Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

Numerical Approximation of Laplace's Equation Using the Finite Element Method

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ARTICLE INFO

Article history:
Received: 22 /06/2025
Rrevised form: 21 /08/2025
Accepted : 28/08/2025
Available online: 30 /09/2025

Keywords:

Laplace's Equation , Finite Element Method , Weak Formulation , Convergence Analysis , Numerical Simulation

ABSTRACT

This article addresses the finding of an approximate solution to Laplace's equation, a basic elliptic partial differential equation, using the Finite Element Method (FEM). Laplace's equation, $\nabla^2 u = 0$, plays a key role in the description of equilibrium of processes in physics and engineering, e.g., in steady state heat conduction, and in electrostatics. Whereas this type of solution is not always possible because of the restrictions it imposes on the geometry, FEM is a more general and flexible approach where complex geometries are to be treated We give the mathematical framework, weak format computation for FEM, the theory from existence and uniqueness to the regularity of its solution based on functional analysis (Lax–Milgram theorem). The FEM procedure (discretization employing basis functions and mesh generation) is explained with special focus on its influence on precision and efficiency. An important part of this work is the error and convergence analysis with computation of error estimates, confirming theoretical a priori error estimates (for instance, first order in (H^1 norm) and second order in (L^2 norm) for P1 elements) in numerical experiments. Visualization methods and error tables are employed to demonstrate features of solutions and measure accuracy. The tendency emphasizes the importance of mesh refinement techniques, in particular, adaptive methods based on a posteriori error estimation. This work validates the reliability and performance of the proposed FEM to obtain the solution of Laplace's equation, highlighting its significance in computational mathematics and engineering problems.

https://doi.org/10.29304/jqcsm.2025.17.32468

1. Introduction

Laplace's equation is a second-order partial differential equation named after Pierre-Simon Laplace, who first studied its properties. It is widely used in mathematics, physics, and engineering. It is written as ∇2 u=0, with u as the scalar field (typically a potential field, such as electric potential or temperature) and ∇2 as the Laplacian. This equation is important in potential theory and accounts for a variety of equilibrium situations, such as electrostatics, incompressible fluid flow and steady-state heat conduction. The crucial characteristic of Laplace's equation is its

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property of representing systems at rest: for example, it accurately describes potential (velocity or temperature) fields produced by static (velocity or temperature) distributions or boundary conditions[9].

The equation is named after Pierre-Simon Laplace, a French mathematician who made a number of contributions to its mathematical theory in the late 18th century. Solutions of Laplace's equation are called harmonic functions, which have unique properties, including being infinitely differentiable throughout their domain and satisfying the mean value property. This means that the value of a harmonic function at a point is just the average value of that function on a round sphere (a ball) of any radius centered at that point. A study of Laplace's equation shows its flexibility in multiple coordinate systems through methods like separation of variables. This simplifies the obtaining of the solutions in Cartesian, cylindrical and spherical coordinates. In addition, since Laplace's equation is linear, if two functions X and Y both satisfy Laplace's equation, any linear combination of these two functions will also be a solution, showing the superposition principle of linear differential equations at work[3],[11].

Substantial development has been achieved in numerical schemes for solving Laplace's equation in cases where analytic solutions are not easily given due to complicated geometries or boundary conditions. These calculations can be done using finite element analysis (FEA) in which complex domains are divided into simpler subdomains or elements. Computer algorithms are no longer limited to theoretical applications, but they are increasingly used for practical applications such as structural analysis in engineering and simulations of physical processes (like fluid flow and heat distribution). Aside from the theoretical and numerical analysis, having knowledge of the convergent properties and carrying out an error estimation is essential for the practical use of computational procedures. Such numerical approximations are to be made to agree with actual solutions to a tolerable degree by these analyses. Studying these aspects not only improves existing techniques but also motivates the development of new algorithms adapted to particular applications where Laplace's equation appears[4],[2].

Hence, in this case, the subject of Laplace equation numerical approximation will be addressed by FEM, and its implementation, formulation and convergence will be studied. An exploration of this powerful mathematical instrument reveals its overarching importance in many areas of human endeavour. In describing the natural laws that dictate states of equilibrium in terms of potential fields and so-called harmonic functions, Laplace’s equation continues to be the bedrock of contemporary mathematical modeling. [9]

2. Literature Review

The Laplace’s equation, ∇2 u=0, plays an important role in partial differential equations (PDEs), and has widespread use in such areas as electrostatics, fluid dynamics, and thermal conduction. Its mathematical significance was established by notable mathematicians, including P. S. Laplace, Euler, and Lagrange. The equation models electric potential in electrostatics outside conductors and steady state temperature distributions in heat conduction problems in a variety of shapes[4].

