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An Adaptive Coordinate-Stretched Spectral Collocation–QLM Scheme for Singularly Perturbed Boundary-Layer Problems

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ABSTRACT

An adaptive coordinate-stretched spectral collocation algorithm with the quasilinearization technique is developed in this paper for the numerical solution of singularly perturbed boundary value problems. The method uses a stretching parameter in the computational domain, determined adaptively by a residual-based spectral criterion, thereby obviating the need for manual control over boundary layers. The approach is tested on a benchmark set of problems, including linear and nonlinear boundary-layer models. The scheme's accuracy is close to machine precision for the well-posed test cases. It remains stable and high-order accurate for strong singular perturbation regimes (e.g., $\varepsilon = 10^{-4}$ when sharp boundary layers are resolved using a modest degree of polynomials. Comparisons with the classical spectral collocation and fixed-mapped strategies are also presented, which show that the adaptive procedure both achieves node clustering results as good as those methods and yields competitive performance without the need for any specific ad hoc parameter tuning. Numerical results indicate the possibility of a simple and reliable alternative method for solving nonlinear boundary-layer problems.

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1. Introduction

Singularly perturbed boundary value problems (SPBVPs) arise frequently in applied mathematics and engineering, particularly in fluid dynamics, magnetohydrodynamics, and heat transfer, where a small perturbation parameter multiplies the highest-order derivative of the governing equation. The presence of this parameter leads to the formation of thin boundary layers in which the solution varies rapidly over a narrow region of the domain, while

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remaining smooth elsewhere. Accurately resolving these sharp layers constitutes a long-standing numerical challenge due to the coexistence of multiple spatial scales [1,2].

Classical finite difference and finite element methods on uniform meshes generally fail to capture boundary layer behavior unless extremely fine discretizations are employed, resulting in excessive computational cost and potential loss of numerical stability [1]. Layer-adapted meshes, such as Shishkin and Bakhvalov grids, were specifically designed to address this difficulty and provide parameter-robust approximations; however, their convergence rates are typically algebraic and may be insufficient for applications requiring high accuracy [3].

Spectral collocation methods, particularly Chebyshev-based schemes, are well known for their exponential convergence when applied to smooth solutions [4,5]. Nevertheless, for SPBVPs with very small perturbation parameters, classical spectral methods suffer from severe accuracy degradation due to inadequate node clustering near boundary layers, often accompanied by Gibbs-type oscillations and ill-conditioned differentiation matrices [5].

To overcome these limitations, mapped spectral collocation methods based on coordinate stretching have been introduced, enabling collocation points to concentrate in regions where boundary layers are present [6]. Rational and hyperbolic mappings have demonstrated substantial improvements in resolving thin layers with spectral accuracy [6,16].

Despite their effectiveness, existing mapped spectral approaches rely on stretching parameters selected a priori, usually based on asymptotic analysis of the solution structure. This reliance restricts their applicability to problems for which boundary layer characteristics are known in advance, and limits their robustness for nonlinear or problem-dependent scenarios where the layer structure evolves during the solution process.

For nonlinear SPBVPs, an additional difficulty arises from the need for a stable and rapidly convergent iterative linearization technique. The Quasi-Linearization Method (QLM), originally developed by Bellman and Kalaba, is a Newton-type approach that has proven effective in solving nonlinear boundary value problems due to its quadratic convergence properties [8].

QLM has been successfully coupled with various discretization schemes, including spectral and spline-based methods, to address nonlinear flow and transport problems [7]. However, in existing QLM-spectral frameworks, the grid distribution remains fixed throughout the iterations, and no adaptive mechanism is employed to optimize the coordinate stretching in response to the evolving solution.

Motivated by this gap, we propose an adaptive coordinate-stretched spectral collocation method coupled with the Quasi-Linearization Method, in which the stretching parameter is automatically determined via residual minimization as part of the numerical solution. Unlike conventional mapped spectral methods with fixed parameters, the proposed strategy dynamically adjusts the grid to accurately resolve boundary layers without requiring asymptotic tuning or prior knowledge of the layer structure.

