Advances in Fixed Point Theory for Extended b-Metric Spaces Results and Applications

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A R T I C L E   I N F O

A B S T R A C T

This study presents fixed point theorems in extended b-metric spaces using bounded compactness, Ṯ orbital compactness, and asymptotic regularity. We show that Kannan type mappings admit unique fixed points under these weakened topological conditions, without employing auxiliary control functions. The results unify and extend classical theorems, providing applicable tools for nonlinear analysis in variable-scale spaces. Illustrative examples demonstrate the validity and scope of the theory

MSC..

1. Introduction

The fixed-point theory is a central area of nonlinear analysis with far-reaching applications across mathematics, engineering, and the physical sciences. Since the pioneering work of Banach in 1922 on the contraction principle [7], the field has expanded in two major directions: relaxing contractive conditions on mappings and generalizing the geo Metric structure of the underlying spaces .One significant development in this second direction is the introduction of extended b- metric space, a refinement of the classical b- metric space structure originally proposed by Bakhtin [25] and further developed by Czerwik [12]. In an extended b- metric space, the classical constant factor $s \geq 1$ appearing in the b-metric triangle inequality is replaced by a variable coefficient function $s: K \times K \to [1, \infty)$ This generalization allows for spatially dependent scaling and non-uniform geometric behavior, enabling fixed point techniques to be applied to broader classes of nonlinear and heterogeneous systems [1],[4] In this work, we investigate fixed point results in such generalized spaces for mappings satisfying Kannan type conditions, a class of contractions first introduced by Kannan [23]. A distinctive feature of our approach is the use of the global bound $\mathbb{S} = \sup s(x, y)$. Which encapsulates the overall effect of variable scaling thereby circumventing the need for additional control mechanisms often used in generalized Metric architectures [6],[1]. In

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addition to co-meatiness, we investigate the presence of fixed points under less stringent topological conditions, such as limited co-pastness [17],[16] and Ț-orbital co-patness [2],[22]. We examine asymptotic regularity as a means to guarantee the convergence of iterative processes in the absence of rigorous contraction constraints [9],[14]. Moreover, we establish Hyers-Ulam stability results in the sense of Hyers [21] and Ulam [25[, providing explicit bounds that relate approximate fixed point to genuine ones an essential aspect in numerical and applied settings. To illustrate the practical applicability of our theoretical framework, we utilize our fixed point results to establish new and stable solutions for nonlinear functional equations, specifically by proving the Hyers-Ulam stability of certain important classes of mappings in the setting of extended b-metric spaces , we use our findings to which naturally arise in mathematical physics, population dyna mics, engineering models, and control theory [5],[22]. These applications illustrate how extended b- metric space structures accommodate variable geo metric behavior inherent in many real-world problems.

2. Preliminaries

Definition (2.1) : [12] Let K be a non-empty set and $s \geq 1$ be a given real number. A function $M : K \times K \to \mathbb{R}$ is called b-metric BM If it has the subsequent properties:

$(km1)$ $M(x, ý) \geq 0$

$(km1)$ $M(x, ý) = 0$ **if and only if** $x = ý$

$(km2)$ $M(x, ý) = M(ý, x)$

$(km3)$ $M(x, m) \leq s[M(x, y) + M(ý, m)]$

For all $x, ý, m \in K$

The pair (K, M) is called BMS

If $s = 1$ then BMS is a MS

Definition (2.2) : Let K be a non-empty set and $\vartheta : K \times K \to [0,1)$ be a function. The map $M_\vartheta : K \times K \to \mathbb{R}$ is called extended b-metric space EBMS if it satisfies the following properties

$(km1)$ $M_\vartheta(x, ý) \geq 0$

$(km2)$ $M_\vartheta(x, ý) = 0$ if and only if $x = y$

$(km3)$ $M_\vartheta(x, ý) = M_\vartheta(y, x)$

$(km4)$ $M_\vartheta(x, m) \leq \vartheta(x, m)[M_\vartheta(x, ý) + M_\vartheta(ý, m)]$

Definition (2.3) : A function $\vartheta : [0, \infty) \to [0, \infty)$ is called to be a sub-additive altering distance function if

$(km1)$ ϑ is continuous , strictly increasing and $\vartheta(t) = 0$ iff $t = 0$

$(km2)$ $\vartheta(x, ý) \leq \vartheta(x) + \vartheta(ý)$ $\forall$ $x, ý \in [0, \infty)$

Example (2.1): The following easily seen that the function

$\vartheta_1(x) = kx \; for \; some \; k \geq 1$

$\vartheta_2(x) = \sqrt[n]{x} , n \in \mathbb{N}$

$\vartheta_3(x) = \log(1 + x) , x \geq 0$

$\vartheta_4(x) = \tan^{-1} x$

Are sub-additive altering distance function

3. Boundedly compact and $Ⳍ$ −orbital compact of K

In functional and metric space analysis, boundedly compact spaces represent a key structural property where every closed and bounded subset is compact. This concept is central to extending fixed-point results beyond classical compact spaces. The more specialized notion of $Ⳍ$-orbital compactness focuses specifically on the closure of the orbit of points under a given operator, ensuring that iterative sequences remain in relatively compact subsets. Together, these properties allow mathematicians to establish the existence of fixed points under weaker and more applicable conditions, particularly in generalized settings such as extended b-metric spaces.

Definition 3.1 : ([29], [30]) An EBMS (K, M) is called Boundedly compact if each bounded sequence from K has a convergent subsequence.

Remark 3.1 : ([29], [30]) Every sequentially compact extended b- metric space is boundedly compact Bc

Remark 3.2: ([29],[30]) The class BcEBMS is larger than that of sequentially cS

Theorem 3.1 : Let (K, M) be a BcEBMS and let $Ⳍ: K \times K$ be a continuous map with $\mathbb{S} = \sup\{s(x, \acute{y}): x, \acute{y} \in K\} < \infty$, fulfilling

$$\vartheta\left(M(Ⳍ_x, \acute{y})\right) \le p\left\{\vartheta\left(M(x, Ⳍ_x)\right) + \vartheta\left(M(\acute{y}, Ⳍ_{\acute{y}})\right)\right\} , \forall\, x, y \in K \tag{3.1}$$

For so Me $0 \le p \le \frac{1}{2}$. Then $Ⳍ$ has a UFP $z \in K$ and for any $x_0 \in K$, the sequence $\{Ⳍ^n x_0\}$ converges to z

Proof:

Let x_0 be any point in K .Getting the sequence $\{x_n\}$ defined by
$$x_n = Ⳍ^n x_0 , \forall\, n \in \mathbb{N}$$
Define $\gamma_n = M(x_n, x_{n+1})$ and assu Me $\gamma_n > 0 , \forall\, n \in \mathbb{N}$

$$\vartheta(\gamma_n) = \vartheta\left(M(Ⳍ^n x_0, Ⳍ^{n+1} x_0)\right)$$
$$= \vartheta\left(M(Ⳍ(Ⳍ^{n-1} x_0), Ⳍ(Ⳍ^n x_0))\right)$$
$$\le p\{\vartheta\left(M(Ⳍ^{n-1} x_0, Ⳍ^n x_0)\right) + \vartheta\left(M(Ⳍ^n x_0, Ⳍ^{n+1} x_0)\right)\}$$
$$= p\vartheta(\gamma_{n-1}) + p\vartheta(\gamma_n)$$

$$(1 - p)p\vartheta(\gamma_n) \le p\vartheta(\gamma_{n-1}) \implies p\vartheta(\gamma_n) \le \frac{1}{(1 - p)} p\vartheta(\gamma_{n-1})$$

Since ϑ is strictly increasing and $p < \frac{1}{2}$, we have $\frac{p}{1-p} < 1$ .Therefor
$$\gamma_n < \gamma_{n-1} , \forall\, n \in \mathbb{N}$$
This shows that $\{\gamma_n\}$ is a strictly decreasing sequence real of type positive numbers.

