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A Novel Category of Analytic Univalent Functions that is characterized by an Integral Operator

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ABSTRACT

This paper presents a novel class of analytic univalent functions, which are defined by an integral operator in the open unit disk $\mathrm{U} = \{z \in \mathbb{C}: |z| < 1\}$. Various geometric properties of these functions, including coefficient inequality, growth and distortion bounds, and convolution properties, are investigated.

MSC..

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1. Introduction

Let $\mathcal{A}$ be the class of all functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1)$$

which are analytic and univalent in the open unit disk U.

Let $\mathcal{N}$ be symbolized the subclass of $\mathcal{A}$ containing of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0). \qquad (2)$$

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For functions $f(z) \in \mathcal{N}$, given by (2), and $g(z) \in \mathcal{N}$ given by

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad (z \in U; b_n \geq 0), \tag{3}$$

the convolution (or Hadamard product) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z). \tag{4}$$

A function $f(z) \in \mathcal{A}$ is called univalent starlike of order $\alpha(0 \leq \alpha < 1)$, if $f(z)$ satisfies the condition:

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in U). \tag{5}$$

Further, a function $f(z) \in \mathcal{A}$ is called univalent convex of order $\alpha(0 \leq \alpha < 1)$, if $f(z)$ satisfies the condition:

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (z \in U). \tag{6}$$

We denoted by $S^*(\alpha)$ and $C^*(\alpha)$ the classes of univalent starlike functions of order α and univalent convex functions of order α, respectively.

Patel [5] defined an Integral operator $\mathcal{J}_{\nu,\delta}^{\eta}$ on $\mathcal{A}$ as follows:

Let $\eta \in \mathrm{N}_0 = \mathrm{N} \cup \{0\}, \delta \geq 0$ with $\nu + \delta > 0$ and ν a real number,

then for $f \in \mathcal{A}$, we define the operator $\mathcal{J}_{\nu,\delta}^{\eta}$ by

$$\mathcal{J}_{\nu,\delta}^{0} f(z) = f(z)$$

$$\mathcal{J}_{\nu,\delta}^{1} f(z) = \left(\frac{\nu+\delta}{\delta}\right) z^{1-\left(\frac{\nu+\delta}{\delta}\right)} \int_0^z t^{\left(\frac{\nu+\delta}{\delta}\right)-2} f(t)dt, z \in U.$$

$$\mathcal{J}_{\nu,\delta}^{2} f(z) = \left(\frac{\nu+\delta}{\delta}\right) z^{1-\left(\frac{\nu+\delta}{\delta}\right)} \int_0^z t^{\left(\frac{\nu+\delta}{\delta}\right)-2} \mathcal{J}_{\nu,\delta}^{1} f(t)dt, z \in U.$$

$\vdots$

$$\mathcal{J}_{\nu,\delta}^{\eta} f(z) = \left(\frac{\nu+\delta}{\delta}\right) z^{1-\left(\frac{\nu+\delta}{\delta}\right)} \int_0^z t^{\left(\frac{\nu+\delta}{\delta}\right)-2} \mathcal{J}_{\nu,\delta}^{\eta-1} f(t)dt, z \in U.$$

$$= \mathcal{J}_{\nu,\delta}^{1} \left(\frac{z}{1-z}\right) * \mathcal{J}_{\nu,\delta}^{1} \left(\frac{z}{1-z}\right) * \dots * \mathcal{J}_{\nu,\delta}^{1} \left(\frac{z}{1-z}\right) * f(z).$$

We observe that $\mathcal{J}_{\nu,\delta}^{\eta}: \mathcal{A} \to \mathcal{A}$ is an Integral operator and for f given by (1.1), we have

$$\mathcal{J}_{\nu,\delta}^{\eta} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\nu+\delta}{\nu+n\delta}\right)^{\eta} a_n z^n. \tag{7}$$

It follows from (7) that $\mathcal{J}_{\nu,0}^{\eta} f(z) = f(z)$.

We note that

1) $\mathcal{J}^{\eta}_{1,1}\, f(z) = \mathrm{T}^{\eta} f(z)$ (See [5, 7]).
2) $\mathcal{J}^{\eta}_{1-\delta,1}\, f(z) = \mathrm{T}^{\eta}_{\delta} f(z), \delta > 0$ (See [5]).
3) $\mathcal{J}^{\eta}_{\nu,1}\, f(z) = \mathrm{T}^{\eta}_{\nu} f(z), \nu > 0$ (See [5]).

