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New subclasses for bi-univalent functions in relation to Sălăgean integro-differential operator

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ABSTRACT

In this present work, two novel classes for bi-univalent functions are introduced using the recently developed Sălăgean integro-differential operator. Estimates for the two initial coefficients of Taylor-Maclaurin series are also provided. Furthermore, the well-known Fekete-Szegö functional is investigated and bounds for each of the initial coefficients, $|e_2|$and $|e_3|$ for the functions belonging to these classes are obtained.

MSC..

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1. Introduction

Investigations on the class of bi-univalent functions began a few decades ago, around 1970 as reported in [1, 2]. In the course of the past decades, the subject matter became intriguing, with numerous articles issued [3]. Relevant estimations of coefficients for particular classes of univalent functions appeared, such as those released in [4 - 7]. Since the start of investigations of complex functions, operators have been widely developed and used. Many previous findings have been made simpler by applying them, and novel results, particularly concerning the starlikeness and convexity of specific functions, have been discovered.

The primary aim of these investigations is the introduction of new classes for analytic functions. The investigations on bi-univalent functions employing operators is additionally an increasingly common strategy nowadays, as evidenced by some of the most recent findings [8]. Also, there is an increasing interest for obtaining the Fekete-Szegö functional in the study of particular classes introduced, as seen in one of the recent investigations [9]. The investigation of coesfficient estimates for functions in certain classes is a subject that dates back to the start of the study of univalent functions. Gronwall's Area Theorem, appeared in 1914 is an important discovery in the theory of univalent functions. It is employed to obtain bounds on the value of the coefficients for the class of meromorphic functions.

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In 1916, Bieberbach resolved a comparable issue regarding the class, and his renowned conjecture mentioned in the same year but only applied in 1984, contributed to the development of different approaches in the geometric theory of the functions of a complex variable. Estimates on the first two Taylor-Maclaurin coefficients typically appear in the investigation of bi-univalent functions, similarly to how they are in the instance of the classes investigated by Gronwall as well as Bieberbach. We expand the investigation and are able to provide estimates on the fourth coefficient for the functions within the classes presented in this work. The operator employed in developing the three novel classes from which estimations of coefficients have been obtained contributes to the novelty of the findings presented in this paper. An operator had been described in [10] as an entirely novel kind of operator developed by combining the differential and integral kinds of the renowned Sălăgean operator.

Let $\mathcal{A}$ denote the class of normalized analytic functions in the open unit disk $\Lambda = \{\hbar \in \mathbb{C} : |\hbar| < 1\}$. Let

$$\phi(\hbar) = \hbar + \sum_{\nu=2}^{\infty} e_\nu \hbar^\nu , (\hbar \in \Lambda) \tag{1}$$

with $\mathcal{S} \subset \mathcal{A}$ comprises of functions univalent in Λ. When a function $k \in \mathcal{A}$ exists and is given by

$$k(\hbar) = \hbar + \sum_{\nu=2}^{\infty} d_\nu \hbar^\nu , (\hbar \in \Lambda),$$

then the Hadamrd product over the two functions is given by

$$\phi(\hbar) * k(\hbar) = \hbar + \sum_{\nu=2}^{\infty} e_\nu d_\nu \hbar^\nu , (\hbar \in \Lambda). \tag{2}$$

The theory of q-calculus is essential in various fields of mathematics, physics, and engineering. Jackson [11, 12] had been the first to employ the q-calculus in certain situations and presented the q-analogues for the traditional derivative and integral operators [13]. For $0 < q < 1$, the power source q-derivative operator for $\phi * k$ is given in [14] as

$$\Pi_q(\phi * k)(\hbar) = \Pi_q\left(\hbar + \sum_{\nu=2}^{\infty} e_\nu d_\nu \hbar^\nu\right)$$
$$= \frac{(\phi * k)(\hbar) - (\phi * k)(q\hbar)}{\hbar[q(q-1) - q^2 + 1]} = 1 + \sum_{\nu=2}^{\infty} [\nu, q] e_\nu d_\nu \hbar^{\nu-1},$$

where

$$[\nu, q] = \frac{q^\nu(q-1) - q^{\nu+1} + 1}{q(q-1) - q^2 + 1} = 1 + \sum_{n=1}^{\nu-1} q^n.$$