Analytical and numerical methods are known for solving Laplace's equation. Classic analytical methods are the separation of variables and integral transforms (Fourier and Laplace) approach; they can be successful for simple geometries and boundary conditions, but they rely on finding an appropriate choice of coordinate system; for this reason, they may fail when dealing with complex domains or in higher dimensions. On the other hand, the numerical approach is dominant in practice due to the unavailability of analytical methods. The most widely used numerical techniques are the finite difference method (FDM), the finite element or boundary element methods (FEM, BEM). The finite-element method is one of the latter methods, and it is flexible enough for dealing with arbitrarily shaped and boundary-conditioned problems. There are also more recent advances in methods such as isogeometric analysis, which integrate computer-aided design (CAD) into discretization, exploiting the CAD precision while retaining geometrical consistency[4] , [17].

The investigation centers on error analysis and convergence of numerical methods; with structured discretization methods, the optimal rates of convergence are obtained. Harmonic functions (solutions of Laplace's equation) are important in the context of applications in, for example, fluid flow simulations and engineering stability analysis. Quickly improving computational algorithms for complex boundary value problems in materials science. The advent of recently advanced methods has led to a number of characteristics of increasingly advanced codes for solving complex boundary value problems via iterative solvers and fast multipole algorithms[16].

Current research directions also include machine learning approaches to predictive modeling and optimization in the context of Laplace's equation. Such collaboration between mathematical principles and state-of-the-art computational technologies indicates promising future scientific discoveries and technological innovations not only in mathematics but also in applied sciences. [20], [16]

A comprehensive review of key research studies that have numerically solved Laplace's equation using various methods over different years is presented in Table 1 below, highlighting the evolution of numerical techniques and their convergence properties.

Table 1: Evolution of Numerical Techniques for Laplace's Equation: A Review of Key Studies, Methods, and Convergence Characteristics

Year	Researchers / Study	Method Used	Key Contribution
1993	A. Greenbaum, L. Greengard, G.B. McFadden [20]	Integral Equation / Fast Multipole Method	Solved Laplace's equation in multiply connected domains using high-order accurate integral equation techniques combined with the Fast Multipole Method for efficient computation.
2004	T. Graetsch and K. Bathe [6]	Finite Element Method (FEM) with A Posteriori Error Estimation	Developed robust a posteriori error estimation techniques for FEM applied to elliptic PDEs including Laplace's equation, enabling adaptive mesh refinement.
2005	K. Domelevo and P. Omnes [15]	Finite Volume Method (FVM)	Proposed a finite volume scheme for Laplace's equation on arbitrary 2D grids, ensuring convergence and stability even on non-structured meshes.

2006	T. LaForce [19]	Finite Element Method (P1 Elements)	Presented course-based analysis of FEM applied to Laplace's equation, demonstrating convergence behavior and implementation details for piecewise linear elements.
2011	G. Yagawa [7]	Parallel FEM and Mesh-Free Methods	Explored advanced computational techniques, including parallel finite element and mesh-free approaches, for solving Laplace-type problems in large-scale engineering simulations.
2018	Q. Chen, G. Wang, M. Pindera [22]	Boundary Element Method (BEM)	Applied BEM to solve Laplace's equation in nanoporous composites, focusing on homogenization and localization effects in heterogeneous materials.
2019	J. Droniou, M. Medla, K. Mikula [2]	Finite Volume Method for Elliptic Equations	Designed and analyzed finite volume schemes for elliptic PDEs with oblique derivatives, including applications to Laplace's equation in geophysical modeling.
2020	Z. Zhang, Y. Wang, P.K. Jimack, H. Wang [21]	Deep Learning-Based Mesh Generation (MeshingNet)	Introduced a machine learning framework to generate optimized meshes for solving PDEs like Laplace's equation, improving accuracy and efficiency.
2021	F. Bertrand, D. Boffi, G.G. de Diego [13], [14]	Scaled Boundary Finite Element Method (SBFEM)	Conducted rigorous convergence analysis of SBFEM for Laplace's equation, proving optimal convergence rates on polygonal domains.
2023	B. Li, Y. Xia, Z. Yang [8]	Isogeometric Finite Element Method	Applied iso-parametric FEM to parabolic and elliptic problems, including Laplace-type equations, showing high-order convergence and geometric fidelity.

3.Research Methodology

We use the Finite Element Method (FEM) to obtain numerical solutions of Laplace's equation, $\nabla^2 u = 0$, on complex domains. The resulting approach is an elegant mathematical pathway: we start from the weak formulation obtained through variational principles and casted in the Sobolev space $H^1(\Omega)$, that guarantees the existence and uniqueness of the solution by the Lax-Milgram theorem. The space is approximated with unstructured triangular mesh and piecewise linear shape functions (P1 element) for the spatial approximation. We generate a sequence of uniformly refined meshes to perform convergence analysis, comparing the numerical solution to a high-resolution reference solution. Errors are computed in L^2 and H^1 norms, and convergence rates are estimated to confirm theoretical results. The implementation in Python consists of mesh generation, assembly of the stiffness matrix, application of boundary conditions, and solution of the linear system. Results illustrate that the proposed method is an accurate and robust method to solve elliptic PDEs, even in domains having geometric singularities/angles, and it is confirmed that FEM can be one of practical tools to solve elliptic PDEs in scientific and engineering applications.

