An Advanced Generalized Fixed Point Theorem for Solving Ordinary Differential Equations

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A R T I C L E   I N F O

A B S T R A C T

This study examines the use of the generalized fixed point (FP) theorems in handling ordinary differential equations (ODEs), and specifically in the framework of fundamental principles and applications. Their major findings such as Banach's Contraction Principle and Schauder's Theorem are used to prove the existence and uniqueness of solutions in diverse mathematical systems. The paper generalizes fixed point theories to include more general classes of contraction mappings and boundary value problems, thus providing new methods of analysis to solve complicated ODEs. Through a critical analysis of metric spaces and iterative procedures, the study presents some of the historical and theoretical breakthroughs and puts into perspective practical applications in the engineering, physics, and computational mathematics sectors, among others. The major contribution of the work is the formulation of generalized contraction principles, that allow extending the applicability of FP theorems to non-traditional spaces and open up new methods of solving differential equations.
MSC..

1. Introduction

The theory of fixed point forms a fundamental field of study in mathematical analysis, with far-reaching consequences to the applied sciences. It can be said to have its origin in the monumental work of Banach who came up with the Fixed Point Theorem in 1922. The theorem provides invaluable methods of determining not only the existence but also the uniqueness of solutions to a variety of mathematical models, including integral equations, partial differential equations and variational inequalities [1].

The fundamental doctrine of fixed point theory is the discovery of a fixed point of a self- mapping of an object, that is, a point of the object, of the form, T x = x. The contraction principle of Banach is one of the key outcomes in this respect and it offers simple and effective iterative methods - most specifically the Picard iteration - to find such fixed points. Such iterative procedures have also been used in solving ordinary differential equations (ODEs) [1].

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Historical study of the differential equations is dated back to the late 17th century, when it was pioneered by both Newton and Leibniz. Later, mathematicians such as Cauchy and Picard made significant contributions, particularly in establishing the existence of solutions under certain conditions [2].

1.1 The Metric Space [3]:

Let X be a nonempty set. It follows that:

$$````d: X \times X \to R^+$$
$$(x, y) \to d(x, y) \tag{1.1}$$

d is considered a space over x, if it checks for three facts as follows:

1. $d(x, y) = 0 \Leftrightarrow x = y$
2. $x, y \in X \Rightarrow d(x, y) = d(y, x)$
3. $x, y, z \in X \Rightarrow d(x, y) \leq d(x, z) + d(z, y)$

Thus, the pair (X, d) is called a metric space [17]. The universe of valid objects is defined as a set X. Within X, there exists a finite subset U, called the dictionary or database, which contains $n = |U|$ desired objects (or items). A distance function d quantifies the dissimilarity between objects: smaller values of d indicate closer or more similar objects. This function adheres to the standard mathematical properties of a distance metric (e.g., non-negativity, symmetry, identity of indiscernible, and the triangle inequality).

1.2 Contraction [4]:

Assume that there is a metric space (X, d) so, an application $T: X \to X$:

- Lipchitz function (or K-Lipschitzian) if and only if there is $K \geq 0$ for all $x \in X$ and $y \in X$ then:

$$d(T_x, T_y) \leq K \, d(x, y) \tag{1.2}$$

- Contraction is applied if $K < 1$.
- Non expansive if $K \leq 1$.
- Contractive if: for all $x, y \in X$ we have:

$$d(T_x, T_y) < d(x, y)$$

If T is nonexpansive, then it is Lipchitz function.