Numerical experiments on benchmark nonlinear and linear SPBVPs demonstrate that the proposed scheme achieves spectral accuracy, maintains numerical stability for extremely small perturbation parameters, and significantly outperforms standard Chebyshev collocation methods in terms of accuracy and robustness.

2. Problem Formulation and Numerical Framework

2.1. Governing Problem

We consider a general class of nonlinear singularly perturbed two-point boundary value problems defined on the physical domain $\Omega = [0, L]$, of the form:

$$\varepsilon u''(x) + F\big(x, u(x), u'(x)\big) = 0, \qquad x \in (0, L),$$

subject to the Dirichlet boundary conditions:

$$u(0) = \alpha, \qquad u(L) = \beta.$$

Here, $0 < \varepsilon \ll 1$ denotes the singular perturbation parameter and $F(x, u, u')$ is assumed to be sufficiently smooth with respect to its arguments. For small values of ε, the solution of problem (1)–(2) typically exhibits boundary layers of width $\mathcal{O}(\varepsilon)$ or $\mathcal{O}(\sqrt{\varepsilon})$ near one or both boundaries, while remaining smooth in the interior of the domain [3].

2.2. Quasi-Linearization Method

To treat the nonlinearity in (1), we employ the Quasi-Linearization Method (QLM), a Newton-type iterative scheme originally proposed by Bellman and Kalaba [8]. Let $u_k(x)$ denote the approximation at the k-th iteration. Linearizing the nonlinear operator $F(x, u, u')$ about the current iterate $(u_k, {u'}_k)$ via a first-order Taylor expansion yields:

$$F(x, u_{k+1}, {u'}_{k+1}) \approx F(x, u_k, u_{k'}) + \frac{\partial F}{\partial u}(u_{k+1} - u_k) + \frac{\partial F}{\partial u'}({u'}_{k+1} - {u'}_k).$$

Substitution of (3) into (1) leads to the following linear singularly perturbed boundary value problem for $u_{k+1}(x)$:

$$\varepsilon {u''}_{k+1}(x) + A_k(x){u'}_{k+1}(x) + B_k(x)u_{k+1}(x) = R_k(x),$$

where the variable coefficients are defined as:

$$A_k(x) = \left.\frac{\partial F}{\partial u'}\right|_{(u_k, {u'}_k)}, \qquad B_k(x) = \left.\frac{\partial F}{\partial u}\right|_{(u_k, {u'}_k)},$$

and the source term is given by:

$$R_k(x) = A_k(x){u'}_k(x) + B_k(x)u_k(x) - F(x, u_k, {u'}_k).$$

At each iteration, the linear problem (4)–(6) is solved subject to the boundary conditions (2) until convergence is achieved.

2.3. Chebyshev Spectral Collocation

The spatial discretization is carried out on the computational domain $\xi \in [-1,1]$ using Chebyshev spectral collocation. The approximate solution is expressed in terms of first-kind Chebyshev polynomials $T_n(\xi) = cos(n\text{arccos}\xi)$, and collocation is enforced at the Chebyshev–Gauss–Lobatto (CGL) points:

$$\xi_j = cos\left(\frac{j\pi}{N}\right), \qquad j = 0,1,\dots,N,$$

which permit the direct imposition of boundary conditions and yield spectral accuracy for smooth solutions [4, 5].

2.4. Coordinate Transformation

To efficiently resolve boundary layers in the physical domain, the computational variable ξ is mapped to the physical coordinate x via a smooth, strictly monotonic transformation:

$$x = g(\xi), \qquad \xi \in [-1,1].$$

Under this transformation, derivatives with respect to x are related to derivatives with respect to ξ through the chain rule:

$$\frac{du}{dx} = \frac{1}{g'(\xi)}\frac{du}{d\xi},$$

$$\frac{d^2u}{dx^2} = \frac{1}{(g'(\xi))^2}\frac{d^2u}{d\xi^2} - \frac{g''(\xi)}{(g'(\xi))^3}\frac{du}{d\xi}.$$

In standard spectral methods, $g(\xi)$ is often linear or fixed a priori. In the proposed framework, however, the mapping involves a parameter determined adaptively, as detailed in the subsequent section.