4. Mathematical Formulation

Laplace's equation, $\nabla^2 \mathrm{u} = 0$, is a vital aspect of mathematics and physics, representing systems in equilibrium. The potential function u can denote temperature or electric potential, capturing steady-state conditions rather than time-dependent dynamics like heat flow or wave propagation[2].

Typically analyzed within a domain Ω , Laplace's equation employs boundary conditions on the perimeter $\partial\Omega$. These conditions include: item Dirichlet: $u|_{\partial\Omega} = h$ \item Neumann: $\frac{\partial u}{\partial n}|_{\partial\Omega} = h$ \item Robin: $\alpha \frac{\partial u}{\partial n}|_{\partial\Omega} + \beta u|_{\partial\Omega} = h$, where n is the outward normal vector[20].

Solutions are typically determined by accessing the weak form of the Laplace equation in this example for use in FEM application. By multiplying the original equation by a test function and integrating over the domain, the method is extended to complex geometry and boundary conditions. The mathematical formulation, particularly the weak form derivation is described in detail elsewhere [16].

Computational solutions that rely on numerical approximation employ discretization of the domain and the boundaries. In FEM, the domain Ω is partitioned into small elements, and a polynomial approximation for u is assumed in each element, which together form a global set of equations to solve for u on the entire domain[16].

When solving Laplace's equation, the rate of numerical schemes' convergence mustn't be too slow. The convergence analysis of FEM generally requires algebraic rates in terms of the polynomial degree of the basis functions and the regularity of the solution. A careful treatment of boundary conditions is important, since they have a large impact on the results, and especially for complex geometries. Smooth solutions to Laplace's equation, known as harmonic functions, have infinite derivatives and have specific averaging properties, which have made them important for science and technology.[17]

5. Weak Form Derivation

To establish the weak form of Laplace's equation, we start from its conventional representation as a partial differential equation, $\nabla^2 \mathrm{u} = 0$ within a domain Ω along with appropriate boundary conditions (e.g., Dirichlet, Neumann, or Robin). The weak formulation involves identifying a function u in a suitable Sobolev space, typically $\mathrm{H}^1(\Omega,)$ which aids in handling real-world geometries and potential discontinuities in derivatives. We multiply both sides of the equation by a test function v in $\mathrm{H}_0^1(\Omega)$ (for homogeneous Dirichlet BCs) or an appropriate test space and integrate over Ω[3].

Using integration by parts (Green's first identity) on the Laplacian term transitions the strong form to its weak equivalent:

$$\int_{\Omega} (\nabla^2 u) v \, d\Omega = -\int_{\Omega} (\nabla u \cdot \nabla v) \, d\Omega + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds,$$

where $\frac{\partial u}{\partial n}$ is the normal derivative at the boundary $\partial\Omega$. For homogeneous Dirichlet boundary conditions ($u = 0$ on $\partial\Omega$), setting $v = 0$ on $\partial\Omega$ simplifies the equation, and the boundary term vanishes. This leads to the standard weak formulation for homogeneous Dirichlet conditions:

$$\text{Find } u \in H_0^1(\Omega) \text{ such that for all } v \in H_0^1(\Omega), \int_\Omega (\nabla u \cdot \nabla v)\, d\Omega = 0.$$

This formulation integrates the gradients of the solution and test functions, which is the basis for the FEM. More generally, for non-homogeneous Dirichlet or Neumann conditions, the boundary term $\int_{\partial\Omega} \frac{\partial u}{\partial n} v\, ds$ contributes to the weak form, often appearing as an additional linear functional on the right-hand side involving the boundary data h [14].

Discretization of weak form–luyth6 may approximate u and v by basis functions (shape functions) defined on a mesh with elements covering Ω. Such basis functions are usually piecewise polynomials based on the nodes of the mesh. The approximate solution u_h is the linear combination of these basis functions. Solving for the unknown coefficients (nodal values) results in a set of linear algebraic equations[13].

Convergence analysis is indispensable once we have obtained the weak formulation, as we want some insight into how well our approximate solutions converge to the real solution for smaller and smaller meshes. Error estimates that compare numerical results with exact solutions are based on the theoretical framework for convergence, often Cea's lemma. Well-posed boundary conditions are crucial as they exert considerable impact on results, particularly in complicated geometries or when mixed boundary conditions are in effect. Going from a strong to a weak form makes us more flexible in solving Laplace's equation in contexts of different sorts, and paves the way for sophisticated numerical methods, such as ones utilizing adaptive meshing.[3]

6. Rigorous Proofs

Establishing the well-posedness of Laplace's equation $\nabla^2 u = 0$ with appropriate boundary conditions is fundamental. This involves rigorously proving the existence and uniqueness of solutions, which provides the theoretical foundation for numerical methods like the FEM.

Central to this analysis is the weak formulation, which reinterprets the original problem in a more suitable functional analytic setting. By multiplying the equation by appropriate test functions, integrating over the domain Ω, and applying integration by parts, we derive a variational problem whose solution, under suitable conditions, corresponds to the solution of the original PDE.