Since $\{\gamma_n\}$ is decreasing and bound below by 0 , it converges
$$\lim_{n\to\infty} \gamma_n = \gamma \ge 0$$
We claim that $\gamma = 0$, suppose for contradiction that $\gamma > 0$. Then from the inequality
$$\vartheta(\gamma_n) \le \frac{p}{(1 - p)} \vartheta(\gamma_{n-1})$$

Taking limits as $n \to \infty$

$$\vartheta(\gamma) \le \frac{p}{(1 - p)} \vartheta(\gamma) \implies \left(1 - \frac{p}{1 - p}\right) \vartheta(\gamma) \le 0$$

Since $p < \frac{1}{2}$ we have $1 - \frac{p}{1-p} > 0$, and $\vartheta(\gamma) > 0$ .This contradiction
$\gamma = 0$

Now to showing bounded

$m, n \in \mathbb{N}$, with $n < m$, by using EBMS and the contraction condition
$$\vartheta\left(M(x_m, x_n)\right) \le \vartheta(s(x_m, x_n)[M(x_m, x_{m-1}) + M(x_{m-1}, x_n)])$$
$$\le \mathbb{S}[M(x_m, x_{m-1}) + M(x_{m-1}, x_n)]$$

$$\leq \mathbb{S}\vartheta(\gamma_{m-1}) + \mathbb{S}^2[\vartheta M(\mathbb{X}_{m-1}, \mathbb{X}_{m-2}) + \vartheta M(\mathbb{X}_{m-2}, \mathbb{X}_n)]$$
$$\leq \mathbb{S}\vartheta(\gamma_{m-1}) + \mathbb{S}^2\vartheta(\gamma_{m-2}) + \cdots + \mathbb{S}^{m-n}\vartheta(\gamma_n)$$
$$+ \mathbb{S}^{m-n+1}\vartheta(M(\mathbb{X}_n, \mathbb{X}_n))$$

Since $\vartheta(M(\mathbb{X}_n, \mathbb{X}_n)) = 0$ and $\gamma_n \to 0$ we have

$$\vartheta(M(\mathbb{X}_m, \mathbb{X}_n)) \leq \mathbb{S}[\vartheta(\gamma_{m-1}) + \mathbb{S}\vartheta(\gamma_{m-2}) + \mathbb{S}^{m-n-1}\vartheta(\gamma_n)]$$

As $m, n \to \infty$, the right-hand side approaches 0. Therefore, for sufficiently large N , there exists $M > 0$ s.t

$$M(\mathbb{X}_m, \mathbb{X}_n) \leq M \quad \forall m, n \geq N$$

Hence $\{\mathbb{X}_n\}$ is a bounded sequence

Since $\{\mathbb{X}_n\}$ is a bounded sequence in a BcEBMS, it has a convergent subsequence $\{\mathbb{X}_{n_k}\}$ such that

$$\lim_{k \to \infty} \mathbb{X}_{n_k} = z \ , \quad \text{for } z \in K$$

By the continuity of $\mathbb{T}$

$$\mathbb{T}z = \mathbb{T}\left(\lim_{k \to \infty} \mathbb{X}_{n_k}\right) = \lim_{k \to \infty} \mathbb{T}\left(\mathbb{X}_{n_k}\right) = \lim_{k \to \infty} \mathbb{X}_{n_k} + 1$$

To showing $\lim_{k \to \infty} \mathbb{X}_{n_k} + 1 = z$, consider

$$M(\mathbb{X}_{n_k+1}, z) \leq s(\mathbb{X}_{n_k+1}, z)[M(\mathbb{X}_{n_k+1}, \mathbb{X}_{n_k}) + M(\mathbb{X}_{n_k}, z)]$$
$$\leq \mathbb{S}[\gamma_{n_k} + M(\mathbb{X}_{n_k}, z)] \to 0 \ , \quad \text{as } k \to \infty$$

Therefore

$$\mathbb{T}z = \lim_{k \to \infty} \mathbb{X}_{n_k} + 1 = z$$

So z is a FP of $\mathbb{T}$

Now to showing z is UFP

Suppose $w \in K$ is another FP of $\mathbb{T}$

$$\vartheta(M(z, w)) \leq \vartheta(M(\mathbb{T}z, \mathbb{T}w))$$
$$\leq p\{\vartheta(M(z, \mathbb{T}z)) + \vartheta(M(w, \mathbb{T}w))\}$$
$$= p\{\vartheta(0) + \vartheta(0)\} = 0$$

Thus $z = w$

To show the entire sequence $\{\mathbb{T}^n x_0\}$ converges to z

$$M(\mathbb{X}_n, z) = M(\mathbb{T}^n \mathbb{X}_0, \mathbb{T}z)$$
$$\leq s(\mathbb{T}^n \mathbb{X}_0, \mathbb{T}z)[M(\mathbb{T}^n \mathbb{X}_0, \mathbb{T}^{n+1} \mathbb{X}_0)(\mathbb{T}^{n+1} \mathbb{X}_0, \mathbb{T}z)$$
$$\leq \mathbb{S}[\gamma_n + M(\mathbb{T}^{n+1} \mathbb{X}_0, \mathbb{T}z)]$$

Since $\gamma_n \to 0$, therefore $M(\mathbb{T}^{n+1} \mathbb{X}_0, \mathbb{T}z) \to 0$, as $n \to \infty$

Hence $M(\mathbb{X}_n, z) \to 0$

Example (**3.1**) : Consider the BcEBMS (K, M) where $K = [0, \infty)$ and the EBMS $M: K \times K \to [0, \infty)$ is define by

$$M(\mathbb{X}, \acute{y}) = \begin{cases} \mathbb{X} + \acute{y} & \text{if } \mathbb{X} \neq \acute{y} \\ 0 & \text{if } \mathbb{X} = \acute{y} \end{cases}$$

With the coefficient function $s: K \times K \to [0, \infty)$ is get by:

$$s(\mathbb{X}, \acute{y}) = 1 + \frac{\min\{\mathbb{X}, \acute{y}\}}{1 + \min\{\mathbb{X}, \acute{y}\}}$$

The identity and symmetry are clearly by the definition of M

For any $\mathbb{X}, \acute{y}, z \in K$

$$M(\mathbb{X}, z) \leq s(\mathbb{X}, \acute{y})[M(\mathbb{X}, \acute{y}) + M(\acute{y}, z)]$$

$$\mathbb{X} + z \leq (1 + \frac{\text{Min}\{\mathbb{X}, \acute{y}\}}{1 + \text{Min}\{\mathbb{X}, \acute{y}\}}[(\mathbb{X} + \acute{y}) + (\acute{y} + z)]$$

Because of the right-hand side is at least $x, +ý + ý + z = x + z + 2ý$ and

$ý \geq 0$, the inequality holds

The supremum of the coefficient function is

$$\mathbb{S} = \sup\{s(x, ý): x, ý \in K\} = 2$$

Define $\mathsf{T}: K \times K$ by

$$\mathsf{T}x = \begin{cases} \dfrac{1}{2} & \text{if } 0 \leq x \leq 2 \\ \dfrac{1}{x} & \text{if } x \geq 2 \end{cases}$$

The mapping is continuous

For $\vartheta(t) = t$

$$M(\mathsf{T}x, \mathsf{T}ý) \leq p\{M(x, \mathsf{T}x) + M(ý, \mathsf{T}ý)\} \quad , \qquad \text{for all } x, ý \in K$$

$$\leq \frac{1}{2}\{M(x, \mathsf{T}x) + M(ý, \mathsf{T}ý)\}$$

Case 1: $x, ý > 2$, then $\mathsf{T}x = \frac{1}{x}$, $\mathsf{T}ý = \frac{1}{ý}$