As $\beta > -1$ and $0 < q < 1$, we obtained the known linear operator $\Upsilon_k^{\beta,q} : \mathcal{A} \to \mathcal{A}$ as

$$\Upsilon_k^{\beta,q} \phi(\hbar) * \mathcal{F}_{q,\beta+1}(\hbar) = \hbar \Pi_q(\phi * k)(\hbar),$$

where the function $\mathcal{F}_{q,\beta+1}(\hbar)$ is given by

$$\mathcal{F}_{q,\beta+1}(\hbar) = \hbar + \sum_{\nu=2}^{\infty} \frac{[(\beta(\beta+1) - \beta^2 + 1), q]_{\nu-1}}{[(\nu(\nu+1) - \nu^2 - 1), q]!} \hbar^\nu. \tag{3}$$

A straightforward calculation indicates that

$$\Upsilon_k^{\beta,q} \phi(\hbar) = \hbar + \sum_{\nu=2}^{\infty} \frac{[\nu, q]!}{[(\beta(\beta+1) - \beta^2 + 1), q]_{\nu-1}} d_\nu e_\nu \hbar^\nu. \tag{4}$$

Utilizing (4), with the subsequent relations for all $\phi \in \mathcal{A}$,

$$[\beta + 1, q] \Upsilon_k^{\beta,q} \phi(\hbar) = [\beta, q] \Upsilon_k^{\beta+1,q} \phi(\hbar) + q^\beta \hbar \Pi_q \left(\Upsilon_k^{\beta+1,q} \phi(\hbar)\right).$$

By applying different unique situations for the coefficients of d_ν, we can derive the subsequent special cases for the operator $\Upsilon_k^{\beta,q}$:

i. As $d_\nu = 1$, we obtained the operator $\mathcal{M}_q^\beta$, which has been studied by Arif et al. [15], given by:

$$\phi(\hbar) = \hbar + \sum_{\nu=2}^{\infty} \frac{[\nu, q]!}{[(\beta(\beta+1) - \beta^2 + 1), q]_{\nu-1}} e_\nu \hbar^\nu.$$

ii. For $d_\nu = \left(\frac{j(j+1)-j^2+1}{j+\nu}\right)^\mu$, $\mu > 0, j \geq 0$, we obtained the operator $\mathcal{F}_{j,q}^{\beta,\mu}$ studied by El-Deeb and Bulboaca [16]:

$$\mathcal{F}_{j,q}^{\beta,\mu}\phi(\hbar) = \hbar + \sum_{\nu=2}^{\infty}\left(\frac{j(j+1)-j^2+1}{j+\nu}\right)^\mu \frac{[\nu,q]!}{[(\beta(\beta+1)-\beta^2+1),q]_{\nu-1}} e_\nu \hbar^\nu.$$

iii. For $d_\nu = \frac{\alpha^{\nu-1}}{(\nu(\nu+1)-\nu^2-1)!}e^{-\alpha}$, $\alpha > 0$, we obtained the q-analogue of the Poisson operator reported in [17] as:

$$\mathcal{B}_q^{\beta,\alpha}\phi(\hbar) = \hbar + \sum_{\nu=2}^{\infty}\frac{\alpha^{\nu-1}}{(\nu(\nu+1)-\nu^2-1)!}e^{-\alpha}\frac{[\nu,q]!}{[(\beta(\beta+1)-\beta^2+1),q]_{\nu-1}} e_\nu \hbar^\nu.$$