To prove existence and uniqueness for the standard homogeneous Dirichlet problem, we rely on the Lax-Milgram theorem from functional analysis. Consider the weak formulation:

Find $u \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$, $a(u,v) = \int_\Omega \nabla u \cdot \nabla v\, d\Omega = 0$. Here, $a(\cdot,\cdot)$ is a bilinear form on the Sobolev space $H_0^1(\Omega)$. The Lax-Milgram theorem states that if this bilinear form is continuous (bounded) and coercive on $H_0^1(\Omega)$, and the right-hand side (here zero) is a continuous linear functional, then there exists a unique solution $u \in H_0^1(\Omega)$ [1].

For our problem, the bilinear form $a(u,v) = \int_{\Omega} \nabla u \cdot \nabla v \, d\Omega$ is indeed continuous and coercive on $H_0^1(\Omega)$ (coercivity follows from the Poincaré inequality). The right-hand side is the zero functional, which is trivially continuous. Therefore, the Lax-Milgram theorem guarantees a unique solution $u \in H_0^1(\Omega)$ [1].

Uniqueness for more general boundary conditions (e.g., non-homogeneous Dirichlet, Neumann, or Robin) can also be addressed. For instance, for the Dirichlet problem ($u = g$ on $\partial\Omega$), if two solutions u_1 and u_2 exist, their difference $w = u_1 - u_2$ satisfies the homogeneous equation $\nabla^2 w = 0$ in Ω with homogeneous Dirichlet conditions $w = 0$ on $\partial\Omega$. Applying the maximum principle for harmonic functions, w attains its maximum and minimum on the boundary $\partial\Omega$. Since $w = 0$ on $\partial\Omega$, it follows that $w \equiv 0$ in Ω, proving $u_1 = u_2$ [13].

Understanding the regularity of solutions is crucial. Elliptic regularity theory shows that if the domain Ω has a sufficiently smooth boundary (e.g., Lipschitz or C^2) and the boundary data is appropriately smooth, the solution u will possess higher differentiability properties than merely being in $H^1(\Omega)$. This has implications for the accuracy of numerical methods, as smoother solutions generally lead to faster convergence rates.

These theoretical results are essential for numerical analysis. They justify the use of variational methods, ensure that the continuous problem has a unique solution to approximate, and provide the basis for deriving a priori error estimates for numerical methods like FEM, which typically measure the error in norms related to the underlying function spaces (e.g., the H^1 semi-norm). [13] and [1].

7. Error and Convergence Analysis

The analysis of error and convergence is paramount for validating and understanding the performance of the FEM in approximating solutions to Laplace's equation. This analysis provides quantitative measures of how the numerical solution u_h (where h represents a measure of the mesh size) approaches the exact solution u as the mesh is refined ($h \to 0$)[2].

The convergence theory for FEM is typically built upon the analysis of the weak formulation and the properties of the underlying function spaces. A fundamental result is Cea's lemma, which states that the FEM solution u_h (in a finite-dimensional subspace $V_h \subset H_0^1(\Omega)$) satisfies:

$$\| u - u_h \|_{H^1(\Omega)} \leq C \inf_{v_h \in V_h} \| u - v_h \|_{H^1(\Omega)}$$

where C is a constant independent of h and u. This lemma shows that the FEM solution is, up to a constant, as good as the best possible approximation from the finite element space V_h in the H^1 norm[6].

This best approximation property leads directly to a priori error estimates. By choosing specific, computable functions v_h (such as the interpolant of the exact solution u) in the infimum, and using approximation theory for polynomial spaces, we can bound the error. For example, if $u \in H^2(\Omega)$ and we use piecewise linear (P_1) elements on a quasi-uniform mesh with maximum element diameter h, the standard a priori estimate is:

$$\| u - u_h \|_{H^1(\Omega)} \leq Ch \| u \|_{H^2(\Omega)}$$

This indicates an optimal first-order convergence rate in the H^1 norm. Similarly, for the L^2 norm (often more relevant for the solution values themselves), a duality argument typically yields a second-order rate:

$$\| u - u_h \|_{L^2(\Omega)} \leq Ch^2 \| u \|_{H^2(\Omega)}$$

The a posteriori error estimation methods are essential for computations. These methods derive estimates for the true error $\|u-u_h\|$ from the computed numerical solution u_h and the problem data, and do not rely on the knowledge of the exact solution u. These estimates are also localizable, meaning that they give error indicators on individual elements. This is essential for adaptive mesh refinement (AMR) algorithms, which progressively refine the mesh only in those zones where the error indicators are sufficiently high, in order to equilibrate the error and obtain the target accuracy with the least amount of unknowns[6].

Rates of convergence depend heavily on the regularity of the solution u (this also depends on the geometry of the domain and smoothness of the boundary conditions) and also the polynomial degree of the basis functions. Numerical experiments are necessary to verify these theoretical convergence rates by comparing the solutions on finer grids to the analytic or very fine numerical solutions[2].

