{\omega}^{\varphi}_{k}(\text{Ƒ}, \delta)_{\mathrm{w},\text{þ}}$.

Where $c = c(m-k)$, $k < m$.

Proof: When $k < m$, Let $m = k+1$, then by symmetric difference of order m, we have for $\text{Ƒ} \in Z^{\text{þ}}_{\alpha,\mathrm{w}}(I)$, $\forall x \in I$, that

$$\Delta^{(m)}_{\frac{\varphi h}{(m)}}(\text{Ƒ}, \mathrm{x})_{\mathrm{w}} = \Delta^{(k+1)}_{\frac{\varphi h}{(k+1)}}(\text{Ƒ}, \mathrm{x})_{\mathrm{w}} .$$

By integrating both sides in relation to the norm in the space $Z^{\text{þ}}_{\alpha,\mathrm{w}}(I)$, we obtain the following:

$$\left\|\Delta^{(m)}_{\frac{\varphi h}{(m)}}(\text{Ƒ}, .)\right\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(I)} = \left\|\Delta^{(k+1)}_{\frac{\varphi h}{(k+1)}}(\text{Ƒ}, .)\right\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(I)} = \left\|\Delta^{(1)}_{\frac{\varphi h}{(1)}}\left(\Delta^{(k)}_{\frac{\varphi h}{(k)}}(\text{Ƒ}, .)\right)\right\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(I)} = \left\|\Delta^{(m-k)}_{\frac{\varphi}{(m-k)}}\left(\Delta^{(k)}_{\frac{\varphi h}{(k)}}(\text{Ƒ}, .)\right)\right\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(I)}$$

By using Lemma (2.1) we get

$$\left\|\Delta^{(m)}_{\frac{\varphi h}{(m)}}(\text{Ƒ}, .)\right\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(I)} \le c(m-k) \left\|\Delta^{(k)}_{\frac{\varphi h}{(k)}}(\text{Ƒ}, .)\right\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(I)} .$$

By taking the supremum of both sides, we obtain that

$$\sup_{0 \le h \le \delta} \left\|\Delta^{(m)}_{\frac{\varphi h}{(m)}}(\text{Ƒ}, .)\right\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(I)} \le c(m-k) \sup_{0 \le h \le \delta} \left\|\Delta^{(k)}_{\frac{\varphi h}{(k)}}(\text{Ƒ}, .)\right\|_{Z^{\text{þ}}_{\alpha,\mathrm{w}}(I)}$$

Based on the definition of the modulus of smoothness in the space $Z^{\text{þ}}_{\alpha,w}(I)$, we conclude the following:

$$\tilde{\omega}^{\varphi}_{m}(\text{Ƒ}, \delta)_{\mathrm{w},\text{þ}} \le c\tilde{\omega}^{\varphi}_{k}(\text{Ƒ}, \delta)_{\mathrm{w},\text{þ}}.$$

Where

$$c = c(m-k)\,,k\ < m\,.$$

It follows that a function Ƒ belonging to the apace $Z^{\text{ƥ}}_{\alpha,\mathrm{w}}(I)$ of order m exhibits greater smoothness than a function of order, where $k < m$

Example 2.6. Let

1) $\text{Ƒ}(\mathrm{x}) = \sqrt{|\mathrm{x}|}\ \ ,\mathrm{w}(\mathrm{x}) = \frac{1}{1+\mathrm{x}^2}\ ,\mathrm{x} \in I\,.$
2) $\text{ƥ} = 4\ ,m = 4\,,k = 2\,,I = [0,1]\,.$

Using the definition of the modulus of smoothness in formula (1.2) when $m = 4$ and again when $k = 2$, we get the following results.

$$\sup_{|h|\le\delta}\left\|\Delta^{(2)}_{\frac{\varphi h}{(2)}}(\text{Ƒ},.)\right\|_{Z^{\text{ƥ}}_{\alpha,\mathrm{w}}(I)} \approx 0.00437$$

and,

$$\sup\left\|\Delta^{(4)}_{\frac{\varphi h}{(4)}}(\text{Ƒ},.)\right\|_{Z^{\text{ƥ}}_{\alpha,\mathrm{w}}(I)} \approx 0.0000944$$

Hence

$$\widetilde{\omega}^{\varphi}_4(\text{Ƒ},\delta)_{\mathrm{w},\text{ƥ}} \le \widetilde{\omega}^{\varphi}_2(\text{Ƒ},\delta)_{\mathrm{w},\text{ƥ}}.$$

Table 1 - The table shows the comparison of the modulus of smoothness at different orders and preserves same relation at different values of *h*.