In addition, suppose $\mathcal{H}$ denotes the class consisting univalent functions in $\mathcal{A}$ in the unit disk Λ. The Koebe One Quarter Theorem [18] confirms with the representation of Λ that every univalent function $\phi \in \mathcal{H}$ contains a disk alongside a radius for 1/4. As a result, every univalent function ϕ has an inverse ϕ^{-1} defined as

$$\hbar = \phi^{-1}(\phi(\hbar)), (\hbar \in \Lambda)$$

with

$$\omega = \phi^{-1}(\phi(\omega)), \left(|\omega| < \rho_0(\phi); \rho_0(\phi) \geq \frac{1}{4}\right), \quad (5)$$

while

$$\phi^{-1}(\omega) = \omega - s_2\omega^2 + (2s_2^2 - s_3)\omega^3 - (5s_2^3 - 5s_2s_3 + s_4)\omega^4 + \cdots. \quad (6)$$

When the two functions ϕ and ϕ^{-1} are univalent in the unit disk Λ, then $\phi \in \mathcal{A}$ can be defined as a bi-univalent functions. Note that for the class of bi-univalent functions in Λ given by $\sum$, that have been normalized through (1), we presume that $\phi(\hbar)$ and $k(\hbar)$ are analytic functions in Λ. The function $\phi(\hbar)$ is said to be subordinate to the function $k(\hbar)$ or just $k(\hbar)$ is considered to be superordinate to $\phi(\hbar)$, if and only exists an analytic Schwarz function $z(\hbar)$ in Λ, with $z(0) = 0$ and $|z(\hbar)| < 1, (\hbar \in \Lambda)$, which means

$$\phi(\hbar) = k(z(\hbar)),$$

Written as

$$\phi \prec k \text{ or } \phi(\hbar) \prec k(\hbar), \quad (\hbar \in \Lambda).$$

Likewise, if the function k is univalent in Λ, we obtain an equivalent $\phi(\hbar) \prec k(\hbar)$ which can be obtained only if $\phi(0) = k(0)$ and $\phi(\Lambda) \subset k(\Lambda)$ [19- 21]. Lewien [22] investigated the class $\sum$ for bi-univalent functions and obtained a coefficient bound of $|e_2| \leq 1.51$ for all $\phi \in \sum$. After that, the works of Lewien [22], Clonie and Branan [23] have aided our estimate for $|e_2| \leq \sqrt{2}$ for all $\phi \in \sum$. Recently, Srivastva et al. [24] re-energized their investigation of bi-univalent and analytic functions that was subsequently conducted through Bulot [25]. Adegaeni and colleagues [26, 27], Goney et al. [28], Srivastva and Wans [29], along with others [30- 36] contributed to the study of bi-univalent functions. We observed that the class of functions developed serves as an alternative for the existing ones. Considering the functions $\hbar$, $\frac{\hbar}{1-\hbar}$, $-\log(1-\hbar)$ and $\frac{1}{2}log\frac{1+\hbar}{1-\hbar}$ that relate to $\sum$, functions in the class of Koebe functions, on the opposite hand, are not in $\sum$. To this day, the coefficient estimation problem for every one of the Taylor-Maclaurin coefficients $|e_\nu|$, $(\nu \in \mathbb{N} = \{1,2,3,4, \dots\dots\}, \nu \geq 3)$ has proven difficult, and it remains an endeavor for advancement [24].

Presenting in this paper a few families of bi-univalent functions established using the recently developed Sălăgean integro-differential operator, obtaining the Fekete–Szegö inequality as well as coefficient bounds for the functions given in the above class.

2. Preliminary

This section contains lemma, definitions and examples relevant to our investigation.