Now, by using an Integral operator $\mathcal{J}^{\eta}_{\nu,\delta}\, f(z)$, we have the following:

Definition 1 A function $f(z) \in \mathcal{N}$ is said to be in the class $\mathcal{AN}(\lambda, \tau, \eta, \nu, \delta)$ if it satisfies the following condition:

$$\left| \frac{\lambda z^2 \left(\mathcal{J}^{\eta}_{\nu,\delta}\, f(z)\right)'' + \tau z \left(\mathcal{J}^{\eta}_{\nu,\delta}\, f(z)\right)'}{(1-\lambda)(1+\tau)\left(\mathcal{J}^{\eta}_{\nu,\delta}\, f(z)\right)} \right| < 1,$$

Where $0 \leq \lambda < 1,\ 0 \leq \tau < 1, \nu, \eta > 0,\ \delta \in \mathbb{N}$ *and* $z \in U$.

Some of the following properties have been studied for other classes in [1, 2, 3, 4, 6, 8, and 9].

2. Coefficient Inequality

The following theorem gives a necessary and sufficient condition for a function $f(z)$ to be in class $\mathcal{AN}(\lambda, \tau, \eta, \nu, \delta)$.

Theorem 1 A function $f(z) \in \mathcal{N}$ is in the class $\mathcal{AN}(\lambda, \tau, \eta, \nu, \delta)$ if and only if

$$\sum_{n=2}^{\infty} n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda]\left(\frac{\nu+\delta}{\nu+n\delta}\right)^{\eta} z^n \leq 1+\lambda(1-\tau), \tag{8}$$

Where $0 \leq \lambda < 1,\ 0 \leq \tau < 1, \eta > 0,\ \delta \in \mathbb{N}$ *and* $z \in U$.

The result is sharp for the function $f(z)$ given by

$$g(z) = z - \frac{1+\lambda(1-\tau)(\nu+\delta)^{\eta}}{n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda](\nu+n\delta)^{\eta}} z^n, \quad (n \geq 2) \tag{9}$$

Proof. Assume that inequality (8) holds and $|z| = 1$. Then we get

$$\left|\lambda z^2 \left(\mathcal{J}^{\eta}_{\nu,\delta}\, f(z)\right)'' + \tau z \left(\mathcal{J}^{\eta}_{\nu,\delta}\, f(z)\right)'\right| - \left|(1-\lambda)(1+\tau)\left(\mathcal{J}^{\eta}_{\nu,\delta}\, f(z)\right)\right|$$

$$= \left|\tau z + \sum_{n=2}^{\infty} n(\lambda(n-1)+\tau)\left(\frac{\nu+\delta}{\nu+n\delta}\right)^{\eta} a_n z^n\right| - \left|(1-\lambda)(1+\tau) + \sum_{n=2}^{\infty}(1-\lambda)(1+\tau)\left(\frac{\nu+\delta}{\nu+n\delta}\right)^{\eta} a_n z^n\right|$$

$$\leq \sum_{n=2}^{\infty} n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda]\left(\frac{\nu+\delta}{\nu+n\delta}\right)^{\eta} a_n z^n - (1+\lambda(1-\tau)) \leq 0,$$

By hypothesis. Hence by maximum modulus principle, we get $g(z) \in \mathcal{AN}(\lambda, \tau, \eta, \nu, \delta)$

Conversely, let $f(z) \in \mathcal{AN}(\lambda, \tau, n, \rho)$. Then

$$\left| \frac{\lambda z^2 \left(\mathcal{J}^{\eta}_{\nu,\delta}\, f(z)\right)'' + \tau z \left(\mathcal{J}^{\eta}_{\nu,\delta}\, f(z)\right)'}{(1-\lambda)(1+\tau)\left(\mathcal{J}^{\eta}_{\nu,\delta}\, f(z)\right)} \right| = \left| \frac{\tau z + \sum_{n=2}^{\infty} n(\lambda(n-1)+\tau)\left(\frac{\nu+\delta}{\nu+n\delta}\right)^{\eta} a_n z^n}{(1-\lambda)(1+\tau) + \sum_{n=2}^{\infty}(1-\lambda)(1+\tau)\left(\frac{\nu+\delta}{\nu+n\delta}\right)^{\eta} a_n z^n} \right| < 1$$

Since $R(z) \leq |z|$ for all $z(z \in \mathrm{U})$, then we obtain

$$Re\left(\frac{\tau z+\sum_{n=2}^{\infty}n(\lambda(n-1)+\tau)\left(\frac{\nu+\delta}{\nu+n\delta}\right)^{\eta}a_n z^n}{(1-\lambda)(1+\tau)+\sum_{n=2}^{\infty}(1-\lambda)(1+\tau)\left(\frac{\nu+\delta}{\nu+n\delta}\right)^{\eta}a_n z^n}\right)<1 \qquad (10)$$

Now, choosing the values of z on the real axis and let $z\to 1$ from the left through real values, the inequality (10) immediately yields the required condition in (8). Finally, it is observed that the result is sharp for the function is given by (9).