$\boldsymbol{h}$	$\left\|\Delta^{(2)}_{\boldsymbol{h}}(\text{Ƒ},\boldsymbol{\omega})\right\|_{\boldsymbol{L}^4}$	$\left\|\Delta^{(4)}_{\boldsymbol{h}}(\text{Ƒ},\boldsymbol{\omega})\right\|_{\boldsymbol{L}^4}$
0.01	0.000068	0.0000000283
0.02	0.000255	0.0000003904
0.03	0.000543	0.000001719
0.04	0.000917	0.000004759
0.05	0.001362	0.00001025
0.06	0.001870	0.00001884
0.07	0.002431	0.00003113
0.08	0.003039	0.00004757
0.09	0.003686	0.00006856
0.10	0.004368	0.00009440

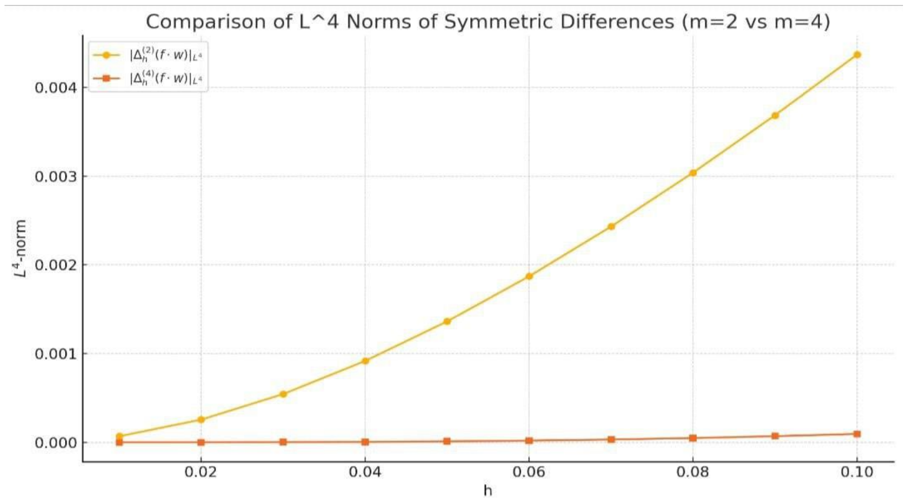

Fig. 2 - A comparison between the modulus of smoothness at orders $m = 2$ and $m = 4$.

Theorem 2.7. Let $\text{F} \in Z^{\text{þ}}_{\alpha,\text{w}}(I)$, $1 \le \text{þ} < \infty$ and $n > m-1$, then for any $n \in N$,$m > \text{þ}$, we obtain

$$\tilde{\omega}^{\varphi}_{m}(\text{F},\delta)_{\text{w},\text{þ}} \le B\, n^{-(m-1)} \tilde{E}_{n-(m-1)}\left(\text{F}^{(m-1)}\right)_{\text{w},\text{þ}}$$

$$+D\,|I|^{m-\frac{1}{\text{þ}}} \sum_{n\ge 2} (1+n^{-\frac{d}{\text{þ}}}) \tilde{E}_n(\text{F})_{\text{w},\text{þ}}.$$

where

$$B = 1 + c\ \ , c = c(m, M, \alpha, k, A_s)\ , D = c(n)B\ \ , d \ge 0.$$

Proof. For any $n \in N$, Let $n = 2^r \ni r = Max\{i{:}\,1,2,\dots,r\ \ , 2^i \le n\}$. Defined the algebraic polynomial $P_n \in \prod_n$ by

$$P_n(\text{x}) = \left(P_n(\text{x}) - P_{2^i}(\text{x})\right) + \left(P_{2^i}(\text{x}) - P_{2^{i-1}}(\text{x})\right) + \cdots + \left(P_{2^{i-(i-1)}}(\text{x}) - P_{2^{i-(i-0)}}(\text{x})\right) + P_{2^{i-(i-0)}}(\text{x}), \forall \text{x} \in I$$

$$P_n(\text{x}) - P_1(\text{x}) = \sum_{i=1}^{r} \left(P_{2^i}(\text{x}) - P_{2^{i-1}}(\text{x})\right),$$