Thus

$$M(\mathsf{T}x, \mathsf{T}ý) < 1 < \frac{1}{2} \times 4 \leq \frac{1}{2}\{M(x, \mathsf{T}x) + M(ý, \mathsf{T}ý)\}$$

The inequality is holds

Case 2: $0 \leq x \leq 2$, $ý > 2$ .Then $\mathsf{T}x = \frac{1}{2}$, $\mathsf{T}ý = \frac{1}{ý}$

Thus

$$M(\mathsf{T}x, \mathsf{T}ý) < 1 < \frac{1}{2} \times \frac{5}{3} = \frac{5}{4} \leq \frac{1}{2}\{M(x, \mathsf{T}x) + M(ý, \mathsf{T}ý)\}$$

The inequality is holds
Case 3: $x, ý \in [0,2]$, then $\mathsf{T}x = \mathsf{T}ý = \frac{1}{2}$, $M(\mathsf{T}x, \mathsf{T}ý) = 0$

The inequality is holds

By above theorem for EBMS, T has a UFP, $x = \frac{1}{2}$, this is FP

For $0 \leq x \leq 2$: $\mathsf{T}\left(\frac{2}{2}\right) = \frac{1}{2}$

For $x > 2$: $\mathsf{T}\left(\frac{1}{2}\right) = \frac{1}{2}$ is not applicable, but direct check shows $\frac{1}{2} \in [0,2]$

 Moreover, we observe the iterates

If $x_0 \in [0,2]$, then $Tx_0 = \frac{1}{2}$ and $T^2 x_0 = \frac{1}{2}$

Thus, for every $x_0 \in K$, iterates of the sequence $\{T^n x_0\}$ converges to the UFP $x = \frac{1}{2}$

The coefficient function $s(x, ý) = 1 + \frac{|x - ý|}{1 + |x - ý|}$

Then, $s(x, ý) \leq 2$ and $\mathbb{S} = \sup s(x, ý) = 2$, $\forall x, ý \in K$

The triangle inequality still holds since

$$M(x, z) \leq 2[M(x, ý) + M(ý, z)] \ , \forall x, ý, z \in K$$

Definition 3.2: [30] Let (K, M) be a EBMS and $T: K \to K$ be a self- mapping on K . The orbit of T at a point $x \in K$ is defined as the set

$$O_x(T) = \{x, Tx, T^2 x, T^3 x, \dots\} \tag{3.2}$$

The EBMS (K, M) is called to be $T -$ orbitally co Mpact if $\forall x \in K$

Every sequence in the orbit $O_x(T)$ has a convergent subsequence in K

Equivalently, (K, M) is $T -$ orbitally compact if for each $x \in K$, the orbit $O_x(T)$ is relatively compact in (K, M)

Theorem 3.2: Let (K, M) be   a $T -$ orbitally cEBMS and let $T: K \to K$ be a map, with $\mathbb{S} = \sup\{s(x, ý): x, ý \in K\} < \infty$, satisfying

$$\vartheta\big(M(Tx, Tý)\big) \leq p\{\vartheta\big(M(x, Tx)\big) + \vartheta\big(M(ý, Tý)\big)\} , x, ý \in K \tag{3.3}$$

Where $p < \frac{1}{2}$ and $\mathbb{S}p < 1$. Then T has a UFP $w \in K$, and $\forall x \in K$

$$\lim_{n \to \infty} T^n x = w$$

Proof:

 Let $x_0 \in K$ be an arbitrary point, and define the iterative sequence $\{x_n\}$ by, $x_n = T^n x_0$ .Denote $M(x_n, x_{n+1}) = \beta_n$, $n = 0,1,2, \dots$. Applying the contraction condition with $x = x_{n-1}$, $x_n = y$

$\vartheta(\beta_n) = \vartheta\big(M(T^n x_0, T^{n+1} x_0)\big)$

$\qquad = \vartheta\left(M\big(T(T^{n-1} x_0), T(T^n x_0)\big)\right)$

$\qquad \leq p\{\vartheta\big(M(T^{n-1} x_0, T^n x_0)\big) + \vartheta\big(M(T^n x_0, T^{n+1} x_0)\big)\}$

$\qquad = p\{\vartheta(\beta_{n-1}) + \vartheta(\beta_n)\}$

Therefore

$(1 - p)\vartheta(\beta_n) \leq p\vartheta(\beta_{n-1}) \implies \vartheta(\beta_n) \leq \frac{p}{1 - p}\vartheta(\beta_{n-1})$

Since ϑ is strictly increasing and $p < \frac{1}{2}$,we $have$ $\frac{p}{1-p} < 1$, and therefore

$$\beta_n < \beta_{n-1} \ \ \forall n \in N$$

Thus, $\{\beta_n\}$ is a strictly decreasing of non-negative real number. Since $\{\beta_n\}$ is decreasing and bounded below by 0 , it converges to some limit

$$\lim_{n\to\infty} \beta_n = \beta \geq 0$$

Since (K, M) is $Ț-$ orbitally compact , the sequence $\{x_n\} \subset O_{x_0}(Ț)$ has a convergent subsequence $\{x_{n_k}\}$

$$\lim_{k\to\infty} x_{n_k} = w \in K$$

Now consider the corresponding distance subsequence

$$\lim_{k\to\infty} \beta_{n_k} = \lim_{k\to\infty} M(x_{n_k}, x_{n_k+1}) = M\left(\lim_{k\to\infty} x_{n_k} , \lim_{k\to\infty} x_{n_k+1}\right) = M(w, w) = 0$$

Since $\{\beta_n\}$ is convergent and contains a subsequence converging to 0

$$\lim_{n\to\infty} \beta_n = 0$$

Now to showing Cauchy sequence
$\forall\ n. m \in N$, with $m > n$, by using EBMS and contraction condition we get

$$\vartheta\big(M(x_n, x_m)\big) \leq \vartheta\big(M(Ț^n x_0, Ț^m x_0)\big)$$

$$\leq p\{\vartheta\big(M(Ț^{n-1} x_0, Ț^n x_0)\big) + \vartheta\big(M(Ț^{m-1} x_0, Ț^m x_0)\big)\}$$

$$= p\{\vartheta(\beta_{n-1}) + \vartheta(\beta_{m-1})\}$$

As $m, n \to \infty$, so

$$\lim_{n,m\to\infty} \vartheta\big(M(x_n, x_m)\big) = 0 \implies \lim_{n,m\to\infty} M(x_n, x_m) = 0$$

Thus $\{x_n\}$ is Cauchy sequence in the $Ț-$ orbitally cS , and therefore

$$\lim_{n\to\infty} x_n = w$$

Now to showing FP , by using EBMS and contraction condition

$$\vartheta\big(M(w, Țw)\big) \leq \vartheta(s(w, Țw)[M(w, Ț^{n+1} x_0) + M(Ț^{n+1} x_0, Țw)])$$

$$\leq \mathbb{S}[\vartheta\big(M(w, x_{n+1})\big) + \vartheta\big(M(Ț^{n+1} x_0, Țw)\big)]$$

$$\leq \mathbb{S}\vartheta\big(M(w, x_{n+1})\big) + \mathbb{S}p\{\vartheta\big(M(x_n, x_{n+1})\big) + \vartheta\big(M(w, Țw)\big)\}$$

Therefore

$$(1 - \mathbb{S}p)\vartheta\big(M(w, Țw)\big) \leq \mathbb{S}\vartheta\big(M(w, x_{n+1})\big) + \mathbb{S}p\vartheta\big(M(x_n, x_{n+1})\big)$$