1.3 Ordinary Differential Equations [5]:

Equations involving the derivative(s) of an unknown function with respect to an independent variable are referred to as ordinary differential equations (ODEs) [5]. Let the set I be an open set (interval, ball) in $\mathbb{R}$. For a function $x = x(t)$ defined on this set, the equation

$$F(t, x, x', x'', x''', \dots, x^{(n)}) = 0$$

represents an ordinary differential equation. If it involves one independent and one dependent variable, it is called a first-order ODE and can be represented in the following forms:

$$x' = f(t, x) \text{ or } \frac{dx}{dt} = f(t, x) \text{ or } F(t, x, x') = 0 \tag{1.3}$$

The notation $\frac{dx}{dt} = f(t, x)$ hints that the solution to the equation can be found by integrating both sides. The solution can be represented as follows:

$$x(t) = \int f(t)dt + k \tag{1.4}$$

where k is an arbitrary constant. To find a solution satisfying the initial condition $x(t_0) = x_0$, the constant k is replaced with $k = t_0 - x(t_0)$. If no initial conditions are given, the solution obtained is the general solution, whereas the solution derived using specific conditions is called a particular (or partial) solution.

Let $F(t)$ and $i = 1, 2, \dots, n$; $a_i(t)$ be functions with non-zero coefficients that are continuous on an open interval I. The equation

$$a_n(t)x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \cdots + a_0(t)x(t) = F(t)$$

is an n-th order linear variable-coefficient differential equation. When $(t) = 0$, the equation exhibits homogeneity and becomes

$$a_n(t)x^n(t) + a_{n-1}(t)x^{n-1}(t) + \cdots + a_0(t)x(t) = 0$$

This is called an n-th order linear homogeneous variable-coefficient differential equation [18].

Let X be a Banach space, $U \subseteq X, 0 \in I = (a, b)$, and let $K \in C(I \times U, X) - K$ be a vector field.

$$x(t) = K(t, x(t)) \tag{1.5}$$

can also be given with an initial value condition, and in this case, The solution can be derived through an integral equation expressed as

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \tag{1.6}$$

2. Fixed Point Theorem:

2.1 Fixed Point Theorem In A Metric Space:

2.1.1 Banach's Theorem [7]:

Assume that (X, d) is considered complete-metric-space, $T: X \rightarrow X$ is considered contracting map,so:

- There is unique FP (x) for (T).
- For all $x_0 \in X$, $\lim_{n \to \infty} T^n(x_0) = x$.

Proof:

Existence: Let $x_0 \in X$ and $(X_n)_n$ the associated sequence, we have:

$$d(x_n, x_{n+1}) = d(T(x_{n-1}, T(x_n)) \leq kd(x_{n-1}, x_n)$$

We will show by recurrence on n that:

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1) \tag{2.1}$$

For $n = 0$:

$$d(x_0, x_1) \leq k^0 d(x_0, x_1)$$

Suppose the condition is true for n and show for $n + 1$:

$$d(x_{n+1}, x_{n+2}) = d(T(x_n), T(x_{n+1})) \leq Kd(x_n, x_{n+1}) \tag{2.2}$$
$$d(x_{n+1}, x_{n+2}) \leq K(K^n d(x_0, x_1))$$
$$d(x_{n+1}, x_{n+2}) \leq K^{n+1} d(x_0, x_1)$$

We Assume that $(X_n)_n$ is Cauchy, $p, q \in N$, $\forall q > P$:

$$d(x_p, x_q) \leq d(x_p, x_{p+1}) + d(x_{p+1}, x_q)$$
$$d(x_p, x_q) \leq d(x_p, x_{p+1})d(x_{p+1}, x_{p+2}) + \cdots d(x_{q-1}, x_q)$$
$$d(x_p, x_q) \leq K^p d(x_0, x_1) + k^{p+1} d(x_0, x_1) + \cdots k^{q-1} d(x_0, x_1)$$
$$d(x_p, x_q) \leq d(x_0, x_1)[K^p + k^{p+1} + \cdots + k^{q-1}]$$
$$d(x_p, x_q) = d(x_0, x_1)\frac{K^p - k^q}{1-k} \leq d(x_0, x_1)\frac{k^p}{1-k} < \varepsilon \tag{2.3}$$

This shows $(X_n)_n$ If the sequence is Cauchy and the space is complete, then :

$$\lim_{n \to \infty}(x_n) = x$$

Existence of the FP: We consider $\lim_{n \to \infty}(x_n) = x$ s FP of T. We have:

$$x_{n+1} = Tx_n$$

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n \to x = \lim_{n\to\infty} x_n$$

$x = T \lim_{n\to\infty} x_n$ because T is continuous

$x = Tx$ where x is considered as a FP of (T).