where $P_{2^i}(\text{x})$ *and* $P_{2^{i-1}}(\text{x})$ are polynomials belong to $\prod_{2^i}$ *and* $\prod_{2^{i-1}}$ respectively and out of $\prod_n$, also P_{2^i} , $P_{2^{i-1}}$ are best approximation in an interval I to $\text{F} \in Z^{\text{þ}}_{\alpha,\text{w}}(I)$ and satisfies when P_n interpolate F in an interval I , that

$$\left\|\text{F} - P_{2^i}\right\|_{Z^{\text{þ}}_{\alpha,\text{w}}(I)} = \inf_{P_n \in \prod_n} \|\text{F} - P_n\|_{Z^{\text{þ}}_{\alpha,\text{w}}(I)} = \tilde{E}_n(\text{F})_{\text{w},\text{þ}} = \tilde{E}_n(\text{F}, \textstyle\prod_n)_{\text{w},\text{þ}} \ .$$

And

$$\left\|\text{F} - P_{2^{i-1}}\right\|_{Z^{\text{þ}}_{\alpha,\text{w}}(I)} = \inf_{P_n \in \prod_n} \|\text{F} - P_n\|_{Z^{\text{þ}}_{\alpha,\text{w}}(I)} = \tilde{E}_n(\text{F})_{\text{w},\text{þ}} = \tilde{E}_n(\text{F}, \textstyle\prod_n)_{\text{w},\text{þ}}$$

By using Corollary (2.2) for $1 \le \text{þ} < \infty$, then

$$\tilde{\omega}^{\varphi}_{m}(\text{F},\delta)_{\text{w},\text{þ}} \le B\, \omega^{\varphi}_{m}(\text{F},\delta)_{\text{w},\text{þ}}.$$

By using (Theorem (1.5.1), (v), [2]) we have for $1 \leq \mathfrak{p} < \infty$ that

$$\omega_m^{\varphi}(\mathrm{F},\delta)_{\mathrm{w},\mathfrak{p}} \leq B\,\delta^{m-1}\omega_m^{\varphi}\left(\mathrm{F}^{(m-1)},\delta\right)_{\mathrm{w},\mathfrak{p}}$$

$$\leq B\,\delta^{(m-1)}\omega_m^{\varphi}\left(\mathrm{F}^{(m-1)}-P_n^{(m-1)},\delta\right)_{\mathrm{w},\mathfrak{p}} + B\,\delta^{m-1}\omega_m^{\varphi}\left(P_n{}^{(m-1)},\delta\right)_{\mathrm{w},\mathfrak{p}}.$$

By (Theorem (1.6.1) [2, 5]) and (Proposition 1.6.5) [3]) we get

$$\omega_m^{\varphi}(\mathrm{F},\delta)_{\mathrm{w},\mathfrak{p}} \leq B\,\delta^{m-1}\left\|\mathrm{F}^{(m-1)}-P_n^{(m-1)}\right\|_{L_{\mathrm{w},\mathfrak{p}}(I)} + B\,\delta^{m-1}\left\|P^{(m)}\right\|_{L_{\mathrm{w},\mathfrak{p}}(I)}$$

$$= I_1 + I_2\,,\ \text{for } n > m-1.$$

$$I_1 = B\,\delta^{m-1}\left\|\mathrm{F}^{(m-1)}-P_n^{(m-1)}\right\|_{L_{\mathrm{w},\mathfrak{p}}(I)}$$

$$\leq B\,\delta^{m-1}\left\|\mathrm{F}^{(m-1)}-P_n^{(m-1)}\right\|_{Z_{\alpha,\mathrm{w}}^{\mathfrak{p}}(I)}.$$

Since P_n the optimal approximation of F *and* F is α −Hölders smooth also $P_n^{(m-1)}$ the optimal approximation of $F^{(m-1)}$, in other words P_n is the optimal approximation to F , not only in values but also in derivatives , hence

$$I_1 \leq B\,\delta^{m-1}\ \tilde{E}_{n-(m-1)}\left(\mathrm{F}^{(m-1)}\right)_{\mathrm{w},\mathfrak{p}}$$

$$= B\,n^{-(m-1)}\,|I|\,\tilde{E}_{n-(m-1)}\left(\mathrm{F}^{(m-1)}\right)_{\mathrm{w},\mathfrak{p}} \tag{2.2}$$

Now,

$$I_2 = B\,\delta^{m-1}\left\|P_n^{(m)}\right\|_{L_{\mathrm{w},\mathfrak{p}}(I)}.$$