Lemma 2.1. [37] If $\phi \in \mathcal{G}$ and $|c_\nu| \leq 2$ for all ν, where $\mathcal{G}$ is the collection of all analytic functions g in the unit disk Λ such that $\Re\{g(\hbar)\} > 0$, the function

$$g(\hbar) = 1 + c_1\hbar + c_2\hbar^2 + c_3\hbar^3 + \cdots.$$

Definition 2.1. Consider a function $\phi(\hbar)$ in the class $\mathcal{M}_{\Sigma}(\beta, q, \lambda, k)$. If $\phi(\hbar) \in \sum$ satisfy certain conditions, then

$$\left| arg\left(\frac{(v(v-1)-v^2+1)\Upsilon_k^{\beta,q}\phi(\hbar)+(v(v-1)-v^2+2v)\Upsilon_k^{\beta+1,q}\phi(\hbar)}{\hbar}\right)\right| < \frac{\lambda\pi}{2}, \tag{7}$$

where $\hbar \in \Lambda,\ v \geq 1, \beta > -1, 0 < q < 1, 0 < \lambda \leq 1$, and

$$\left| arg\left(\frac{(v(v-1)-v^2+1)\Upsilon_k^{\beta,q}g(u)+(v(v-1)-v^2+2v)\Upsilon_k^{\beta+1,q}g(u)}{u}\right)\right| < \frac{\lambda\pi}{2}, \tag{8}$$

in which $g(u)$ is a function defined by

$$g(u) = u - s_2u^2 + [2s_2^2 - s_3]u^3 - [5s_2^3 - 5s_2s_3 + s_4]\omega^4 + \cdots, (g = \phi^{-1}). \tag{9}$$

Throughout this work, we will employ the following notation frequently for $\mathcal{H}_v$,

$$\mathcal{H}_v = \left((v(v-1)-v^2+1)\frac{[v,q]!}{[(\beta(\beta+1)-\beta^2+1),q]_{v-1}} + v\frac{[v,q]!}{[(\beta(\beta+1)-\beta^2+2),q]_{v-1}}\right),$$

where $v = 2,3,4, \dots$.

Example 2.1. If $v = 0$, we obtain bi-functions of order λ:

$$\left|arg\left(\frac{\Upsilon_k^{\beta,q}\phi(\hbar)}{\hbar}\right)\right| < \frac{\lambda\pi}{2}, \quad \left|arg\left(\frac{\Upsilon_k^{\beta,q}g(u)}{u}\right)\right| < \frac{\lambda\pi}{2}, \quad 0 < \lambda \leq 1.$$

Example 2.2. If $v = 1$ and $\beta = q = 0$, we obtain the following bi-functions of order λ:

$$\left|arg\left(\frac{\Upsilon_k\phi(\hbar)}{\hbar}\right)\right| < \frac{\lambda\pi}{2}, \quad \left|arg\left(\frac{\Upsilon_k g(u)}{u}\right)\right| < \frac{\lambda\pi}{2}, \quad 0 < \lambda \leq 1.$$

Definition 2.2. If a function $\phi(\hbar)$ of the form (1) is in the class $\mathcal{Y}_{\Sigma}(\beta, q, \eta, k)$ and satisfy the conditions for $\phi(\hbar) \in \sum$, then

$$\Re\left\{\frac{(v(v-1)-v^2+1)\Upsilon_k^{\beta,q}\phi(\hbar)+(v(v-1)-v^2+2v)\Upsilon_k^{\beta+1,q}\phi(\hbar)}{\hbar}\right\} > \eta, \tag{10}$$

where $\hbar \in \Lambda, v \geq 1, \beta > -1, 0 < q < 1, 0 < \eta \leq 1$ and

$$\Re\left\{\frac{(v(v-1)-v^2+1)\Upsilon_k^{\beta,q}g(u)+(v(v-1)-v^2+2v)\Upsilon_k^{\beta+1,q}g(\hbar)}{u}\right\} > \eta, \tag{11}$$

in which g is given by (9).