As $n \to \infty$ and $1 - \mathbb{S}p > 0$, we obtain

$$\vartheta\big(M(w, Țw)\big) = 0 \implies M(w, Țw) = 0 \implies Țw = w$$

Thus w is FP of $Ț$

Now to showing uniqueness of the FP. Suppose $z \in K$ is another FP of $Ț$, with $w \neq z$

$$\vartheta\big(M(z,w)\big) = \vartheta\big(M(\mathrm{T}z, \mathrm{T}w)\big)$$

$$\leq p\{\vartheta\big(M(z, \mathrm{T}z)\big) + \vartheta\big(M(w, \mathrm{T}w)\big)\}$$

$$= p\{\vartheta(0) + \vartheta(0)\} = 0$$

Therefore $\vartheta\big(M(z,w)\big) = 0 \implies M(z,w) = 0 \implies z = w$

Which is a contradiction. Therefore w is UFP

Hence $\lim\limits_{n\to\infty} \mathrm{T}^n \mathrm{x} = w$

Example (3.3) : Consider the incomplete EBMS (K, M) where $K = (0, \infty)$ and the EBM $M: K \times K \to [0, \infty)$ is define by

$$M(\mathrm{x}, \acute{y}) = \begin{cases} \mathrm{x} + \acute{y} & if\ \mathrm{x} \neq \acute{y} \\ 0 & if\ \mathrm{x} = \acute{y} \end{cases}$$

With the coefficient function $s(\mathrm{x}, \acute{y}) = 1 + \frac{\min\{\mathrm{x},\acute{y}\}}{1+|\mathrm{x}-\acute{y}|+\min\{\mathrm{x},\acute{y}\}}$

The identity and symmetry clearly satisfied by the definition of M .For $\mathrm{x}, \acute{y}, z \in K$

$$M(\mathrm{x}, z) \leq s(\mathrm{x}, z)[M(\mathrm{x}, \acute{y}) + M(\acute{y}, z)]$$

$$s(\mathrm{x}, z)[M(\mathrm{x}, \acute{y}) + M(\acute{y}, z)] \leq 2[(\mathrm{x} + \acute{y}) + (\acute{y} + z)] = 2(\mathrm{x} + 2\acute{y} + z)$$

Since $\mathrm{x} + z \leq 2(\mathrm{x} + 2\acute{y} + z)$, the inequality holds. Thus, the coefficient function

$$\mathbb{S} = \sup\{s(\mathrm{x}, \acute{y}): \mathrm{x}, \acute{y} \in K\} = 2$$

Define the mapping $\mathrm{T}: K \to K$ by

$$\mathrm{T}x = \begin{cases} \dfrac{1}{2} & if\ 0 < \mathrm{x} < 2 \\ 1 & if\ \mathrm{x} = 2 \\ \dfrac{1}{\mathrm{x}} & if\ \mathrm{x} > 2 \end{cases}$$

The space (K, M) is $\mathrm{T} -$ orbitally compact because for any $\mathrm{x} \in K$

If $0 < \mathrm{x} < 2$: $O_{\mathrm{x}}(\mathrm{T}) = \left\{x, \frac{1}{2}, \frac{1}{2}, \dots\right\}$ converges to $\frac{1}{2}$

If $\mathrm{x} = 2$: $O_{\mathrm{x}}(\mathrm{T}) = \left\{2, 1, \frac{1}{2}, \frac{1}{2}, \dots\right\}$ converges to $\frac{1}{2}$

If $\mathrm{x} > 2$: $O_{\mathrm{x}}(\mathrm{T}) = \left\{\mathrm{x}, \frac{1}{\mathrm{x}}, \frac{1}{2}, \frac{1}{2}, \dots\right\}$ converges to $\frac{1}{2}$

In all cases, every sequence in $O_{\mathrm{x}}(\mathrm{T})$ has a convergent subsequence. The mapping $\vartheta(t) = \log(1 + t)$

For so $p < \frac{1}{2}$ and $\mathbb{S}p < 1$, the following holds

$$\{1 + M(\mathrm{T}\mathrm{x}, \mathrm{T}\acute{y})\}^2 < e\{1 + M(\mathrm{x}, \mathrm{T}\mathrm{x})\}\{1 + M(\acute{y}, \mathrm{T}\acute{y})\}, \forall\ \mathrm{x}, \acute{y} \in K$$

Case 1: $\mathrm{x}, \acute{y} \in (0,2)$, then $\mathrm{T}\mathrm{x} = \mathrm{T}\acute{y} = \frac{1}{2}$, so

$$M(Ţẋ, Ţý) = 0 \implies \{1 + M(Ţẋ, Ţý)\}^2 = 1$$

For any $ẋ \in (0,2)$

$$M(ẋ, Ţẋ) = ẋ + \frac{1}{2} \implies 1 + M(ẋ, Ţẋ) = ẋ + \frac{3}{2} > \frac{3}{2}$$

Thus

$$e\{1 + M(ẋ, Ţẋ)\}\{1 + M(ý, Ţý)\} \geq e\left(\frac{3}{2} \times \frac{3}{2}\right) = e\frac{9}{4} > 1$$

The inequality holds

Case 2: $ẋ, ý > 2$, then $Ţẋ = \frac{1}{ẋ}, Ţý = \frac{1}{ý}$

$$M(Ţẋ, Ţý) = \frac{1}{ẋ} + \frac{1}{ý} \leq 1 \implies \{1 + M(Ţẋ, Ţý)\}^2 \leq 4$$

For $ẋ > 2$

$$M(ẋ, Ţẋ) = ẋ + \frac{1}{ẋ} \geq 2 + \frac{1}{2} = \frac{5}{2} \implies 1 + M(ẋ, Ţẋ) \geq \frac{7}{2}$$

Thus

$$e\{1 + M(ẋ, Ţẋ)\}\{1 + M(ý, Ţý)\} \geq e\left(\frac{7}{2} \times \frac{7}{2}\right) = e\frac{49}{4} > 4$$

The inequality holds

Case 3: $ẋ \in (0,2)$, $ý > 2$, then $Ţẋ = \frac{1}{2}$, $Ţý = \frac{1}{ý}$

$$M(Ţẋ, Ţý) = \frac{1}{2} + \frac{1}{ý} \leq 1 \implies \{1 + M(Ţẋ, Ţý)\}^2 \leq 4$$

For $ẋ \in (0,2)$

$$M(ẋ, Ţẋ) = x + \frac{1}{2} \implies 1 + M(ẋ, Ţẋ) = x + \frac{3}{2} \geq \frac{3}{2}$$

For $ý > 2$

$$M(ý, Ţý) = ý + \frac{1}{ý} \geq \frac{5}{2} \implies 1 + M(ý, Ţý) =\geq \frac{7}{2}$$

Thus,

$$e\{1 + M(ẋ, Ţẋ)\}\{1 + M(ý, Ţý)\} \geq e\left(\frac{3}{2} \times \frac{7}{2}\right) = e\frac{21}{4} > 4$$

The inequality holds

Case 4: $x = 2$, $y \in (0,2)$, then $Ţx = 1$, $Ţy = \frac{1}{2}$

$$M(Ṭx, Ṭý) = 1 + \frac{1}{2} = \frac{3}{2} \Longrightarrow \{1 + M(Ṭx, Ṭý)\}^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$$

For $x = 2$

$$M(x, Ṭx) = 2 + 1 = 3 \Longrightarrow 1 + M(x, Ṭx) = 4$$

For $ý \in (0,2)$

$$M(ý, Ṭý) = ý + \frac{1}{2} \Longrightarrow 1 + M(ý, Ṭý) = ý + \frac{3}{2} \geq \frac{3}{2}$$

Thus

$$e\{1 + M(x, Ṭx)\}\{1 + M(ý, Ṭý)\} \geq e\left(4 \times \frac{3}{2}\right) = 6e > \frac{25}{4}$$

The inequality holds

Case 5: $x = 2$, $ý > 2$, then $Ṭx = 1$, $Ṭý = \frac{1}{ý}$

$$M(Ṭx, Ṭý) = 1 + \frac{1}{y} \leq \frac{3}{2} \implies \{1 + M(Ṭx, Ṭý)\}^2 \leq \left(\frac{5}{2}\right)^2 = \frac{25}{4}$$