By using ((proposition (3.2.2) [4],[6]) we get

$$I_2 \leq c(n)B\,\delta^{m-1}\,|I|^{m-\frac{1}{\mathfrak{p}}}\,\|P_n\|_{L_{\mathrm{w},\mathfrak{p}}(I)}\ ,m,n \in N$$

$$= D\,\delta^{m-1}\,|I|^{m-\frac{1}{\mathfrak{p}}}\,\|P_n - P_1\|_{L_{\mathrm{w},\mathfrak{p}}(I)}, D = B\,c(n)$$

$$\leq D\,\delta^{m-1}\,|I|^{m-\frac{1}{\mathfrak{p}}}\sum_{i=1}^{r}\left\|P_{2^i}-P_{2^{i-1}}\right\|_{L_{\mathrm{w},\mathfrak{p}}(I)}$$

$$\leq D\,n^{-(m-1)}\,|I|^{m-\frac{1}{\mathfrak{p}}}\sum_{i=1}^{r}\left[\,\left\|\mathrm{F}-P_{2^i}\right\|_{L_{\mathrm{w},\mathfrak{p}}(I)} + \left\|\mathrm{F}-P_{2^{i-1}}\right\|_{L_{\mathrm{w},\mathfrak{p}}(I)}\right]$$

$$\leq D\,n^{-(m-1)}\,|I|^{m-\frac{1}{\mathfrak{p}}}\sum_{i=1}^{r}\left[\,\left\|\mathrm{F}-P_{2^i}\right\|_{Z_{\alpha,\mathrm{w}}^{\mathfrak{p}}(I)} + \left\|\mathrm{F}-P_{2^{i-1}}\right\|_{Z_{\alpha,\mathrm{w}}^{\mathfrak{p}}(I)}\right].$$

To investigate the behavior of the approximation error associated with hierarchical polynomial approximations of a function F , we consider error contributions from successive approximation levels. specifically, we examine the cumulative error arising from polynomial approximation at levels 2^i and 2^{i-1} , and observe that their combined contribution at each step can be estimated by

$$I_2 \leq D\ 2^{-r(m-1)}|I|^{m-\frac{1}{\text{þ}}} \sum_{i=1}^{r} [\tilde{E}_{2^i}(\text{F})_{\text{w,þ}} + \tilde{E}_{2^{i-1}}(\text{F})_{\text{w,þ}}].$$

This quantity represents the total error due to two consecutive approximation levels.

By aggregating such pairwise error contributions at to level 2^r , we obtainan an upper bound on the overall error accumulated across all intermediate levels. More formally, we have the inequality:

$$I_2 \leq D\ 2^{-r(m-1)}|I|^{m-\frac{1}{\text{þ}}} \sum_{i=1}^{r} 2^i\ \tilde{E}_{2^i}\left(\text{F}, \prod\nolimits_{2^i}\right)_{\text{w,þ}}.$$

Such a formulation is particularly useful in multilevel numerical schemes, where selective control over error contributions can lead to more efficient and adaptive approximation strategies

$$I_2^{\text{þ}} \leq D\ 2^{-(m-1)r\text{þ}}\ |I|^{\left(m-\frac{1}{\text{þ}}\right)\text{þ}} \sum_{i=1}^{r} 2^{i\text{þ}}\ \tilde{E}_{2^i}\left(\text{F}, \prod\nolimits_{2^i}\right)_{\text{w,þ}}^{\text{þ}}.$$

By using the following inequality (see [7]):

$2^{i\text{þ}v} \leq \sum_{k=2^{i-1}+1}^{2^i} (k+1)^{v\text{þ}-1}$ hence for $k < m-1$ we get

For $v = 1$, we get

$$I_2^{\text{þ}} \leq D\ 2^{-(m-1)r\text{þ}}\ |I|^{\left(m-\frac{1}{\text{þ}}\right)\text{þ}} \sum_{i=1}^{r} \sum_{k=2^{i-1}+1}^{2^i} (k+1)^{\text{þ}-1}\ \tilde{E}_{2^i}\left(\text{F}, \prod\nolimits_{2^i}\right)_{\text{w,þ}}^{\text{þ}}.$$

We have $k < m-1 < n$ then $k+1 < m < n+1$, $(k+1)^{\text{þ}-1} < (n+1)^{\text{þ}-1}$, then

$$I_2^{\text{þ}} \leq D\ |I|^{\left(m-\frac{1}{\text{þ}}\right)\text{þ}} \sum_{k=2}^{n} n^{-(m-1)\text{þ}}(k+1)^{\text{þ}-1}\ \tilde{E}_n\left(\text{F}, \prod\nolimits_{2^i}\right)_{\text{w,þ}}^{\text{þ}}$$

$$\leq D\ |I|^{\left(m-\frac{1}{\text{þ}}\right)\text{þ}} \sum_{\forall n\geq 2} n^{-(m-1)\text{þ}}\ (n+1)^{\text{þ}-1} \tilde{E}_n(\text{F})_{\text{w,þ}}^{\text{þ}}$$