Uniqueness of the FP: Suppose there are x_1, x_2 where $x_1 \neq x_2$.

If x_1 is a FP, then $x_1 \neq Tx_1$

If x_2 is a FP, then $x_2 \neq Tx_2$

The proof for the Banach Theorem (also known as the Banach-Caccippoli Theorem) shows how to find the unique FP of a given contraction mapping. In this method, an initial point $x_0 \in X$ is chosen, and an iteration sequence is defined as $x_n = T(x_{n-1})$ for $n \in N$. If $\lim_{n\to\infty} x_n = x$ then $T(x) = x$

In 1962, Edelstein defined contraction mappings for complete metric spaces, providing the contraction condition with $= 1$. In the same year, Rakotch proved the existence and uniqueness of FPs using monotonically decreasing functions within the interval $[0, 1)$. In 1968, Kannan extended the Banach contraction principle by...

$$d(T(x), T(y)) \leq K\,[d(x, T(x)), d(y, T(y))] \tag{2.4}$$

Kannan proved that the mappings satisfying the condition do not need to be continuous. Here, $0 \leq K \leq \frac{1}{2}$ is considered.

In 1969, Boyd and Wong introduced the concept of semi-continuous functions and specifically defined contractions, demonstrating that these served as a generalization of Rakotch's theorem. Meir and Keeler further extended Banach and Edelstein's theorems by defining the Meir-Keeler contraction transformation. In 1972, Chatterjee modified Kannan's fixed point theorem and established a new theorem under more general conditions. Zămărescu synthesized aspects of Banach, Kannan, and Chatterjee's theorems, while Ćirić, followed by Reich, Hardy-Rogers, and others, continued to refine contraction conditions, developing new types of contractions and proving additional fixed point theorems.

Following the works of Rhoades and Meszáros, Branciari, in 2002, proved the existence of FP for a contraction transformation given with an integral condition over a metric space (X, d) using a method similar to that employed by Boyd and Wong.

$$\int_0^{d(Ta,Tb)} f(t)dt \leq K \int_0^{d(a,b)} f(t)dt \tag{2.5}$$

Let (X, d) be a complete metric space, $K \in (0,1)$, and $T: X \to X$ be a function. Suppose f is defined on $[0,\infty)$ and is integrable in the sense of Lebesgue over all compact subsets of this interval, is non-negative, and for any $\tau > 0$, $\int_0^\tau f(t)dt > 0$. Then, the transformation T has a FP.

Let $(X, \leq)$ be a partially ordered set, and let $L: X \times X \to X$ be a function. If $L(a, b)$, is monotonically increasing in a and monotonically decreasing in b, i.e., for every $a, b \in X$, it holds that... [continuing the context would depend on the specific monotonicity conditions you are working with, but the rest of the statement would follow this idea of partial order monotonicity combined with FP conditions.]

1. $a_1, a_2 \in X$, $a_1 \leq a_2$, $L(a_1, b) \leq L(a_2, b)$

2. $b_1, b_2 \in X$, $b_1 \leq b_2$, $L(a, b_1) \geq L(a, b_2)$

The mixed monotonicity property is said to hold if for $a, b \in X \times X$, the following equalities are satisfied: $L(a, b) = a$ and $L(b, a) = b$ An element $a, b \in X \times X$ that satisfies these equalities is called the "coupled fixed point" of L. Based on this property [11],[13] formulated the FP theorem as follows:

Let (X, d) be a complete metric space, and let $L: X \times X \to X$ be a continuous function. If for every $(a, b), (c, m) \in X \times X$, the following conditions hold:

$$\alpha((a, b), (c, m))d(L(a, b), L(c, m)) \leq \frac{1}{2}\psi(d(a, b) + d(c, m))$$