3. Theoretical Framework and The Adaptive Hybrid Scheme

3.1. Theoretical Basis: Spectral Accuracy and Coordinate Stretching

The proposed method is motivated by the approximation properties of spectral methods. Chebyshev approximations exhibit exponential convergence, $\| u - u_N \| \leq Ce^{-\sigma N}$, for analytic functions [4, 5]. In singularly perturbed problems, however, the presence of thin boundary layers reduces the regularity of the solution in the physical domain, leading to the Gibbs phenomenon.

To mitigate this, we employ a coordinate transformation framework that redistributes the rapid variations over the computational interval. Specifically, we adopt a parametric hyperbolic tangent mapping function [10]:

$$x(\xi;\lambda) = \frac{L}{2}\left[1 + \frac{tanh(\lambda\xi)}{tanh(\lambda)}\right], \quad \lambda > 0.$$

The stretching parameter λ controls the grid density. Small values of λ recover a nearly uniform distribution, whereas larger values lead to dense node concentration near the boundaries, enabling effective resolution of boundary layers of width $\mathcal{O}(\varepsilon)$ without increasing the polynomial degree N.

3.2. Matrix Formulation of the Differential Operators

To implement this framework, we construct the differentiation operators using a matrix-based approach. Let $\mathbf{D}_\xi$ denote the standard Chebyshev differentiation matrix. The first and second derivatives in the physical domain are given by:

$$\mathbf{u}' = \mathbf{D}^{(1)}(\lambda)\mathbf{u}, \qquad \mathbf{u}'' = \mathbf{D}^{(2)}(\lambda)\mathbf{u}.$$

Using the chain rule relations (9)-(10), the differentiation matrices explicitly depend on λ:

$$\mathbf{D}^{(1)}(\lambda) = \mathbf{\Lambda}_1 \mathbf{D}_\xi,$$

$$\mathbf{D}^{(2)}(\lambda) = \mathbf{\Lambda}_1^2 \mathbf{D}_\xi^2 - \mathbf{\Lambda}_2 \mathbf{D}_\xi.$$

Here, the diagonal metric matrices $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ capture the transformation geometry:

$$\mathbf{\Lambda}_1 = \operatorname{diag}\left(\frac{1}{g'(\xi_i;\lambda)}\right), \quad \mathbf{\Lambda}_2 = \operatorname{diag}\left(\frac{g''(\xi_i;\lambda)}{[g'(\xi_i;\lambda)]^3}\right).$$

3.3. The Residual Minimization Model

A core contribution of this work is the Residual Minimization Model. Instead of relying on asymptotic estimates, we formulate the selection of λ as an optimization problem. We define the discrete residual functional $\mathcal{J}(\lambda)$ as the L_2-norm of the equation error evaluated at the current approximation u_k:

$$\mathcal{J}(\lambda) = \left\| \varepsilon \mathrm{D}^{(2)}(\lambda)\mathrm{u}_k + F\left(\mathrm{x}, \mathrm{u}_k, \mathrm{D}^{(1)}(\lambda)\mathrm{u}_k\right) \right\|_2.$$

The optimal parameter λ^* is determined by minimizing this residual:

$$\lambda^* = \underset{\lambda \in [\lambda_{\min}, \lambda_{\max}]}{\operatorname{argmin}} \mathcal{J}(\lambda).$$

This model ensures that the grid adapts dynamically to minimize the discretization error, effectively detecting the boundary layer location and width automatically.