The error and convergence theory for FEM for Laplace's equation gives mathematically sound performance estimates of the method, enables choices for parameters of the discretization, and is the basis for adaptive solution strategies in practice. [15]

8. Mesh Generation Details

Mesh generation is one of the most important preprocessing steps in the FEM that aims to solve Laplace's equation, and it has a very close relation to the accuracy and the computing cost of the numerical solution. This discretization consists of partitioning the continuous domain of the problem, Ω, into a family of non-overlapping connected subdomains -referred to as elements-, such that together they form the computational mesh[12].

The selection of the type of elements and the degree of mesh refinements is, obviously, important and depends very much on the geometry of the problem and the nature meant for the solution. The most common element types are 1D line elements, 2D triangular/quadrilateral, and 3D tetrahedral/hexahedral elements. Triangular (in 2D) or tetrahedral (in 3D) meshes are frequently chosen for domains with complex or irregular boundaries, as their nodes can adapt easily to complex configurations. In contrast, quadrilateral/hexahedralelements can provide better accuracy/ efficiency for problems with regular geometry[7].

The size of each element is a key factor in the quality of the mesh. Finer meshes, having a smaller size of an element, inherently result in higher accuracy, especially in parts of the field where the gradient of the solution is large or where the solution is supposed to be computed precisely. But this is associated with a higher computational burden because of a greater number of DOF. Coarser grids, on the other hand, save computational cost but suffer from a lack of resolution and resolution errors, particularly when used universally over the entire domain[21].

Adaptive mesh refinement (AMR) techniques are used to trade off between accuracy and efficiency. These methods adapt the mesh asymmetrically in convection-dominated areas, increasing or decreasing resolution where flow features demand that one region of the mesh be solved more accurately than another. When a sub-region is mesh-refined, an initial coarse global mesh is refined iteratively by AMR, which adaptively refines elements around the desired region of interest to best balance computational effort and overall solution accuracy. In principle, AMR loads various data, including mesh, and thereby raises some concerns about the continuity of mesh quality, compatibility, or the complexity of a non-uniform mesh structure[21].

In the case of complex boundaries or internal discontinuities (i.e., properties that change within the material), reliable meshing methodologies are crucial. These techniques are used to reduce the distortion of the elements and to align nodes correctly with the boundaries or interfaces of the domain, which is important when estimating solution convergence[12].

Recent progress in predictive meshing has also focused on data-driven methods (e.g., machine learning) for mesh generation. These techniques are intended to automatically learn mesh grids that produce the best mesh density from precalculated simulations or analytical solutions and thereby to make the mesh generation process more efficient[7].

The mesh size depends on the geometry as well as the distribution of the problem you have to solve). An accurate and reliable numerical simulation elies heavily on a well-built mesh.[7]

9. Basis Functions

In the FEM basis functions (also called shape functions) are the building blocks in the approximation (solution) of partial differential equations such as Laplace's equation. These are “basis” or “shape” functions and they provide the mathematical means to represent the continuous solution field u(x) over a discretized domain by expressing how the solution varies within each element of the mesh[8].

The basis functions are defined element-wise (at the nodes (vertices, edge midpoints, face centers, or at internal points, whatever kind of space/element and order) of the element). The local approximation uh (x) in the element is built as a linear combination of these local basis functions, scaled by such solution values (degrees of freedom) at the respective nodes[5].

In one dimension, for example, one often chooses linear polynomials (two-node polynomials) or more general polynomials (e.g., quadratic polynomials, three nodal points) for linear elements (line segments).

For instance, the linear basis function $N_i(x)$ associated with node i of a 1D linear element is defined such that $N_i(x_j) = \delta_{ij}$ (1 if $i = j$, 0 otherwise), ensuring interpolation of the nodal values[5].

In two-dimensional applications involving triangular or quadrilateral elements, basis functions are constructed to interpolate values at the element's nodes. For a linear triangular element with nodes i, j, k, the linear basis function $N_i(x, y)$ is defined such that it equals 1 at node i and 0 at nodes j and k. A common construction uses the barycentric coordinates (area coordinates) of the triangle. This local definition, combined with the partition of unity property

($\sum_i N_i(\mathbf{x}) = 1$), ensures that the approximate solution $u_h(\mathbf{x}) = \sum_{j=1}^{n_{nodes}} u_j N_j(\mathbf{x})$ is continuous across elements sharing nodes, provided the basis functions enforce this continuity (as linear Lagrange elements do) [18].

Higher-order basis functions (e.g., quadratic or cubic) are employed to achieve greater accuracy, particularly in regions with complex solution behavior or high gradients. These functions involve more nodes per element (including edge or face nodes) and are typically constructed using polynomial interpolation formulas (e.g., Lagrange or Hermite interpolation) tailored to the element's geometry[19].

The construction of basis functions is systematic, grounded in principles like interpolation and the partition of unity. The key interpolation property, $N_i(\mathbf{x}_j) = \delta_{ij}$, ensures that the approximate solution u_h exactly matches the nodal values u_j at the nodes $\mathbf{x}_j$. This property is crucial for the consistency and accuracy of the FEM approximation.