Example 2.3. If $v = 0$, we obtain the following bi-functions of order η:

$$\Re\left\{\frac{\Upsilon_k^{\beta,q}\phi(\hbar)}{\hbar}\right\} > \eta, \quad \Re\left\{\frac{\Upsilon_k^{\beta,q}g(u)}{u}\right\} > \eta, \quad 0 \leq \eta < 1.$$

Example 2.4. If $v = 1$ and $\beta = q = 0$, we obtain the bi-functions of order η as:

$$\Re\left\{\frac{\Upsilon_k\phi(\hbar)}{\hbar}\right\} > \eta, \quad \Re\left\{\frac{\Upsilon_k g(u)}{u}\right\} > \eta, \quad 0 \leq \eta < 1.$$

3. Main Results

In this section, we will perform certain basic mathematical computations and obtain new significant results in this field.

Theorem 3.1. If the function $\phi(\hbar)$ given in (1) belongs to the class $\mathcal{M}_{\Sigma}(\beta, q, \lambda, k)$, for $\upsilon \geq 1, \beta > -1, 0 < q < 1, 0 < \lambda \leq 1$. Then

$$|e_2| \leq \frac{(\lambda(\lambda+1) - \lambda^2 + \lambda)}{\sqrt{(\lambda(1-\lambda) + \lambda^2 + \lambda)\mathcal{H}_3 + (\lambda(\lambda-1) - \lambda^2 + 1)\mathcal{H}_2^2}}, \tag{12}$$

and

$$|e_3| \leq \frac{(\lambda(1-\lambda) + \lambda^2 + \lambda)}{\mathcal{H}_3}. \tag{13}$$

Proof. Considering (7) and (8), we have

$$\frac{(\nu(\nu-1) - \nu^2 + 1)\Upsilon_k^{\beta,q}\phi(\hbar) + (\nu(\nu-1) - \nu^2 + 2\nu)\Upsilon_k^{\beta+1,q}\phi(\hbar)}{\hbar} = (\mathcal{J}(\hbar))^{\lambda} \tag{14}$$

and

$$\frac{(\nu(\nu-1) - \nu^2 + 1)\Upsilon_k^{\beta,q}g(u) + (\nu(\nu-1) - \nu^2 + 2\nu)\Upsilon_k^{\beta+1,q}g(\hbar)}{u} = (\mathcal{P}(u))^{\lambda}, \tag{15}$$

where

$$\mathcal{J}(\hbar) = 1 + \mathcal{J}_1\hbar + \mathcal{J}_2\hbar^2 + \mathcal{J}_3\hbar^3 + \cdots$$

and

$$\mathcal{P}(u) = 1 + \mathcal{P}_1 u + \mathcal{P}_2 u^2 + \mathcal{P}_3 u^3 + \cdots$$

are in $\mathcal{G}$.
Adding the coefficients in (14) and (15), we have

$$\mathcal{H}_2 e_2 = (\lambda(\lambda+2) - \lambda^2 - \lambda)\mathcal{J}_1, \tag{16}$$

$$\mathcal{H}_3 e_3 = (\lambda(\lambda+2) - \lambda^2 - \lambda)\mathcal{J}_2 + \frac{(\lambda^2(1-\lambda) + \lambda^3 - \lambda)}{2}\mathcal{J}_1^2, \tag{17}$$

$$-\mathcal{H}_2 e_2 = (\lambda(\lambda+2) - \lambda^2 - \lambda)\mathcal{P}_1, \tag{18}$$

$$\mathcal{H}_3(2e_2^2 - e_3) = (\lambda(\lambda+2) - \lambda^2 - \lambda)\mathcal{P}_2 + \frac{(\lambda^2(1-\lambda) + \lambda^3 - \lambda)}{2}\mathcal{P}_1^2, \tag{19}$$

where

$$\mathcal{H}_\nu = \left((1-\upsilon)\frac{[\nu,q]!}{[(\beta(\beta+1) - \beta^2 + 1), q]_{\nu-1}} + \upsilon\frac{[\nu,q]!}{[(\beta(\beta+1) - \beta^2 + 2), q]_{\nu-1}}\right).$$