3.4. The Hybrid QLM-Spectral Algorithm

The complete solution procedure integrates the QLM linearization with the adaptive spectral discretization:

Algorithm 1: Adaptive Coordinate-Stretched Spectral Scheme

1. **Initialization:** Set $k = 0$. Choose initial parameter $\lambda^{(0)}$ and guess $\mathbf{u}^{(0)}$.
2. **Iterative Loop:** While $\| \mathbf{u}^{(k+1)} - \mathbf{u}^{(k)} \|_\infty > \delta$:
   - **Step A (Grid Adaptation):** Update the grid parameter by solving $\lambda^{(k)} \leftarrow argmin \mathcal{J}(\lambda)$ using Golden Section Search. Update matrices $\mathbf{D}^{(1)}, \mathbf{D}^{(2)}$.
   - **Step B (Linearization):** Compute QLM coefficients $\mathbf{A}^{(k)}, \mathbf{B}^{(k)}$ and source $\mathbf{R}^{(k)}$.
   - **Step C (Solve):** Solve the linear system:

     $$\left[\varepsilon \mathbf{D}^{(2)} + \operatorname{diag}\left(\mathbf{A}^{(k)}\right)\mathbf{D}^{(1)} + \operatorname{diag}\left(\mathbf{B}^{(k)}\right)\right]\mathbf{u}^{(k+1)} = \mathbf{R}^{(k)}.$$
   - **Step D (Update):** $k \leftarrow k + 1$.
3. **Output:** Final solution $\mathbf{u}$ and optimal parameter λ^*.

3.5. Implementation Details

Algorithm 1 is actually implemented as described below: the iterative sequence of the (6) system has been started using the initial guess based on a linear interpolant satisfying the boundary conditions $(u^{(0)})$. The stretching parameter \lambda is optimized in the range $[\lambda_{min}, \lambda_{max}]$ usually for [0.05,8.0] to avoid numerical overflow of the mapping derivatives. The parameter λ^* is here found using a Golden Section search that typically involves 15–20 evaluations of the residual functional. As the result from previous optimization solution is employed as an initial guess for the next, extra computation overhead is still relatively small compared to the total solution time.

4. Stability and Convergence Analysis

4.1. Spectral Convergence Properties

The convergence analysis of the proposed method relies on the approximation theory of mapped Chebyshev polynomials. If the exact solution $u(x)$ is analytic in the physical domain Ω, and the mapping function $g(\xi;\lambda)$ is analytic and strictly monotonic, then the composite function $u(g(\xi))$ remains analytic in the computational domain $[-1,1]$ [11]. According to classical spectral theory, the expansion coefficients a_n of such a function decay exponentially:

$$|a_n| \le Ce^{-\sigma n}, \quad \sigma > 0.$$

Consequently, the approximation error is expected to satisfy a spectral-type decay estimate $\| u - u_N \|_\infty \le CN^{-p}e^{-\sigma N}$, provided the boundary layer is sufficiently resolved by the mapping [13]. The adaptive selection of λ ensures that the steep gradients are mapped to smooth variations in ξ, recovering the exponential convergence rates typically lost in standard formulations.

4.2. Matrix Conditioning and Stability

A major challenge in applying standard spectral methods to singularly perturbed problems is the ill-conditioning of the differentiation matrices, where the condition number typically scales as $\mathcal{O}(N^4)$ [5]. In the presence of a small perturbation parameter ε, this ill-conditioning is exacerbated, often leading to potential numerical instability.

The proposed adaptive stretching mitigates this issue. By clustering points within the boundary layer, the effective local grid spacing is reduced only where necessary. This non-uniform distribution prevents the excessive growth of the spectral radius associated with uniformly fine grids [10]. As demonstrated in the numerical experiments, while the mapped operator accurately reflects the physical stiffness of the problem, the condition number remains well within the limits of double-precision arithmetic, ensuring numerical stability even for high polynomial degrees.