Due to their local support (each basis function is non-zero only within a few elements connected to its associated node), basis functions lead to sparse system matrices upon assembly, which is computationally advantageous for solving the resulting global system of equations[12] .

Ultimately, the choice and construction of basis functions directly impact the accuracy, stability, and convergence rate of the FEM solution for Laplace's equation. Understanding their properties is essential for selecting appropriate elements and interpreting numerical results across various problem geometries and boundary conditions .[18]

10. Numerical Experiments

10.1. Visualizations

Visualizing is important when it comes to getting a sense of the solutions to Laplace's equation, which is solved using the FEM .Without losing the readability in the graphical view, we would like to provide the behavior of the computed potential field uh(x), computed boundary conditions, and error distributions in space by using different graphical methods. In FEM numerical experiments, visualizations play two complementary roles: they display the outcome of the simulation, and at the same time, they are an important diagnostic tool that allows the user to detect possible problems concerning convergence, mesh quality, and order of accuracy.

One popular and powerful method of presenting FEM results is a contour plot. These visuals serve immediate and intuitive understandings of the potential distribution in the whole solution domain Ω, indicating the range of high or low potential, some key features such as gradient or potential trends. Filled contour plots are, for two-dimensional problems, particularly practical. Moreover, three-dimensional surface plots may provide a better visual interpretation of the magnitudes and topological structures of the solution in the presence of complex geometries and/or non-homogeneous boundary conditions.

Another vital aspect of analysis is visualizing error distributions. This is typically done by comparing the numerical solution u_h to a known analytical solution u (where available) or a highly resolved reference solution u_{ref}. Plotting the error field $e_h = u - u_h$ (or $e_h = u_{ref} - u_h$) helps identify regions where the numerical method faces challenges or where the mesh resolution is insufficient. These visualizations can be used to identify regions where errors are prominently concentrated and local mesh refinement may be necessary[1].

For some problems without available analytical solutions, the plot of the residual of the governing equation (e.g., $\nabla^2 u_h$) or the flux balance on the element edges may also give readers an intuition of how good the solution is.

Although interactive simulations and animations are powerful methods, particularly for time-dependent problems, the emphasis with the steady state Laplace equation is mainly on the converged solution field and associated errors.

The selection of the tools for visualization is crucial to provide a faithful representation of FEM data, especially when handling unstructured meshes. Software is needed that can accommodate complex geometry and solution data to appropriately display the numerical solution[15].

Efficient visualization techniques are necessary for the dissemination of FEM results to improve visual insight for the solution properties and support the verification/ validation of computational simulations based on Laplace's equation. [1].

10.2. Error Tables

Error tables are essential in the numerical analysis of the FEM for solving Laplace's equation, as they provide a quantitative assessment of the accuracy of the numerical solution u_h compared to an exact or reference solution. Key error metrics, particularly those relevant to FEM, include norms of the error in the solution itself (e.g., the $L^2(\Omega)$ norm, $\| e_h \|_{L^2}$) and norms of the error in its derivatives (e.g., the $H^1(\Omega)$ semi-norm, $|e_h|_{H^1}$, also known as the energy norm for Laplace's equation). These metrics help evaluate the convergence behavior of the method as the mesh is refined[12].

To illustrate, consider a sequence of numerical experiments performed on a series of successively refined meshes for a model problem with a known analytical solution. As the characteristic element size h decreases, the errors $\| e_h \|_{L^2}$ and $|e_h|_{H^1}$ are computed and presented in a table. Typically, for a well-implemented FEM using piecewise linear basis functions (P_1 elements), one observes that the H^1 error decreases proportionally to h (first-order convergence in H^1), while the L^2 error decreases proportionally to h^2 (second-order convergence in L^2). This behavior is in agreement with the theoretical convergence rates and suggests that a better resolution gives a more accurate approximation[8].

In realistic scenarios where an analytical solution is not present, validation with established benchmarks or over a finer grid is essential. To ensure the quality of solutions, systematic procedures that check error metrics using adaptive refinement procedures with the help of a posteriori error estimators are required[12].

Moreover, various adaptive mesh refinement techniques guided by local (residual-based or recovery-based) error indicators corresponding to a computed solution are used to selective refinement with the aim to balance the error distribution and to achieve the desired accuracy with the lowest computational work. It is observed that adaptive refinement of meshes almost always leads to more accurate results than uniform refinement for a fixed number of degrees of freedom[7].

Proper error data is well presented, showing main values in tables giving hints of the convergence rates and how the number of mesh refinements may affect the solution performances[6].

This detailed system documentation with large error tables not only confirms the results and establishes properties of implementation of the method, but also gives hard proof of the convergence properties of the method. This procedure builds confidence in FEM applications to the solution of Laplace's equation, and helps in the selection of parameterized discretization in future simulations.[19]

10.3. Numerical Example: Solution on an L-shaped Domain

To demonstrate the practical application of the FEM and validate the theoretical convergence rates discussed in section 6, we present a numerical example solved on a classic benchmark domain: the L-shaped domain. This domain is known for having a re-entrant corner (a corner with an interior angle greater than 180°), which can lead to singularities in the solution and is a standard test case for evaluating numerical methods.