Using (16) and (18) as examples, we have

$$\mathcal{J}_1 = -\mathcal{P}_1. \tag{20}$$

Now, to obtain the estimate for $|\mathcal{J}_1|$, utilizing the estimates for $|e_2|$ and $|e_3|$, we begin by adding (17) and (19) which yields

$$2\mathcal{H}_3 e_2^2 = (\lambda(\lambda+2) - \lambda^2 - \lambda)(\mathcal{J}_2 + \mathcal{P}_2) + \frac{(\lambda^2(1-\lambda) + \lambda^3 - \lambda)}{2}(\mathcal{J}_1^2 + \mathcal{P}_1^2)$$

Utilizing (20) in the previous equation, we now have

$$2\mathcal{H}_3 e_2^2 = (\lambda(\lambda+2) - \lambda^2 - \lambda)(\mathcal{J}_2 + \mathcal{P}_2) + (\lambda(\lambda^2 + \lambda) - \lambda^3 - \lambda)\mathcal{J}_1^2.$$

By simplifying $e_2 = \frac{\lambda\mathcal{J}_1}{\mathcal{H}_2}$ comparing with (16), we arrived at

$$\mathcal{J}_1^2 = \frac{(\mathcal{J}_2 + \mathcal{P}_2)\mathcal{H}_2^2}{2(\lambda(\lambda+2) - \lambda^2 - \lambda)\mathcal{H}_3 + (\lambda(\lambda+1) - \lambda^2 - 1)\mathcal{H}_2^2}. \tag{21}$$

Employing $|\mathcal{J}_2| \leq 2$ and $|\mathcal{P}_2| \leq 2$, we get

$$|\mathcal{J}_1| \leq \frac{2\mathcal{H}_2}{\sqrt{2(\lambda(\lambda+2) - \lambda^2 - \lambda)\mathcal{H}_3 + (\lambda(\lambda+1) - \lambda^2 - 1)\mathcal{H}_2^2}}.$$

Consequently,

$$|e_2| \le \frac{(\lambda(1-\lambda)+\lambda^2+\lambda)}{\sqrt{(\lambda(1-\lambda)+\lambda^2+\lambda)\mathcal{H}_3+(\lambda(\lambda+1)-\lambda^2-1)\mathcal{H}_2^2}}.$$

For determining a bound on $|e_3|$ that we may define e_3 as a function of $\mathcal{J}_1, \mathcal{J}_2, \mathcal{P}_1$ as well as $\mathcal{P}_2$. In order to do so, remove (19) compared to (17) to get

$$2\mathcal{H}_3(e_3-e_2^2) = (\lambda(\lambda+2)-\lambda^2-\lambda)(\mathcal{J}_2-\mathcal{P}_2)+\frac{(\lambda(\lambda^2+\lambda)-\lambda^3-\lambda)}{2}(\mathcal{J}_1^2-\mathcal{P}_1^2)$$

Employing (20) in the previous equation, we arrived at

$$2\mathcal{H}_3 e_3 = 2\mathcal{H}_3 e_2^2+(\lambda(\lambda+2)-\lambda^2-\lambda)(\mathcal{J}_2-\mathcal{P}_2). \tag{22}$$

Simplifying $e_2 = \frac{\lambda \mathcal{J}_1}{\mathcal{H}_2}$ and comparing to (16), as well as applying (21), we obtain

$$2\mathcal{H}_3 e_3 = 2\mathcal{H}_3\left(\frac{(\lambda(\lambda+2)-\lambda^2-\lambda)\mathcal{J}_1}{\mathcal{H}_2}\right)^2+(\lambda(\lambda+2)-\lambda^2-\lambda)(\mathcal{J}_2-\mathcal{P}_2)$$