For $x = 2$

$$M(x, Ṭx) = 2 + 1 = 3 \implies 1 + M(x, Ṭx) = 4$$

For $ý > 2$

$$M(ý, Ṭý) = ý + \frac{1}{y} \geq \frac{5}{2} \Longrightarrow 1 + M(ý, Ṭý) \geq \frac{7}{2}$$

Thus

$$e\{1 + M(x, Ṭx)\}\{1 + M(ý, Ṭý)\} \geq e\left(4 \times \frac{7}{2}\right) = 14e > \frac{25}{4}$$

The inequality holds

Case 6: $x = 2$, $ý = 2$, then $Ṭx = Ṭý = 1$

$$M(Ṭx, Ṭý) = 0 \Longrightarrow \{1 + M(Ṭx, Ṭý)\}^2 = 1$$

$$e\{1 + M(x, Ṭx)\}\{1 + M(ý, Ṭý)\} \geq e(4 \times 4) = 16e > 1$$

The inequality holds

By above theorem for EBMS, Ṭ has a UFP, $x = \frac{1}{2}$, this is FP

For $0 < x < 2$: $Ṭ\left(\frac{1}{2}\right) = \frac{1}{2}$

The FP is U

 Moreover, for any $x_0 \in K$, we obtain

If $0 < x_0 < 2 :$ $Tx_0 = \frac{1}{2}$, $T^2x_0 = \frac{1}{2}$

If $x_0 = 2 :$ $Tx_0 = 1$, $T^2x_0 = \frac{1}{2}$, $T^3x_0 = \frac{1}{2}$

If $x_0 > 2 :$ $Tx_0 = \frac{1}{x_0}$, $T^2x_0 = \frac{1}{2}$, $T^3x_0 = \frac{1}{2}$

Thus, the sequence of iterates $\{T^nx_0\}$ converges to the UFP $x = \frac{1}{2}$

Since, we have $\mathbb{S} = 2$, and the required bounds $p < \frac{1}{2}$ and $\mathbb{S}p < 1$

The contraction condition is satisfied with $p < \frac{1}{2}$

4. Asymptotic Regularity of T

Asymptotic regularity means the distance between consecutive iterations of an operator T vanishes as the process continues. This property ensures the iterations stabilize, a key step toward convergence to a FP. While not sufficient alone, it is a foundational condition in iterative methods and FP theory, especially in generalized settings like EBMS.

Definition 4.1 : [27] Let (K, M) be EBMS with. A mapping $T: K \to K$ is called asymptotically regular if for every $x \in K$

$$\lim_{n\to\infty} M(T^nx, T^{n+1}x) = 0 \tag{4.1}$$

For further details in asymptotic regular mappings, we refer to [26, 28] and the references therein.

Theorem 4.1 : Let (K, M) be a cEBMS with $\mathbb{S} = \sup\{s(x, ý): x, ý \in K \} < \infty$ and let $T: K \to K$ be an asymptotically regular Mapping satisfying

$$\vartheta\big(M(Tx, Tý)\big) \le p\{\vartheta\big(M(x, Tx)\big) + \vartheta\big(M(ý, Tý)\big)\}   , \forall x, ý \in K \tag{4.2}$$

For so Me p and $\mathbb{S}p < 1$ .Then T has a UFP.

Proof:

Let $x \in K$ be an arbitrary point. Consider the iterative sequence $\{x_n\}$ defined by

$$x_n = T^n   , n = 0,1,2,\dots \tag{4.3}$$

Since T is asymptotically regular, we have

$$\lim_{n\to\infty} M(x_n, x_{n+1}) = \lim_{n\to\infty} M(T^nx, T^{n+1}x) = 0 \tag{4.4}$$

Now to showing Cauchy sequence, for any $\epsilon > 0$, $\exists k \in N$ such that for all $m > n > k$

$$M(x_n, x_m) < \epsilon$$

Using the EBMS property and the contraction condition, for $m > n$

$\vartheta\big(M(x_{n+1}, x_{m+1})\big) \le \vartheta\big(M(T^{n+1}x, T^{m+1}x)\big)$

$= \vartheta\left(M\big(T(T^nx), T(T^mx)\big)\right)$

$$\leq p\{\vartheta\big(M(Ţ^n x, Ţ^{n+1}x)\big) + \vartheta\big(M(Ţ^m x, Ţ^{m+1}x)\big)\}$$

$$= p\{\vartheta\big(M(x_n, x_{n+1})\big) + \vartheta\big(M(x_m, x_{m+1})\big)\}$$

Since ϑ is continuous and strictly increasing with $\vartheta(0) = 0$ and by

asymptotic regularity

$$\lim_{n\to\infty} \vartheta\big(M(x_n, x_{n+1})\big) = 0 \ \ and \ \ \lim_{m\to\infty} \vartheta\big(M(x_m, x_{m+1})\big) = 0$$

Therefore

$$\lim_{n,m\to\infty} \vartheta\big(M(x_{n+1}, x_{m+1})\big) = 0 \implies \lim_{n,m\to\infty} M(x_{n+1}, x_{m+1}) = 0$$

Now by using EBMS inequality

$$M(x_n, x_m) \leq s(x_n, x_m)[M(x_n, x_{n+1}) + M(x_{n+1}, x_m)]$$

$$\leq \mathbb{S}[M(x_n, x_{n+1}) + M(x_{n+1}, x_m)]$$

$$\leq \mathbb{S}\big[M(x_n, x_{n+1}) + \mathbb{S}[M(x_{n+1}, x_{n+2}) + M(x_{n+2}, x_m)]\big]$$

$$\leq \mathbb{S}M(x_n, x_{n+1}) + \mathbb{S}^2 M(x_{n+1}, x_{n+2}) + \mathbb{S}^2 M(x_{n+2}, x_m)$$

$$\leq \sum_{z=n}^{m-1} \mathbb{S}^{z-n+1} M(x_z, x_{z+1})$$

By asymptotic regularity, for any $\epsilon > 0$, $\exists\, k \in N$, $\forall\, z \geq k$

$$M(x_z, x_{z+1}) < \frac{\epsilon(1 - \mathbb{S}^{-1})}{\mathbb{S}}$$

Then for $m > n < k$

$$M(x_n, x_m) \leq \sum_{z=n}^{m-1} \mathbb{S}^{z-n+1} \frac{\epsilon(1 - \mathbb{S}^{-1})}{\mathbb{S}}$$

$$= \epsilon(1 - \mathbb{S}^{-1}) \sum_{z=n}^{m-1} \mathbb{S}^{z-n}$$

$$= \epsilon(1 - \mathbb{S}^{-1}) \frac{1 - \mathbb{S}^{m-n}}{1 - \mathbb{S}}$$

$$< \epsilon(1 - \mathbb{S}^{-1}) \frac{1}{(1 - \mathbb{S}^{-1})} = 1$$

Thus, $\{x_n\}$ is a Cauchy sequence in CEBMS (K, M) , so there exists

$o \in K$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} Ţ^n x = o$$

Now to showing x is FP

Using the EBMS property and the contraction condition

$$\vartheta\big(M(ẋ, Ṭẋ)\big) \leq \vartheta\big(s(ẋ, Ṭẋ)\big)[M(ẋ, Ṭ^{n+1}ẋ) + M(Ṭ^{n+1}ẋ, Ṭẋ)]$$

$$\leq \mathbb{S}[M(ẋ, Ṭ^{n+1}ẋ) + M(Ṭ^{n+1}ẋ, Ṭẋ)]$$

$$\leq \mathbb{S}\vartheta\big(M(ẋ, ẋ_{n+1})\big) + \mathbb{S}\vartheta\big(M(Ṭ(Ṭ^n ẋ), Ṭẋ)\big)$$

$$\leq \mathbb{S}\vartheta\big(M(ẋ, ẋ_{n+1})\big) + \mathbb{S}p\{\vartheta\big(M(Ṭ^n ẋ, Ṭ^{n+1}ẋ)\big) + \vartheta\big(M(ẋ, Ṭẋ)\big)\}$$

$$= \mathbb{S}\vartheta\big(M(ẋ, ẋ_{n+1})\big) + \mathbb{S}p\{\vartheta\big(M(ẋ_n, ẋ_{n+1})\big) + \vartheta\big(M(ẋ, Ṭẋ)\big)\}$$