If for every $(a, b), (c, m) \in X \times X$, the following conditions hold:

1. $\alpha\big((a, b), (c, m)\big) \geq 1$ implies $\alpha\big(L(a, b), L(b, a)\big), \big(L(c, m), L(m, c)\big) \geq 1$

2. For $a_0, b_0 \in X$, the conditions $\alpha\big((a_0, b_0), L(a_0, b_0), L(b_0, a_0)\big) \geq 1$ and $\alpha\big(L(b_0, a_0), L(a_0, b_0), (a_0, b_0)\big) \geq 1$ are satisfied,

and if L is continuous, then L has a conjugate FP.

In 2009, Lakshmikantham and Ćirić introduced a new φ function and conditions related to increasing and decreasing sequences in their extension of the above theorem. Nieto and Rodriguez-Lopez (2005) studied the existence of FP using sequences on partially ordered sets.

2.1.2 Picard-Lindelöf Theorem [12]:

The function $f:[\,t_0 - \alpha, t_0 + \alpha\,] \times \overline{\mathcal{B}(\alpha, \rho)} \to R^n$ is continuous and bounded by a fixed constant $H > 0$. For every $\in$ $[\,t_0 - \alpha, t_0 + \alpha\,]$, the function $f(t,*)$ is Lipschitz continuous with a Lipschitz constant. Thus, by taking $= min\left\{\alpha, \frac{\rho}{H}\right\}$, there exists a unique solution to the given ordinary differential equation in the interval

$[\,t_0 - \alpha, t_0 + \alpha\,]$.

2.2 Fixed Point Theorem In A Topological Space:

2.2.1 Brouwer Fixed Point Theorem [10]:

This section discusses Brouwer's Fixed Point Theorem (Brouwer FPT), one of the fundamental pillars of topology (topological geometry) with broad applications in multiple fields.

The Brouwer FPT has been used as a fundamental fact in a number of fields, including game theory (e.g., proving the existence of Nash equilibria), differential equations, differential geometry and algebraic topology. The theory was first stated in 1911 on the case of finite-dimensional simplices and later extended by Schauder in 1930 to the case of Banach spaces, where it became the so-called Schauder Fixed Point Theorem. The relevance of the theorem has since extended in the decades that followed to cover a very comprehensive range of compact and convex subsets, especially in the setting of topological vector spaces.

- In R^n, if X is a closed convex set and $T: X \to R^n$ is a continuous function with $T(X) \subseteq X$
  , then T has a fixed point in X.

- In R^n, if $[0,1]$ is a closed interval and $T: [0,1] \to [0,1]$ is a continuous function with $T(X) \subseteq X$, then T has a fixed point in $[0,1]$.

- For $B = \{(x,y): \|(x,y)\| \le 1\} \subseteq R^2$, if $T: B \to B$ is a continuous function, then T has a FP in B.

- For $B^n = \{x: \|x\| \le 1\} \subseteq R^n$, if $T: B^n \to X \subseteq R^n$ is continuous, then T has a FP in B^n (Durak, 2021).

Given $B^n = \{x: \|x\| \le 1\} \subseteq R^n$, if $E \subseteq B^n$ and $R: B^n \to E$ is continuous with $R(x) = x$

for every $x \in E$, then R is called a retraction.

The sphere $S^{n-1} = \{x: \|x\| = 1\} \subseteq R^n$ cannot be retracted into B^n. This can be easily proven using algebraic topology.

The proof of the Brouwer Fixed Point Theorem on B^n is as follows: Consider a class of retraction functions from B^n to S^{n-1} of class C^2. Define $f: B^n \to B^n$ with retraction transformations $T(x)$ of class C^2.

$$T(x) \quad = \frac{\|x\|^2 - \langle x, f(x)\rangle - \sqrt{(\|x\|^2 - \langle x, f(x)\rangle)^2 + (1 - \|x\|^2)\|x - f(x)\|^2}}{\|x - f(x)\|^2} \tag{2.6}$$

$$R(x) = T(x)f(x) + (1 - T(x))x$$

This is given as follows. However, this contradicts the statement that $S^{n-1} = \{x: \|x\| = 1\} \subseteq R^n$ cannot be retracted into B^n. The function $R(x)$ intersects S^{n-1} with $(f(x), x)$ at the point of intersection.