Now, we have $n+1 \approx n\ as\ n \to \infty$ then $(n+1)^{\text{þ}-1} \approx n^{\text{þ}-1}$ hence

$$n^{-m\text{þ}+\text{þ}}\ (n+1)^{\text{þ}-1} \approx n^{-m\text{þ}+\text{þ}} n^{\text{þ}-1}$$

We have $m > \text{þ}$ then $-m < -\text{þ}$ then $n^{-m} < n^{-\text{þ}}$ and $n^{-m\text{þ}} < n^{-\text{þ}^2}$ hence $n^{-m\text{þ}}\ n^{\text{þ}} < n^{-\text{þ}^2}\ n^{\text{þ}}$ and $n^{-m\text{þ}}\ n^{\text{þ}}\ n^{\text{þ}-1} < n^{-\text{þ}^2}\ n^{\text{þ}}\ n^{\text{þ}-1} = n^{-\text{þ}^2+2\text{þ}-1}$, that is

$$n^{-(m-1)\text{þ}}\ (n+1)^{\text{þ}-1} \leq n^{-(\text{þ}^2-2\text{þ}+1)}.$$

Therefor

$$I_2^{p} \leq D\ |I|^{\left(m-\frac{1}{\text{þ}}\right)\text{þ}} \sum_{n\geq 2} n^{-d}\ \ \tilde{E}_n(\text{F})_{\text{w,þ}}^{\text{þ}}\ \ , d = \text{þ}^2 - 2\text{þ} + 1\ \ and\ d \geq 0$$

$$I_2\ \ \leq D\ |I|^{m-\frac{1}{\text{þ}}} \sum_{n\geq 2} n^{\frac{-d}{\text{þ}}}\ \tilde{E}_n(\text{F})_{\text{w,þ}} \tag{2.3}$$

Where $n+1 > m > \text{þ}, n, m \in N$, $1 \leq \text{þ} < \infty$ we set form (2.2) and (2.3), that

$$\omega_m^{\varphi}(\mathrm{F},\delta)_{\mathrm{w},\mathrm{\beta}} \leq I_1 + I_2$$

$$\leq B\ \ n^{-(m-1)} \tilde{E}_{n-(m-1)}(\mathrm{F}^{(m-1)})_{\mathrm{w},\mathrm{\beta}} + D\ |I|^{m-\frac{1}{\mathrm{\beta}}} \sum_{n\geq 2} n^{\frac{-d}{\mathrm{\beta}}}\ \widetilde{E_n}\ (\mathrm{F})_{\mathrm{w},\mathrm{\beta}}.$$

Where

$$d \geq 0\ , B = 1 + c\ \ , c = c(m, M, \alpha, k, A_s)\ , D = c(n)B.$$

References

[1] J. G. Mthawer and N. Z. A. Al-Sada, “On approximation by Lagrange polynomials in weighted space,” *AIP Conf. Proc.*, vol. 3264, no. 1, (2025), Art. no. 050092.

[2] N. Z. Abd Al-Sada, *On positive and copositive approximation in $L_{\psi,p}(I)$ spaces* $0 < p < 1$, Ph.D. thesis, Mustansiriyah Univ., Baghdad, Iraq, 2015.

[3] R. A. DeVore and G. G. Lorentz, *Constructive Approximation*. Berlin: Springer-Verlag, (1993).

[4] J. G. Mthawer and N. Z. A. Al-Sada, “On the error of the two-sides comonotonic approximation,” *Nonlinear Functional Analysis and Applications*, vol. 29, no. 4, (2024), pp. 1109–1123.

[5] N. Z. A. Al-Sada, “Copositive spline approximation in weighted space,” *J. Interdiscip. Math.*, vol. 25, no. 4, (2022), pp. 1171–1186.

[6] N. Z. A. Al-Sada, “Copositive approximation by Hermits extension,” *J. Interdiscip. Math.*, vol. 25, no. 4, pp. 1187–1200, 2022.

[7] N. Zuhair, “Pointwise estimates for finding the error of best approximation by spline, positive algebraic polynomials and copositive,” *Iraqi J. Sci.*, vol. 63, no. 11, (2022), pp. 4967–4986.