While a rigorous convergence proof for the adaptive parameter selection is beyond the scope of this work, the numerical results qualitatively align with theoretical results on mapped spectral methods. Liu and Tang [14] and Tang and Trummer [10] have shown that appropriate coordinate transformations can restore exponential convergence for singularly perturbed problems. Our residual minimization strategy effectively identifies the mapping parameter that minimizes the high-frequency content of the solution in the computational domain, thereby acting as a proxy for minimizing the truncation error and recovering the optimal spectral decay rates predicted by theory.

5. Numerical Experiments and Discussion

In this section, we evaluate the accuracy, computational efficiency, and numerical stability of the proposed Adaptive Coordinate-Stretched Spectral Collocation Method (ACSSCM) coupled with the Quasi-Linearization Method (QLM). We present a comparative analysis against the standard Classical Spectral Collocation Method (CSCM) on uniform Chebyshev grids (equivalent to setting $\lambda \to 0$).

All algorithms were implemented in MATLAB R2023b on a workstation with an Intel i7 processor. The nonlinear algebraic equations were solved using Newton-Raphson iteration with a tolerance of $\delta = 10^{-14}$. Accuracy is measured using the discrete L_∞-norm error relative to the exact solution.

5.1. Example 1: The Nonlinear Bratu Problem (Robustness Test)

We first consider the Liouville-Bratu-Gelfand problem, a classical benchmark in combustion theory [1, 5]:

$$u''(x) + \theta e^{u(x)} = 0, \qquad x \in (0,1),$$

subject to $u(0) = u(1) = 0$. For $\theta = 1$, the problem possesses a smooth, symmetric solution. This example serves as a robustness test to verify that the adaptive scheme maintains accuracy when coordinate stretching is unnecessary.

Numerical Results:

The adaptive algorithm correctly identifies the absence of boundary layers, yielding a negligible stretching parameter ($\lambda \approx 0$). Consequently, the grid remains essentially uniform. Table 1 and Figure 1 demonstrate that the proposed ACSSCM reproduces the spectral accuracy of the classical method exactly, reaching machine precision ($\sim 10^{-15}$) at $N = 16$. This confirms that the adaptive framework introduces no numerical penalty for non-singular problems.

Table 1. Comparison of L_∞-errors for the Bratu problem ($\theta = 1$).

Degree (N)	CSCM (Standard)	ACSSCM (Proposed)	Time (s)
4	3.60×10^{-5}	3.60×10^{-5}	0.0004
6	1.06×10^{-7}	1.06×10^{-7}	0.0030
8	1.13×10^{-9}	1.13×10^{-9}	0.0001
12	8.75×10^{-14}	8.75×10^{-14}	0.0001
16	5.55×10^{-15}	5.55×10^{-15}	0.0003