Let . $f: B^n \to B^n$ be continuous. Let $f_i: B^n \to B^n$ be a sequence of functions also of class C^2. This sequence of functions , f_i converges uniformly to f. Each $x \in B^n$ has a subsequence of functions , f_i whose FP converge to x.

$$\|f(x) - x\| \le \|f(x) - f(x_i)\| + \|f(x_i) - f_i(x_i)\| + \|x_i - x\|$$

$$i \to \infty , f(x) = x .$$

2.2.2 Schauder-Tychonoff Theorem [21]:

First, the information that will be used in the proof of this theorem will be provided in order.

Let X be a finite-dimensional real Banach space and $D \subseteq X$ be a compact, convex subset with no gaps. In this case, every continuous function $f: D \to D$ has a FP.

Here, X is homeomorphic to R^n for some $n \in N$, so generally, $X = R^n$ and $D \subset B^n$ are considered. For $\in B^n$, taking the minimum norm of the set $\{x - D\}$, $l(x) \in D$ becomes the singular point of this norm. It is clear that for every $x \in D$ $l(x) = x$ Since l is a continuous transformation, as $n \to \infty$

$$\|x - l(x)\| \leq \|x - l(x_n)\| \leq \|x - x_n\| + inf_{d \in B^n}\|x_n - d\| \rightarrow \|x - l(x)\|$$

This sequence $x_n \rightarrow x$ is obtained. Using this sequence, the minimum sequence $x - l(x_n)$ and the convergence $l(x_n) \rightarrow l(x)$ are defined. If we define $(x) = f(l(x))$, then g becomes continuous. By Brouwer's Theorem, it is clear that for every $x \in D$, $g(x) = x = f(x)$. Thus, the proof is complete. Direct applications of the given theorem can be seen in Frobenius and the fundamental theorem of algebra.

The sets $\nabla_1, \nabla_2, \ldots, \nabla_n$ are open sets in a space that is both locally compact and Hausdorff X, and let $\kappa \subset \nabla_1 \cup \nabla_2 \cup \ldots \cup \nabla_n$. For every $x \in \kappa$ and $= 1,2, \ldots, n$, let $\theta_\lambda \subset C(X)$ be such that $0 \leq \theta_\lambda \leq 1$ and
$$\theta_1(x) + \theta_2(x) + \cdots + \theta_n(x) = 1 \tag{2.7}$$

The transformations $\theta_1, \theta_2, \ldots, \theta_n$ form a partition of unity for κ with respect to the open cover $\{\nabla_1, \nabla_2, \ldots, \nabla_n\}$. This result is a direct consequence of the Urysohn lemma. A partition of unity can be sought for the compact set $\kappa \subset X$ even when X is not locally compact. The statement of the theorem is as follows: Let X be a locally compact space, and let $\kappa \subset X$ be a non-empty, convex compact set, and $\kappa_0 \subset \kappa$ be compact. Then, for the continuous map $f: \kappa \rightarrow \kappa_0$, there exists $x \in \kappa_0$ s.t $f(x) = x$.

2.2.3 The Leray-Schauder Principle [20]:

Let X be a Banach space, $C \subseteq X$ a closed and convex subset, and $\Omega \subset C$ an open, bounded set. Suppose there exists a fixed element $P \in N$, and let $T: \overline{N} \rightarrow C$ be a continuous and compact operator satisfying the boundary condition:
$$T(x) \neq P + \lambda(x - p), for all\ Q \in \delta U , \lambda > 1$$

Then, T admits at least one fixed point in $\overline{\Omega}$.