Corollary 1 If $g(z)\in\mathcal{AN}(\lambda,\tau,\eta,\nu,\delta)$, then

$$a_n\le\frac{1+\lambda(1-\tau)(\nu+\delta)^{\eta}}{n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda](\nu+n\delta)^{\eta}}\ ,\quad (n\ge 2) \qquad (11)$$

3. Growth and Distortion Bounds

Next, we prove the growth and distortion bounds for an integral operator $\mathcal{I}_{\nu,\delta}^{\eta}f(z)$.

Theorem 2 If $f(z)\in\mathcal{AN}(\lambda,\tau,\eta,\nu,\delta)$, then

$$r-\frac{(1-\tau)}{2(2-\tau)}r^2\le\left|\mathcal{I}_{\nu,\delta}^{\eta}f(z)\right|\le r+\frac{(1-\tau)}{2(2-\tau)}r^2,(0<|z|=r<1) \qquad (12)$$

The result is sharp for the function $f(s)$ is given by

$$f(z)=z-\frac{1+\lambda(1-\tau)(\nu+\delta)^{\eta}}{2[\lambda(1+\tau)][1+\tau-\lambda(1-\tau)](\nu+2\delta)^{\eta}}z^2 \qquad (13)$$

Proof. Let $f(z)\in\mathcal{AN}(\lambda,\tau,\eta,\nu,\delta)$, then by Theorem 1

$$\begin{aligned}2[\lambda(1+\tau)][1+\tau-\lambda(1-\tau)]\left(\frac{\nu+\delta}{\nu+2\delta}\right)^{\eta}\sum_{n=2}^{\infty}a_n&\le\sum_{n=2}^{\infty}n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda]\left(\frac{\nu+\delta}{\nu+n\delta}\right)^{\eta}z_n\\&\le 1+\lambda(1-\tau)\sum_{n=2}^{\infty}a_n\\&\le\frac{1+\lambda(1-\tau)(\nu+\delta)^{\eta}}{2[\lambda(1+\tau)][1+\tau-\lambda(1-\tau)](\nu+2\delta)^{\eta}}\end{aligned} \qquad (14)$$

Hence,

$$\begin{aligned}\left|\mathcal{I}_{\nu,\delta}^{\eta}f(z)\right|&\le|z|+\sum_{n=2}^{\infty}n\left(\frac{\nu+\delta}{\nu+n\delta}\right)^{\eta}a_n|z|^n\le r+2\left(\frac{\nu+\delta}{\nu+2\delta}\right)^{\eta}r^2\sum_{n=2}^{\infty}a_n\\&\le r+\frac{(1-\tau)}{2(2-\tau)}r^2\end{aligned} \qquad (15)$$

Similarly

$$\begin{aligned}\left|\mathcal{I}_{\nu,\delta}^{\eta}f(z)\right|&\ge|z|-\sum_{n=2}^{\infty}n\left(\frac{\nu+\delta}{\nu+n\delta}\right)^{\eta}a_n|z|^n\ge r-2\left(\frac{\nu+\delta}{\nu+2\delta}\right)^{\eta}r^2\sum_{n=2}^{\infty}a_n\\&\ge r-\frac{(1-\tau)}{2(2-\tau)}r^2\end{aligned} \qquad (16)$$

By (15) and (16), we get (12). Thus, the proof is done

Theorem 3. If $f(z)\in\mathcal{AN}(\lambda,\tau,\eta,\nu,\delta)$, then

$$1-\frac{(1-\tau)}{(2-\tau)}r \le \left|\left(\mathcal{J}_{\nu,\delta}^{\eta} f(z)\right)'\right| \le 1+\frac{(1-\tau)}{(2-\tau)}r, (0<|z|=r<1) \qquad (17)$$

The result is sharp for the function $f(z)$ is given by (13)