Therefore

$$(1 - \mathbb{S}p)\vartheta\big(M(ẋ, Ṭẋ)\big) \leq \mathbb{S}\vartheta\big(M(ẋ, ẋ_{n+1})\big) + \mathbb{S}p\vartheta\big(M(ẋ_n, ẋ_{n+1})\big)$$

As $n \to \infty$

Therefore, the right-hand side approaches 0 . Since $1 - \mathbb{S}p > 0$

we obtain

$$\vartheta\big(M(ẋ, Ṭẋ)\big) = 0 \implies M(ẋ, Ṭẋ) = 0 \implies Ṭẋ = ẋ$$

Thus ẋ is FP of Ṭ

Now to showing FP is (U)

Suppose $w \in K$ is another FP of Ṭ , with $ẋ \neq w$, then

$$\vartheta\big(M(ẋ, w)\big) = \vartheta\big(M(Ṭẋ, Ṭw)\big)$$

$$\leq p\{\vartheta\big(M(ẋ, Ṭẋ)\big) + \vartheta\big(M(w, Ṭw)\big)\}$$

$$= p\{\vartheta(0) + \vartheta(0)\} = 0$$

This implies

$$\vartheta\big(M(ẋ, w)\big) = 0 \implies M(ẋ, w) = 0 \implies ẋ = w$$

which contradicts our assumption that $ẋ \neq w$

Therefore, the FP is U

Example 4. 1 : Consider the (CEB MS) (K, M) where $K = [0,1]$ and the EBM, $M: K \times K \to [0,1]$ is defined by $M(ẋ, ý) = |ẋ - ý|$ with the coefficient function $s: K \times K \to [0, \infty)$ is given by

$$s(ẋ, ý) = 1 + \frac{|ẋ - ý|}{1 + |ẋ - ý| + \text{Min}\{ẋ, ý\}}$$

The identity and symmetry are clearly satisfied by definition of M

For any $ẋ, ý, z \in K$

$$M(ẋ, z) \leq s(ẋ, z)[M(ẋ, ý) + M(ý, z)]$$

Therefore $|\chi - z| \le |\chi - \acute{y}| + |\acute{y} - z|$

Since $s(\chi, z) \ge 1$, the EBM inequality holds

The supremum of the coefficient function is

$$\mathbb{S} = \sup\{s(\chi, \acute{y}) : \chi, \acute{y} \in K \} = 2$$

Define the mapping $\Upsilon: K \to K$ by $\Upsilon\chi = \frac{\chi}{2}$, $\chi \in K$

Asymptotic regularity for any $x \in K$, $and\ n \in N$

$$M(\Upsilon^n \chi, \Upsilon^{n+1}\chi) = \left|\frac{\chi}{2^n} - \frac{\chi}{2^{n+1}}\right| = \frac{|\chi|}{2^{n+1}} \to 0 \, , as\ n \to \infty$$

Thus, Υ is asymptotically regular

The contraction condition with $\vartheta(t) = \log(1 + t)$

For some p with $\mathbb{S}p < 1$, the following holds

$$\{1 + M(\Upsilon\chi, \Upsilon\acute{y})\}^2 < e\{1 + M(\chi, \Upsilon\chi)\}\{1 + M(\acute{y}, \Upsilon\acute{y})\} \, , \forall \chi, \acute{y} \in K$$

For any $x, y \ \in K$

$$M(\Upsilon\chi, \Upsilon\acute{y}) = \left|\frac{\chi}{2} - \frac{\acute{y}}{2}\right| = \frac{|\chi - \acute{y}|}{2}$$

$$M(\chi, \Upsilon\chi) = \left|\chi - \frac{\chi}{2}\right| = \frac{|\chi|}{2}$$

$$M(\acute{y}, \Upsilon\acute{y}) = \left|\acute{y} - \frac{\acute{y}}{2}\right| = \frac{|\acute{y}|}{2}$$

Left-hand said of the inequality

$$\{1 + M(\Upsilon\chi, \Upsilon\acute{y})\}^2 = \left(1 + \frac{|\chi - \acute{y}|}{2}\right)^2$$

Right-hand said of inequality

$$e\{1 + M(\chi, \Upsilon\chi)\}\{1 + M(\acute{y}, \Upsilon\acute{y})\} = e\left(1 + \frac{|\chi|}{2}\right)\left(1 + \frac{|\acute{y}|}{2}\right)$$

Since $\chi, \acute{y} \in [0,1]$, $we\ have$

$$|\chi - \acute{y}| = 1$$

$$\left(1 + \frac{1}{2}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4} = 2.25$$

When $\chi = \acute{y} = 0$

$$e(1 + 0)(1 + 0) = 2 \approx 2.718 > 2.25$$

For all other values of $x, \acute{y} \in [0,1]$, therefore the inequality holds for all $x, \acute{y} \in K$

By above theorem, T has a UFP $Tx = x \Longrightarrow x = 0$

Thus $x = 0$ is the UFP of T

For any $x_0 \in K$, the sequence of iterates is

$$\{T^n x_0\} = \left\{x_0, \frac{x_0}{2}, \frac{x_0}{4}, \frac{x_0}{8}, \dots\right\}$$

which convergent to 0 , the UFP

Consider the EBMS (K, M) , were

$$M(x, \acute{y}) = |x - \acute{y}|^2$$

with the coefficient function

$$s(x, \acute{y}) = 1 + \frac{|x - \acute{y}|}{1 + |x - \acute{y}|}$$

Therefore, $\mathbb{S} = \sup s(x, \acute{y}) = 3$

T is asymptotically regular since

$$M(T^n x, T^{n+1} x) = \left|\frac{x}{2^n} - \frac{x}{2^{n+1}}\right|^2 = \frac{x^2}{2^{2n+1}} \to 0$$

The contraction condition becomes

$$\{1 + M(Tx, T\acute{y})\}^2 < e\{1 + M(x, Tx)\}\{1 + M(\acute{y}, T\acute{y})\}$$

which is satisfied for appropriate $p < \frac{1}{2\mathbb{S}} = \frac{1}{6}$

And is satisfied by T and we get the same result as before

Theorem 4. 2: Let (K, M) be a CEBMS with $\mathbb{S} = \sup\{s(x, \acute{y}) : x, \acute{y} \in K\} < \infty$ and let $T : K \to K$ be an asymptotically regular map satisfying

$$\vartheta\big(M(Tx, T\acute{y})\big) \leq p\{\vartheta\big(M(x, \acute{y})\big) + \vartheta\big(M(x, Tx)\big) + \vartheta\big(M(\acute{y}, T\acute{y})\big)\} \qquad (4.5)$$

For some p with $\mathbb{S}p < 1$. Then T has a UFP

Proof:

Let $x \in K$ be an arbitrary point. Consider the iterative sequence $\{x_n\}$ is define by

$$x_n = T^n x \text{ for } n = 0,1,2, \dots$$

Since T is asymptotically regular, we have

$$\lim_{n\to\infty} M(x_n, x_{n+1}) = \lim_{n\to\infty} M(T^n x, T^{n+1} x) = 0$$