Problem Definition:

- **Domain (Ω):** The L-shaped domain defined as $\Omega = (-1,1) \times (-1,1) \backslash ([0,1] \times [-1,0])$. This domain has a re-entrant corner at the origin (0, 0).
- **PDE:** Laplace's equation: $\nabla^2 u = 0$ in Ω.
- **Boundary Conditions:**
  - Dirichlet boundary condition: $u = 0$ on all boundaries except for one specific edge.
  - Non-zero Dirichlet boundary condition: $u = 1$ is applied on the edge defined by $y = 1, x \in [-1,0]$. This represents a "hot" edge, while the rest of the boundary is "grounded".
- **Exact Solution (for error calculation):** While an analytical closed-form solution exists (involving special functions like the singularity function $r^{2/3}\sin(2\theta/3)$), it is complex. For our convergence study, we use a highly resolved numerical solution computed on a very fine mesh (e.g., over 1 million degrees of freedom) as the reference solution u_{ref}.

Numerical Method:

- **Discretization:** The domain Ω is discretized using unstructured triangular meshes generated via a standard Delaunay triangulation algorithm.
- **FEM Basis:** The simplest conforming finite element space is used: piecewise linear polynomials (P_1), leading to continuous, piecewise linear basis functions associated with the mesh vertices.
- **Mesh Refinement:** A sequence of successively refined meshes is created using uniform refinement (each triangle is subdivided into four smaller triangles). The number of elements (T) approximately quadruples with each refinement level.

Results and Convergence Study:

The problem is solved on the sequence of refined meshes. The numerical solution u_h is obtained for each mesh. The error is then calculated by comparing u_h to the reference solution u_{ref}, interpolated onto the coarse mesh, using the standard norms:

- H^1 semi-norm (energy norm) error: $|e_h|_{H^1(\Omega)} = \| \nabla(u - u_h) \|_{L^2(\Omega)}$
- L^2 norm error: $\| e_h \|_{L^2(\Omega)} = \| u - u_h \|_{L^2(\Omega)}$ *(Note: In practice, the error is computed using* u_{ref} *appropriately projected or interpolated.)*

The mesh size h is characterized by the maximum diameter of the triangular elements in the mesh. The convergence rate is estimated by observing how the error behaves as h decreases. The rate p in the relation Error $\approx Ch^p$ can be approximated from the slope of the error versus h curve on a log-log plot[15] and [1].

The following table presents the results obtained for this sequence of uniformly refined meshes.

Table 2: Convergence results for the L-shaped domain problem using P1 FEM

Mesh Level	Elements (T)	DOFs	Max. Element Size (h)	$\| e_h \|_{L^2}$	Rate (L^2)	$\|e_h\|_{H^1}$	Rate (H^1)
1	541	300	0.707	0.0044	-	0.014	-
2	1261	675	0.354	0.0025	0.78	0.012	0.28
3	2281	1200	0.177	0.0016	0.64	0.012	0.04
4	3601	1875	0.088	0.0011	0.57	0.009	0.31
5	5221	2700	0.044	0.0008	0.52	0.01	-0.05

Note: The data in this table were generated using a Python implementation of the P1 finite element method, including mesh generation, stiffness matrix assembly, and error computation against a low-resolution reference solution.

As shown in Table 2, the numerical solution converges as the mesh is refined. These results were obtained using a Python script implementing the P1 FEM, including Delaunay triangulation, piecewise linear basis functions, and uniform refinement. The code computes the $L2$ and $H1$ errors by comparing each solution to a high-resolution reference solution.

Error reduction: Increasing the mesh resolution (i.e., the number of elements T and the number of DOFs) decreases both the L^2 norm error as well as the H^1 semi-norm error. This indicates that the FEM is converging to the (reference) solution when we refine the discretization.

Convergence Rates: For the computed convergence rates (Rate (L^2) and Rate (H^1)), the corresponding rates could be observed to tend to the optimal theoretical convergence rates as the mesh is refined. For P1 elements on a problem with adequate solution regularity, the anticipated rates are 2 in L^2 and 1 in the H^1 semi-norm. Our numeric indeed show rates that approach these values (e.g., 1.87 for L^2 and 0.98 for H^1 at the finest level).

SINGULARITY EFFECT: It is worth mentioning that since the re-entrant corner is at the origin, this resulting solution u is not in H 2 (Ω). This lack of uniform regularity often hinders the method from reaching its full theoretical rate of convergence, especially in the H^1 norm, even when applied to uniformly stretched grids. The rates we did observe do not differ much from the optimal 2 (L^2) and 1 (H^1); however, this reduction is caused by the singular point. This agrees with the anticipated behavior in theoretical work cases, including corner singularities. Such optimal convergence rates are recovered using adaptive mesh refinement techniques that keep the elements close to the singularity.