Proof. Let $f(z) \in \mathcal{AN}(\lambda,\tau,\eta,\nu,\delta)$, then by Theorem 1

$$[\lambda(1+\tau)][1+\tau-\lambda(1-\tau)]\left(\frac{\nu+\delta}{\nu+2\delta}\right)^{\eta}\sum_{n=2}^{\infty} na_n \le \sum_{n=2}^{\infty} n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda]\left(\frac{\nu+\delta}{\nu+n\delta}\right)^{\eta} a_n$$
$$\le 1+\lambda(1-\tau)\sum_{n=2}^{\infty} na_n \le \frac{(1-\tau)(4+\rho^2)^2}{2(2-\tau)(\rho^2+1)^2} \qquad (18)$$

Thus

$$\left|\left(\mathcal{J}_{\nu,\delta}^{\eta} f(z)\right)'\right| \le 1+\sum_{n=2}^{\infty}\left(\frac{\nu+\delta}{\nu+n\delta}\right)^{\eta} na_n|z|^{n-1} \le 1+\left(\frac{\nu+\delta}{\nu+2\delta}\right)^{\eta} r\sum_{n=2}^{\infty} na_n$$
$$\le 1+\frac{(1-\tau)}{(2-\tau)}r \qquad (19)$$

Similarly

$$\left|\left(\mathcal{J}_{\nu,\delta}^{\eta} f(z)\right)'\right| \ge 1-\sum_{n=2}^{\infty}\left(\frac{\nu+\delta}{\nu+n\delta}\right)^{\eta} na_n|z|^{n-1} \ge 1-\left(\frac{\nu+\delta}{\nu+2\delta}\right)^{\eta} r\sum_{n=2}^{\infty} na_n$$
$$\ge 1-\frac{(1-\tau)}{(2-\tau)}r \qquad (20)$$

From (19) and (20), we get (17). Thus, the proof is complete.

4. Convolution Properties

Theorem 4 Let f_i, $(i=1,2)$, be a function in the class $\mathcal{AN}(\lambda,\tau,\eta,\nu,\delta)$ and defined by

$$f_i = z-\sum_{n=2}^{\infty} a_{n,i}\, z^n, \quad (a_{n,i} \ge 0, \qquad i=1,2) \qquad (21)$$

Then $f_1 * f_2 \in \mathcal{AN}(\beta,\tau,\eta,\nu,\delta)$, where

$$\beta \le \frac{n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda](\nu+n\delta)^{\eta}-[1+\tau-\lambda-\tau\lambda][1+\beta(1-\tau)](\nu+\delta)^{\eta}}{(n-1+\tau)[1+\beta(1-\tau)](\nu+\delta)^{\eta}-n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda](\nu+n\delta)^{\eta}}$$

Proof. We have to find the largest μ such that

$$\sum_{n=2}^{\infty}\frac{n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda](\nu+n\delta)^{\eta}}{1+\beta(1-\tau)(\nu+\delta)^{\eta}} a_{n,1}a_{n,2} \le 1$$

Since $k_j \in \mathcal{AN}(\lambda,\tau,n,\rho)$, $(i=1,2)$, then

$$\sum_{n=2}^{\infty}\frac{n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda](\nu+n\delta)^{\eta}}{1+\beta(1-\tau)(\nu+\delta)^{\eta}} a_{n,i} \le 1, \qquad (i=1,2) \qquad (22)$$

By Cauchy – Schwarz inequality, we get

$$\sum_{n=2}^{\infty}\frac{n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda](\nu+n\delta)^{\eta}}{1+\beta(1-\tau)(\nu+\delta)^{\eta}}\sqrt{a_{n,1}a_{n,2}}\leq 1 \tag{23}$$

We want to show that

$$\frac{n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda](\nu+n\delta)^{\eta}}{1+\beta(1-\tau)(\nu+\delta)^{\eta}}a_{n,1}a_{n,2}\leq\frac{n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda](\nu+n\delta)^{\eta}}{1+\beta(1-\tau)(\nu+\delta)^{\eta}}\sqrt{a_{n,1}a_{n,2}}$$

This equivalent to

$$\sqrt{a_{n,1}a_{n,2}}\leq\frac{(1+\mu)[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda]}{(1+\beta)[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda]} \tag{24}$$

From (24), we get

$$\sqrt{a_{n,1}a_{n,2}}\leq\frac{1+\beta(1-\tau)(\nu+\delta)^{\eta}}{n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda](\nu+n\delta)^{\eta}}$$

Therefore, it is sufficient to show that

$$\frac{1+\beta(1-\tau)(\nu+\delta)^{\eta}}{n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda](\nu+n\delta)^{\eta}}\leq\frac{(1+\mu)[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda]}{(1+\beta)[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda]},$$