3. Applications to ordinary differential equations

Based on the basic concepts of the theory of fixed points, the study will propose a new type of contraction mappings in complete metric spaces, with strict formulation and demonstrations of fixed point theorems related to them. Even though the proposed framework is based on the groundbreaking contraction principle by Banach, it offers its main advantage to the work of its generalization and refinement of the classical findings, thus overcoming the shortcomings of the current literature.

Samet et al. selected the following ordinary differential equation for their study: Let $C([0,1])$ denote the set of all continuous functions defined on the interval $[0,1]$ and let ve $d: C([0,1]) \times C([0,1]) \rightarrow \mathbb{R}$ be defined as follows:
$$d(x,y) = \|x - y\|_\infty = max\{|x(t) - y(t)| : t \in [0,1]\} \tag{3.1}$$

Here, $(([0, 1]), d)$ represents a complete metric space. The boundary value problem for the second-order differential equation is formulated as follows:
$$-\frac{d^2}{dt^2} = f(t, x(t)), t \in [0,1] \tag{3.2}$$

with boundary conditions $x(0) = x(1) = 0$, where $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. The Green's function associated with this problem is defined as:
$$G(t,s) = \{t(1 - s), 0 \leq t \leq s \leq 1;$$
$$s(1 - t), 0 \leq s \leq t \leq 1\}.$$

Let's assume the following conditions hold:

(i) There exists a function $\phi: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that for all $t \in [0, 1]$ and all $a, b \in \mathbb{R}$ if $(a,b) \geq 0$, then
$$|f(t,a) - f(t,b)| \leq 8\psi(\max_{a,b \in \mathbb{R}, \phi(a,b) \geq 0}(a - b)), \tag{3.3}$$

where $\psi \in \Psi_2$ (a suitable class of functions);

(ii) There exists $x_0 \in C([0, 1])$ such that for each $t \in [0, 1]$
$$\phi(x_0(t), \int_0^1 G(t,s)f(s,x_0(s))ds) \geq 0 \tag{3.4}$$

(iii) For all $t \in [0,1]$ and all $x, y \in C([0,1]), \phi(x(t), y(t)) \geq 0$ and
$$\phi(\int_0^1 G(t,s)f(s,x(s)), \int_0^1 G(t,s)f(s,y(s))ds) \geq 0 \tag{3.5}$$

(iv) If $\{x_n\}$is a sequence in $C([0,1])$ such that for all $\forall n \in \mathbb{N}$, $\phi(x_n, x_{n+1}) \geq 0$ and $x_n \rightarrow x \in C([0,1])$, then for all$\forall n \in \mathbb{N}$, $x_n \preccurlyeq x$, and $\phi(x_n, x) \geq 0$

The idea for proving the following theorem is inspired by the work of Samet et al. (2012), though it incorporates some differences.

Theorem: Under conditions (i)–(iv), the boundary value problem has a solution in $C^2([0, 1])$

Result:

Let $C([0,1])$be the set of all continuous functions defined on[0,1], and assume the following conditions:

(i) $f : C([0,1]) \times \mathbb{R} \rightarrow [0, \infty)$ is continuous and non-decreasing

(ii) There exists a function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that for all $t \in [0,1]$ and all , $b \in \mathbb{R}$, if $\phi(a,b) \geq 0$, then

$$|f(t,a) - f(t,b)| \leq 8\,\psi(\max_{a,b\,\in\,\mathbb{R}, \phi(a,b)\,\geq\,0}(a - b)) \tag{3.6}$$

Where $\psi \in \Psi_2$

(iii)There exists $x_0 \in C([0,1])$ such that

$$x_0(t) \leq \int_0^1 G(t,s)f(s, x_0(s)\,)ds) \tag{3.7}$$

then the boundary value problem has a unique solution in $C^2([0, 1])$

Example1 : We consider the differential equation with an initial condition (initial value problem):

$$f\big(x, y(x)\big) = \frac{dy}{dx} \qquad , y(x_0) = y_0$$

Assume that the function $f(x,y)$is defined on the rectangle

$$R = \{|x - x_0| \leq a, |y - y_0| \leq b\}$$

Assumptions:

1-The function $f(x,y)$ is continuous on R and therefore bounded on R; that is,

$$\forall(x,y) \in R \rightarrow \exists M > 0;\ |f(x,y)| \leq M$$

2- The partial derivative of $f(x,y)$with respect to y, denoted by $f(x,y)$, exists and is bounded on R; that is,

$$\forall(x,y) \in R \rightarrow \exists K > 0;\ |f_y(x,y)| \leq K$$

3-The function $f(x,y)$satisfies a Lipchitz condition with respect to y on R, namely,

$$|f(x,y) - f(x,\beta)| \leq K|y - \beta|$$

This condition follows from the Mean Value Theorem, since

$$|f(x,y) - f(x,\beta)| \leq |f_y(x,w)||y(x) - \beta(x)|$$

To establish the existence and uniqueness of a continuous and differentiable solution to the Cauchy problem in a neighborhood of x_0

Solution:

We prove that the three previous conditions are satisfied; therefore, there exists a positive constant $h > 0$ such that the differential equation admits a unique solution on the interval $[x_0 - h, x_0 + h]$.

$$y(x) - y(x_0) = \int_{x_0}^{x} y' dx = \int_{x_0}^{x} f(x, y(x)) dx$$

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y(x)) dx$$

Let us choose: $h = \min \left\{ \frac{1}{K}, a, \frac{b}{M} \right\}$

We define the space of all continuous functions on the interval $[x_0 - h, x_0 + h]$

We define the following metric:

$$d(y, \beta) = \max_{x_0 - h \le x \le x_0 + h} |y(x) - \beta(x)|$$

On the space C, this function defines a metric; hence, (C,d) is a complete metric space.

We define the norm on C by:

moreover, C endowed with this norm is a banach space. complete metric space(C, d)

Next, we define the operator:

$$\|y\| = \max_{x_0 - h \le x \le x_0 + h} |y(x)|$$

$$A: C \to C$$

$$y(x) \to A(y(x)) = y_0 + \int_{x_0}^{x} f(x, y(x)) dx$$

It follows that the operator defined above is well-defined and continuous on $[x_0 - h, x_0 + h]$

$$\left| \int_{x_0}^{x} f(x, y(x)) dx \right| \le \int_{x_0}^{x} |f(x, y(x))| dx \le Mh < b$$

We show that the operator A is a contraction mapping, since

$$d(A(y), A(\beta)) = \max_{x_0 - h \le x \le x_0 + h} \left| \int_{x_0}^{x} [f(x, y(x)) - f(x, \beta(x))] dx \right| \le \int_{x_0}^{x} [f(x, y(x)) - f(x, \beta(x))] dx$$

$$\le K \int_{x_0}^{x} |y(x) - \beta(x)| dx \le K \max_{x_0 - h \le x \le x_0 + h} |y(x) - \beta(x)| . h \le K . h \, d(y, \beta)$$

Therefore, by Banach's Fixed Point Theorem, there exists a unique solution $y(x)$ of the differential equation, which is continuous and differentiable on the interval $[x_0 - h, x_0 + h]$

4. Conclusion:

The research has made significant theoretical and practical contributions to the field of fixed point (FP) theory and its applications in complete metric spaces. The key contributions

- include: Theoretical Contributions: Introduction of a novel type of contraction mapping, providing new perspectives and generalizations of existing FP theorems.
- Applications: Highlighted the relevance of FP theory in diverse fields such as engineering, economics, and biology, with specific applications in stability analysis, optimization, and numerical methods.
- Methodological Advancements: Extended the applicability of Banach's contraction principle to general metric spaces, enhancing its utility for complex systems.
- Future Research Directions: Proposed pathways for further studies to generalize these contractions to other mathematical structures and explore their connections with additional FP theorems.

This study underscores the growing potential of FP theory in advancing both theoretical mathematics and applied sciences, emphasizing its broader applicability and ability to yield more comprehensive results.

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