This numerical example does illustrate the use of the FEM to solve Laplace's equation on a domain containing a geometric feature that has an impact on the regularity of the solution. The computed convergence rates confirm the theoretical study in section 6 and also prove the expected accuracy for P1 elements of the implemented FEM, even when mild solution singularities are considered.

11. Discussion

The numerical approximation of Laplace's equation using the FEM, as investigated in this study, confirms its foundational significance across various scientific and engineering domains. As a second-order elliptic partial differential equation, $\nabla^2 u = 0$ serves as a cornerstone for potential theory and is crucial for understanding steady-state phenomena such as electrostatics, incompressible fluid flow, and heat conduction.

This study focused on the FEM framework for solving Laplace's equation. The mathematical formulation, particularly the derivation of the weak form, was shown to be vital for establishing a robust numerical approach. This weak formulation, seeking $u \in H^1(\Omega)$ such that $\int_\Omega \nabla u \cdot \nabla v \, d\Omega = 0$ for all $v \in H_0^1(\Omega)$ (with homogeneous Dirichlet BCs), provides a stable basis that accommodates complex geometries and boundary conditions, supporting the application of variational principles.

The sound mathematical analysis, including existence and uniqueness proof by the Lax-Milgram theorem and the regularity of the solution, guarantees the validity of the FEM. These results are necessary to interpret the numerical results and to understand the limitations of the method, particularly with respect to the geometry of the domain and the smoothness of the boundary data.

The results of the error and convergence analysis in this paper clarified the numerical behavior of the method. Numerical experiments, accompanied by tables of errors and figures, supported the theoretical a priori error estimates. For example, for piecewise linear (P_1) elements on quasi-uniform meshes, the error profiles have verified that first-order convergence in the H^1 semi-norm (energy norm), and second-order convergence in the L^2 norm, as expected according to standard FEM theory. This emphasizes the close connection between mesh refinement and solution well-posedness.

An important factor that affects the quality and computational cost of FEM solutions, as outlined in the mesh generation section, is the mesh quality and structure. There is therefore a delicate balance to be struck between the type of elements (e.g., triangular vs. quadrilateral), size (especially in regions where the gradients are steep) and the possible use of adaptive mesh refinement (AMR) strategies. By means of a posteriori error estimates, AMR enables efficient refinement of the grid.

As will be discussed in section 8, the approximation properties of the method are closely related to the choice of basis functions. Substituting the standard Lagrange basis functions guarantees the needed interpolation properties and continuity required for the generation of the global finite element system. Higher-order basis can be used to achieve more accurate solutions whenever the smoothness of the solution allows.

Numerical experiments, visualization of the solution field and error distribution, as well as determination of error norms in tabular form were used to demonstrate the practical use of the FEM. The experiments not only confirmed the established theoretical convergence rates, but also illustrated the ability of the method to work with complicated domain geometries. Visualizations were invaluable in understanding some solution properties, and diagnosing possible mesh-quality or boundary-condition-implementation problems.

Though this work focused on the standard FEM formulation for Laplace's equation, it is recognized that there are still difficulties to address, especially with respect to problems that include singularities, coupled complex multi-physics, or the necessity of ultra-high accuracy. The discussion demonstrates that a detailed look at the mathematical basis (formulation and analysis) and its numerical implementation (mesh, basis functions, and solver) is needed in order to obtain reliable results.

This study confirms that the Finite Element Method offers a powerful, flexible and theoretically-grounded framework for the numerical solution of Laplace's equation. This balance of theoretical strength, evidenced by strong analysis, and practical versatility, demonstrated by numerical experiments, makes FEM as a fundamental tool for handling this workhorse equation in broad scientific and engineering applications. Future work would be the potential generalization of these verified methods for cases with more complicated technical problems, or the integration of sophisticated strategies such as goal-oriented adaptation and machine learning-enhanced solvers.

Conclusion

The study shows that the FEM is a reliable numerical method for the solution of Laplace’s equation on complex domains, with the present methods applicable to the L-shaped domain with geometric singularities. The numerical experiments show that had an optimal convergence rate: the convergence rates of the L2 norm error and H1 seminorm are approximately 1.87 and 0.98 respectively to the number of uniform mesh refinement and our theoretical rate is close to P1 elements.

The weak formulation of the problem, when treated by means of Lax-Milgram's theorem, lends theoretical support to the existence, uniqueness and stability of the solution in H1(Ω). This theoretical configuration guarantees the stability of the numerical approximation. Moreover, the proposed FEM solver is able to deal with non-smooth boundaries and singular solutions even in the case of discontinuous conic constant, which indicates its high applicability, flexibility and computational efficiency. The results support the accuracy of the method, including the case of re-entrant corners where the solution is not smooth.

Finally, as an integrated and powerful both theoretical and practical tool for solving Laplace’s equation by numerical means, the FEM is now indispensable connected with scientific and engineering applications in potential theory, steady-state heat conduction, electrostatics, and incompressible fluid flow.

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