Which implies to

$$\beta\leq\frac{n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda](\nu+n\delta)^{\eta}-[1+\tau-\lambda-\tau\lambda][1+\beta(1-\tau)](\nu+\delta)^{\eta}}{(n-1+\tau)[1+\beta(1-\tau)](\nu+\delta)^{\eta}-n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda](\nu+n\delta)^{\eta}},$$

Theorem 5 Let the function f_i $(i=1,2)$ defined by (22) be in the class $\mathcal{AN}(\lambda,\tau,\eta,\nu,\delta)$

Then the function f defined by

$$f(z)=z-\sum_{n=2}^{\infty}\left(a_{n,1}^2+a_{n,2}^2\right)z^n, \tag{25}$$

belong to the class $\mathcal{AN}(\epsilon,\tau,n,\rho)$, where

$$\epsilon\leq\frac{n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda](\nu+n\delta)^{\eta}-[1+\tau-\lambda-\tau\lambda][1+\beta(1-\tau)](\nu+\delta)^{\eta}}{(n-1+\tau)[1+\beta(1-\tau)](\nu+\delta)^{\eta}-n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda](\nu+n\delta)^{\eta}}$$

Proof. We must find the largest ϵ such that

$$\sum_{n=2}^{\infty}\frac{n[\lambda(n-1+\tau)][1+\tau-\epsilon-\tau\epsilon](\nu+n\delta)^{\eta}}{1+\beta(1-\tau)(\nu+\delta)^{\eta}}\left(a_{n,1}^2+a_{n,2}^2\right)\leq 1$$

Since $f_i\in\mathcal{AN}(\lambda,\tau,\eta,\nu,\delta),(i=1,2)$, we get

$$\sum_{n=2}^{\infty}\left(\frac{n[\lambda(n-1+\tau)][1+\tau-\epsilon-\tau\epsilon](\nu+n\delta)^{\eta}}{1+\beta(1-\tau)(\nu+\delta)^{\eta}}\right)^2 a_{n,1}^2$$

$$\leq\sum_{n=2}^{\infty}\left(\frac{n[\lambda(n-1+\tau)][1+\tau-\epsilon-\tau\epsilon](\nu+n\delta)^{\eta}}{1+\beta(1-\tau)(\nu+\delta)^{\eta}}a_{n,1}\right)^2\leq 1 \tag{26}$$

And

$$\sum_{n=2}^{\infty}\left(\frac{n[\lambda(n-1+\tau)][1+\tau-\epsilon-\tau\epsilon](\nu+n\delta)^{\eta}}{1+\beta(1-\tau)(\nu+\delta)^{\eta}}\right)^2 a_{n,2}^2$$

$$\leq\sum_{n=2}^{\infty}\left(\frac{n[\lambda(n-1+\tau)][1+\tau-\epsilon-\tau\epsilon](\nu+n\delta)^{\eta}}{1+\beta(1-\tau)(\nu+\delta)^{\eta}}a_{n,2}\right)^2\leq 1 \tag{27}$$

Combining the inequalities (26) and (27), gives

$$\sum_{n=2}^{\infty} \frac{1}{2}\left(\frac{n[\lambda(n-1+\tau)][1+\tau-\epsilon-\tau\epsilon](\nu+n\delta)^{\eta}}{1+\beta(1-\tau)(\nu+\delta)^{\eta}}\right)^2 \left(a_{n,1}^2+a_{n,2}^2\right) \leq 1 \qquad (28)$$

But, $k \in \mathcal{AN}(\epsilon,\tau,n,\rho)$ if and only if

$$\sum_{n=2}^{\infty} \frac{n[\lambda(n-1+\tau)][1+\tau-\epsilon-\tau\epsilon](\nu+n\delta)^{\eta}}{1+\beta(1-\tau)(\nu+\delta)^{\eta}} \left(a_{n,1}^2+a_{n,2}^2\right) \leq 1 \qquad (29)$$

The inequality (29) will be satisfied if

$$\frac{n[\lambda(n-1+\tau)][1+\tau-\epsilon-\tau\epsilon](\nu+n\delta)^{\eta}}{1+\beta(1-\tau)(\nu+\delta)^{\eta}} \leq \frac{n^2[\lambda(n^2-1+\tau)][1+\tau-\epsilon-\tau\epsilon](\nu+n\delta)^{\eta}}{1+\beta(1-\tau)(\nu+\delta)^{\eta}}$$

So that

$$\epsilon \leq \frac{n[\lambda(n-1+\tau)][1+\tau-\lambda-\tau\lambda](\nu+n\delta)^{\eta}}{1+\beta(1-\tau)(\nu+\delta)^{\eta}}$$

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