$$= \frac{(\lambda(1+\lambda)+\lambda^2-\lambda)\mathcal{H}_3(\mathcal{J}_2+\mathcal{P}_2)}{(\lambda(1-\lambda)+\lambda^2+\lambda)\mathcal{H}_3+(\lambda(\lambda-1)-\lambda^2+1)\mathcal{H}_2^2}+(\lambda(\lambda+2)-\lambda^2-\lambda)(\mathcal{J}_2-\mathcal{P}_2)$$

$$= \lambda\left(\frac{((\lambda(2-\lambda)+\lambda^2+2\lambda)\mathcal{H}_3+(\lambda(\lambda-1)-\lambda^2+1)\mathcal{H}_2^2)\mathcal{J}_2-(\lambda(\lambda-1)-\lambda^2+1)\mathcal{H}_2^2\mathcal{P}_2}{(\lambda(1-\lambda)+\lambda^2+\lambda)\mathcal{H}_3+(\lambda(\lambda-1)-\lambda^2+1)\mathcal{H}_2^2}\right).$$

Employing $|\mathcal{J}_2| \le 2$ and $|\mathcal{P}_2| \le 2$, we obtain

$$|e_3| \le \frac{(\lambda(1-\lambda)+\lambda^2+\lambda)}{\mathcal{H}_3}.$$

And replace $\mathcal{M}_\Sigma(\beta,q,\lambda,k)$ with $\mathcal{M}_\Sigma(\beta,q,1,k)$in Corollary 3.1.

Corollary 3.1. If $\phi(\hbar) \in \mathcal{M}_\Sigma(\beta,q,\lambda,k)$, it follows that

$$|e_2| \le \frac{2}{\sqrt{2\mathcal{H}_3}}, \quad |e_3| \le \frac{2}{\mathcal{H}_3}.$$

Theorem 3.2. If $\phi(\hbar) \in \mathcal{A}$ is an element of the class $\mathcal{Y}_\Sigma(\beta,q,\eta,k)$, with $0<q<1, 0<\eta\le 1, \upsilon \ge 1$ and $\beta > -1$. Then it follows that

$$|e_2| \le \sqrt{\frac{2(\eta(\eta-1)-\eta^2+1)}{\mathcal{H}_3}}, \tag{23}$$

and

$$|e_3| \le \frac{2(\eta(\eta-1)-\eta^2+1)}{\mathcal{H}_3}. \tag{24}$$

Proof. Considering (10) and (11), with $\mathcal{J}(\hbar)$ and $\mathcal{P}(u)$ in $\mathcal{G}$, we have

$$\frac{(\nu(\nu-1)-\nu^2+1)\Upsilon_k^{\beta,q}\phi(\hbar)+\upsilon\Upsilon_k^{\beta+1,q}\phi(\hbar)}{\hbar} = \eta+(\eta(\eta-1)-\eta^2+1)\mathcal{J}(\hbar) \tag{25}$$

and

$$\frac{(\nu(\nu-1)-\nu^2+1)\Upsilon_k^{\beta,q}g(u)+\upsilon\Upsilon_k^{\beta+1,q}g(u)}{u} = \eta+(\eta(\eta-1)-\eta^2+1)\mathcal{P}(u), \tag{26}$$

where

$$\mathcal{J}(\hbar) = 1+\mathcal{J}_1\hbar+\mathcal{J}_2\hbar^2+\mathcal{J}_3\hbar^3+\cdots$$

and

$$\mathcal{P}(u) = 1+\mathcal{P}_1 u+\mathcal{P}_2 u^2+\mathcal{P}_3 u^3+\cdots$$

are in $\mathcal{G}$.