We need to showing that $\{x_n\}$ is Cauchy sequence. For $m > n$, using the contraction condition

$$\vartheta\big(M(\mathrm{x}_{n+1}, \mathrm{x}_{m+1})\big) = \vartheta\big(M(\mathrm{T}^{n+1}\mathrm{x}, \mathrm{T}^{m+1}\mathrm{x})\big)$$

$$= \vartheta\left(M\big(\mathrm{T}(\mathrm{T}^n\mathrm{x}), \mathrm{T}(\mathrm{T}^m\mathrm{x})\big)\right)$$

$$\le p\{\vartheta\big(M(\mathrm{T}^n\mathrm{x}, \mathrm{T}^m\mathrm{x})\big) + \vartheta\big(M(\mathrm{T}^n\mathrm{x}, \mathrm{T}^{n+1}\mathrm{x})\big) + \vartheta\big(M(\mathrm{T}^m\mathrm{x}, \mathrm{T}^{m+1}\mathrm{x})\big)\}$$

$$= p\{\vartheta\big(M(\mathrm{x}_n, \mathrm{x}_m)\big) + \vartheta\big(M(\mathrm{x}_n, \mathrm{x}_{n+1})\big) + \vartheta\big(M(\mathrm{x}_m, \mathrm{x}_{m+1})\big)\}$$

Let $\epsilon_n = \vartheta\big(M(\mathrm{x}_n, \mathrm{x}_{n+1})\big) + \vartheta\big(M(\mathrm{x}_m, \mathrm{x}_{m+1})\big)$ .By asymptotic regularity

$$\lim_{n\to\infty} \epsilon_n = 0$$

Thus, for sufficiently large m, n

$$\vartheta\big(M(\mathrm{x}_{n+1}, \mathrm{x}_{m+1})\big) \le p\vartheta\big(M(\mathrm{x}_n, \mathrm{x}_m)\big) + p\epsilon_n$$

A standard argument shows that there exists $k > 0$ such that

$$\vartheta\big(M(\mathrm{x}_{n+1}, \mathrm{x}_{m+1})\big) \le \frac{kp}{1-p}\epsilon_n \to 0, \qquad as\ n \to \infty$$

Now using EBM inequality

$$M(\mathrm{x}_n, \mathrm{x}_m) \le s(\mathrm{x}_n, \mathrm{x}_m)[M(\mathrm{x}_n, \mathrm{x}_{n+1}) + M(\mathrm{x}_{n+1}, \mathrm{x}_m)]$$

$$\le \mathbb{S}[M(\mathrm{x}_n, \mathrm{x}_{n+1}) + M(\mathrm{x}_{n+1}, \mathrm{x}_m)]$$

$$\le \mathbb{S}\big[M(\mathrm{x}_n, \mathrm{x}_{n+1}) + \mathbb{S}[M(\mathrm{x}_{n+1}, \mathrm{x}_{n+2}) + M(\mathrm{x}_{n+2}, \mathrm{x}_m)]\big]$$

$$\le \mathbb{S}M(\mathrm{x}_n, \mathrm{x}_{n+1}) + \mathbb{S}^2 M(\mathrm{x}_{n+1}, \mathrm{x}_{n+2}) + \mathbb{S}^2 M(\mathrm{x}_{n+1}, \mathrm{x}_m)$$

We obtain

$$M(\mathrm{x}_n, \mathrm{x}_m) \le \sum_{k=n}^{m-1} \mathbb{S}^{k-n+1} M(\mathrm{x}_k, \mathrm{x}_{k+1})$$

By asymptotic regularity and the convergence of $\vartheta\big(M(\mathrm{x}_{n+1}, \mathrm{x}_{m+1})\big)$ to 0

It follows that $\{\mathrm{x}_n\}$ is Cauchy sequence. Since (K, M) is complete there exists $z \in K$ such that

$$\lim_{n\to\infty} \mathrm{x}_n = \lim_{n\to\infty} \mathrm{T}^n\mathrm{x} = z$$

Now to showing z is FP

By using EBM property and the contraction condition

$$\vartheta\big(M(z, \mathrm{T}z)\big) \le \vartheta(s(z, \mathrm{T}z)[M(z, \mathrm{T}^{n+1}\mathrm{x}) + M(\mathrm{T}^{n+1}\mathrm{x}, \mathrm{T}z)])$$

$$\le \mathbb{S}[M(z, \mathrm{T}^{n+1}\mathrm{x}) + M(\mathrm{T}^{n+1}\mathrm{x}, \mathrm{T}z)]$$

$$\le \mathbb{S}\vartheta\big(M(z, \mathrm{x}_{n+1})\big) + \mathbb{S}\vartheta\big(M(\mathrm{T}(\mathrm{T}^n\mathrm{x}), \mathrm{T}z)\big)$$

$$\le \mathbb{S}\vartheta\big(M(z, \mathrm{x}_{n+1})\big) + \mathbb{S}p\{\vartheta\big(M(\mathrm{T}^n\mathrm{x}, z)\big) + \vartheta\big(M(\mathrm{T}^n\mathrm{x}, \mathrm{T}^{n+1}\mathrm{x})\big) + \vartheta\big(M(z, \mathrm{T}z)\big)\}$$

$$= \mathbb{S}\vartheta\big(M(z, \mathrm{x}_{n+1})\big) + \mathbb{S}p\{\vartheta\big(M(\mathrm{x}_n, z)\big) + \vartheta\big(M(\mathrm{x}_n, \mathrm{x}_{n+1})\big) + \vartheta\big(M(z, \mathrm{T}z)\big)\}$$

Thus,

$$(1 - \mathbb{S}p)\vartheta\big(M(z, \mathrm{T}z)\big) \leq \mathbb{S}\vartheta\big(M(z, x_{n+1})\big) + \mathbb{S}p\{\vartheta\big(M(x_n, z)\big) + \vartheta\big(M(x_n, x_{n+1})\big)\}$$

As $n \to \infty$ all the right-hand said approaches. Since $1 - \mathbb{S}p > 0$

$$\vartheta\big(M(z, \mathrm{T}z)\big) = 0 \implies M(z, \mathrm{T}z) = 0 \implies \mathrm{T}z = z$$

Thus, z is FP of T

Now to showing the FP is U

Suppose $w \in K$ is another FP of T with $z \neq w$

$$\vartheta\big(M(z, w)\big) = \vartheta\big(M(\mathrm{T}z, \mathrm{T}w)\big)$$

$$\leq p\{\vartheta\big(M(z, w)\big) + \vartheta\big(M(z, \mathrm{T}z)\big) + \vartheta\big(M(w, \mathrm{T}w)\big)\}$$

$$= p\{\vartheta\big(M(z, w)\big) + \vartheta(0) + \vartheta(0)\}$$

$$= p\vartheta\big(M(z, w)\big)$$

This implies

$$(1 - p)\vartheta\big(M(z, w)\big) \leq 0 \implies \vartheta\big(M(z, w)\big) = 0 \implies M(z, w) = 0 \implies z = w$$

Which are contradiction

Therefore, the FP is U

5. Applications

This section builds on the limited compactness of K, T-orbital compactness, and asymptotic regularity of T to discuss the stability of fixed points with minor deviations. In this part, we use the HUS framework to show that approximation FP behave similarly to accurate ones in the EBMS scenario.

Definition 5.1: Let (K, M) be a CEBMS and $\mathrm{T}: K \to K$ be a map. The FP problem

$$x = \mathrm{T}x^* \tag{5.1}$$

is say HUS iff $\forall x \in K$ fulfill the disparity.

$$K(x, \mathrm{T}x) \leq \epsilon \tag{5.2}$$

And this disparity

$$K(\mathrm{T}x, x) \leq \epsilon \tag{5.3}$$

Where $\epsilon > 0$, there is a solution $x^* \in K$ and a constant $W > 0$

autonomous from x & x^* s.t

$$K(x^*, x) \leq W\epsilon \tag{5.4}$$

And

$$\tag{5.5}$$

$$K(x, x^*) \leq W\epsilon$$

Theorem (5.1) : Let (K, M) be a $\intercal$-orbitally cEBMS, and let $\intercal: K \to K$ be a map with $\mathbb{S} = \sup\{s(x, \acute{y}): x, \acute{y} \in K\} < \infty$ satisfying:

$$M(\intercal x, \intercal \acute{y}) \leq p\{M(x, \intercal x) + M(\acute{y}, \intercal \acute{y})\} \quad \forall x, \acute{y} \in K$$

where $p < \frac{1}{2}$ and $Sp < 1$. Then $\intercal$ has a unique fixed point $z \in K$.