Adding the coefficients in (25) and (26), we have

$$\mathcal{H}_2 e_2 = (\eta(\eta-1)-\eta^2+1)\mathcal{J}_1, \tag{27}$$

$$\mathcal{H}_3 e_3 = (\eta(\eta-1)-\eta^2+1)\mathcal{J}_2, \tag{28}$$

$$-\mathcal{H}_2 e_2 = (\eta(\eta-1)-\eta^2+1)\mathcal{P}_1, \tag{29}$$
$$\mathcal{H}_3(2e_2^2 - e_3) = (\eta(\eta-1)-\eta^2+1)\mathcal{P}_2, \tag{30}$$

where

$$\mathcal{H}_\nu = \left((\nu(\nu-1)-\nu^2+1)\frac{[\nu,q]!}{[\beta+1,q]_{\nu-1}} + (\nu(\nu-1)-\nu^2+2\nu)\frac{[\nu,q]!}{[\beta+2,q]_{\nu-1}} \right).$$

From (27) and (29), we obtain

$$\mathcal{J}_1 = -\mathcal{P}_1. \tag{31}$$

Now, using (28) and (31), we get

$$2\mathcal{H}_3 e_2^2 = (\eta(\eta-1)-\eta^2+1)(\mathcal{J}_2+\mathcal{P}_2).$$

Substituting $e_2 = \frac{(\eta(\eta-1)-\eta^2+1)\mathcal{J}_1}{\mathcal{H}_2}$ in (27) and simplifying, we obtain

$$\mathcal{J}_1^2 = \frac{(\mathcal{J}_2+\mathcal{P}_2)\mathcal{H}_2^2}{2(\eta(\eta-1)-\eta^2+1)\mathcal{H}_3}. \tag{32}$$

Employing $|\mathcal{J}_2| \le 2$ and $|\mathcal{P}_2| \le 2$, we get

$$|\mathcal{J}_1| \le \sqrt{\frac{2}{(\eta(\eta-1)-\eta^2+1)\mathcal{H}_3}}\mathcal{H}_2.$$

Consequently

$$|e_2| \le \sqrt{\frac{2(\eta(\eta-1)-\eta^2+1)}{\mathcal{H}_3}}.$$

Next, putting a limit on $|e_3|$ using the values of $\mathcal{J}_1, \mathcal{J}_2, \mathcal{P}_1$ and $\mathcal{P}_2$, we subtract (30) from (28) and obtain

$$2\mathcal{H}_3(e_3 - e_2^2) = (\eta(\eta-1)-\eta^2+1)(\mathcal{J}_2-\mathcal{P}_2)\,.$$

Considering (27) and (32), we obtain

$$2\mathcal{H}_3 e_3 = 2(\eta(\eta-1)-\eta^2+1)\mathcal{J}_2.$$

Considering $|\mathcal{J}_2| \le 2$, we have

$$|e_3| \le \frac{2(\eta(\eta-1)-\eta^2+1)}{\mathcal{H}_3}.$$

And replace $\mathcal{M}_\Sigma(\beta,q,\lambda,k)$ with $\mathcal{M}_\Sigma(\beta,q,0,k)$in Corollary 3.2.

Corollary 3.2. If $\phi(\hbar) \in \mathcal{Y}_\Sigma(\beta,q,\eta,k)$, then

$$|e_2| \le \sqrt{\frac{2}{\mathcal{H}_3}}, \quad |e_3| \le \frac{2}{\mathcal{H}_3}.$$

4. Conclusions

The unique results obtained correspond to the coefficient estimates for the two classes of functions developed in this work. The classes developed in this article is associated with a novel integro-differential operator known as the Sălăgean integro-differential operator. Since the sole purpose of this investigation is to obtain the coefficient estimates for the classes of functions, this work is significant and has provided research opportunities for further investigations. Moreover, the study investigates and introduced two subclasses for novel bi-univalent functions associated to the operator $\Upsilon_k^{\beta,q}$. The coefficient bounds for functions in these classes are determined over $|e_2|$ and $|e_3|$. Obtaining estimates for the bound of $|e_\nu|$, $(\nu \in \mathbb{R} - \{1,2,3,4\})$ to satisfy the classes developed in this work remains a challenging task. Particular applications of coefficient estimates may yield possibly intriguing new results. The findings of this article could motivate further research into integro-differential operators, which are used to introduce novel classes of bi-univalent functions.

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