For every $x \in X$, the sequence $\{\intercal^n x\}$ converges to z.
If $\acute{y} \in K$ satisfies $M(\acute{y}, \intercal \acute{y}) \leq W\varepsilon$ for so Me $\epsilon > 0$, then

$$M(\acute{y}, z) \leq \frac{\epsilon}{1 - 2\mathbb{S}p}$$

Proof.

Let $\acute{y}_0 \in K$ and define $x_n = T^n x_0$. Let $\gamma_n = d(x_n, x_{n+1})$

$$\gamma_n = M(\intercal x_{n-1}, \intercal x_n) \leq p\{M(x_{n-1}, \intercal x_{n-1}) + M(x_n, \intercal x_n)\} = p(\gamma_{n-1} + \gamma_n)$$

Thus $(1 - p)\gamma_n \leq p\gamma_{n-1}$, implying:

$$\gamma_n \leq \frac{p}{1-p}\gamma_{n-1} = q\gamma_{n-1}$$

where $q = \frac{p}{1-p} < 1$ since $p < \frac{1}{2}$.

Since K is $\intercal$ -orbitally compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\} \to z$. For any $m > n$:

$$M(x_n, x_m) \leq s(x_n, x_m)[M(x_n, x_{n+1}) + M(x_{n+1}, x_m)] \leq [\gamma_n + M(x_{n+1}, x_m)]$$

Using induction and $q < 1$, $\{x_n\}$ is Cauchy and converges to z.

$$M(z, \intercal z) \leq s(z, \intercal z)[M(z, \intercal^{n+1} x_0) + M(\intercal^{n+1} x_0, \intercal z)]$$

$$\leq \mathbb{S}[M(z, x_{n+1}) + M(\intercal x_n, \intercal z)]$$

$$\leq \mathbb{S}[M(z, x_{n+1}) + p(M(x_n, x_{n+1}) + M(z, \intercal z))]$$

$$(1 - \mathbb{S}p)M(z, \intercal z) \leq \mathbb{S}M(z, x_{n+1}) + \mathbb{S}p\gamma_n \to 0$$

Thus $M(z, \intercal z) = 0$. If w is another FP

$$M(z, w) = M(\intercal z, \intercal w) \leq p(M(z, \intercal z) + M(w, \intercal w)) = 0$$

Let $\acute{y} \in X$ with $M(\acute{y}, \intercal \acute{y}) \leq \epsilon$.

$$M(\acute{y}, z) = M(\acute{y}, \intercal z) \leq s(\acute{y}, z)[M(\acute{y}, \intercal \acute{y}) + M(\intercal \acute{y}, \intercal z)] \leq \mathbb{S}[\epsilon + M(\intercal \acute{y}, \intercal z)]$$

$$M(\intercal \acute{y}, \intercal z) \leq p(M(\acute{y}, \intercal \acute{y}) + M(z, \intercal z)) \leq p\epsilon$$

$$M(\acute{y}, z) \leq \mathbb{S}[\epsilon + p\epsilon] = \mathbb{S}(1 + p)\epsilon$$

Since $\mathbb{S}p < 1$, $\mathbb{S}(1 + p) \leq \frac{1}{1 - 2\mathbb{S}p}$ when $p < \frac{1}{2\mathbb{S}}$, we obtain by definition (5.1)

$$M(\acute{y}, z) \leq \frac{1}{1 - 2\mathbb{S}p} \cdot \epsilon$$

Therefore

$$W = \frac{1}{1 - 2\mathbb{S}p}$$

Theorem (5.2): Let (K, M) be a CEBMS with $\mathbb{S} = \sup\{s(x, ý): x, ý \in K\} < \infty$ and let $T: K \to K$ be an asymptotically regular map holds:

$$M(Tx, Tý) \leq p\{M(x, Tx) + M(ý, Tý)\} \quad \forall x, ý \in K$$

for so Me p with $\mathbb{S}p < 1$. Then T has a unique fixed point $z \in K$.

For every $x \in K$, li $M_{n \to \infty} T^n x = z$. If $ý \in K$ satisfies $M(ý, Tý) \leq \epsilon$ for so Me $\epsilon > 0$, then

$$M(ý, z) \leq \frac{1}{1 - 2\mathbb{S}p} \cdot \epsilon$$

Proof.

Let $x \in K$ and define $x_n = T^n x$. Since T is asymptotically regular:

$$\lim_{n \to \infty} M(x_n, x_{n+1}) = 0$$

For $m > n$,

$$M(x_{n+1}, x_{m+1}) = M(Tx_n, Tx_m)$$
$$\leq p\{M(x_n, x_{n+1}) + M(x_m, x_{m+1})\}$$

As $n, m \to \infty$, the right-hand side $\to 0$, so $\{x_n\}$ is Cauchy. By complete, $\exists z \in X$ such that $x_n \to z$.

$$M(z, Tz) \leq s(z, Tz)[M(z, T^{n+1}x) + M(T^{n+1}x, Tz)]$$
$$\leq \mathbb{S}[M(z, x_{n+1}) + M(Tx_n, Tz)]$$
$$\leq \mathbb{S}[M(z, x_{n+1}) + p(M(x_n, x_{n+1}) + M(z, Tz))]$$

Thus

$$(1 - Sp)M(z, Tz) \leq \mathbb{S}M(z, x_{n+1}) + \mathbb{S}pM(x_n, x_{n+1}) \to 0$$

So $M(z, Tz) = 0 \Rightarrow Tz = z$.

If $w \neq z$ with $Tw = w$, then

$$M(z, w) = M(Tz, Tw) \leq p(M(z, Tz) + M(w, Tw)) = 0$$

Hence z is unique.

Let $ý \in K$ satisfy $M(ý, Tý) \leq \varepsilon$.

$$M(ý, z) = M(ý, Tz) \leq s(ý, z)[M(ý, Tý) + M(Tý, Tz)]$$
$$\leq \mathbb{S}[\epsilon + M(Tý, Tz)]$$
$$M(Tý, Tz) \leq p\{M(ý, Tý) + M(z, Tz)\} = p\epsilon$$
$$M(ý, z) \leq \mathbb{S}[\epsilon + p\epsilon] = \mathbb{S}(1 + p)\epsilon$$

Since $\mathbb{S}p < 1$, $\mathbb{S}(1 + p) \leq \frac{1}{1 - 2\mathbb{S}p}$ when $p < \frac{1}{2\mathbb{S}}$.

$$\mathbb{S}(1 + p) \leq \frac{1}{1 - 2\mathbb{S}p}$$

Therefore, by definition (5.1)

$$M(ý, z) \leq \frac{\epsilon}{1 - 2\mathbb{S}p}$$

$$W = \frac{1}{1 - 2\mathbb{S}p}$$

6. Conclusion

In conclusion, this study has advanced fixed-point theory by establishing novel results in the context of extended b-metric spaces. We introduced and utilized key concepts such as Ţ-orbital compactness and asymptotic regularity to prove the existence and uniqueness of fixed points under generalized contractive conditions. The practical significance of our theoretical framework was demonstrated by applying the main theorems to prove the Hyers-Ulam stability of certain functional equations. These findings not only extend the classical theory but also provide robust tools applicable to nonlinear problems in analysis and applied mathematics. Future research may explore further generalizations of the metric structure or applications to more complex integral and differential equations.

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