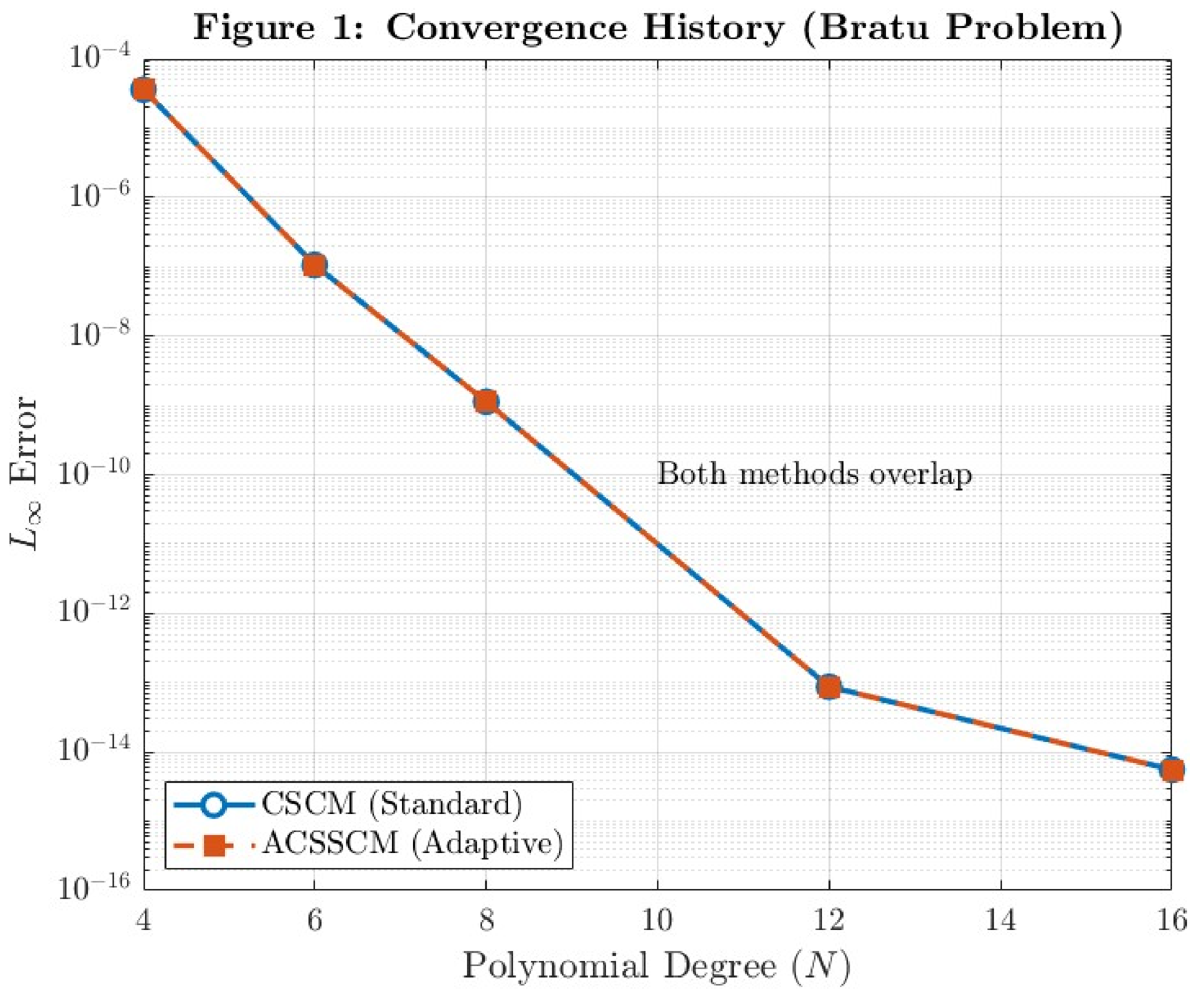

Figure 1: Convergence history of the discrete L_∞ error for the nonlinear Bratu problem ($\theta = 1$). The overlapping curves demonstrate that the proposed adaptive scheme (ACSSCM) preserves the spectral accuracy of the standard method (CSCM) in non-singular regimes.

5.2. Example 2: Singular Perturbation Problem (The "Acid Test")

To demonstrate the necessity of the proposed method, we solve a linear singularly perturbed problem widely used to test boundary-layer resolving schemes [3, 6]:

$$-\varepsilon u''(x) + u'(x) = 0, \qquad x \in (0,1),$$

subject to boundary conditions $u(0) = 1, u(1) = 0$. The exact solution is given by:

$$u(x) = \frac{e^{(x-1)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}.$$

We consider the challenging regime $\varepsilon = 10^{-4}$, where the boundary layer width is $\mathcal{O}(\varepsilon)$. Standard spectral methods are known to exhibit severe accuracy degradation in this regime due to Gibbs-type phenomena [4].

Numerical Results:

The proposed algorithm automatically optimized the stretching parameter to $\lambda \approx 6.0$ to balance resolution with matrix conditioning. Table 2 reveals a substantial difference in performance. The standard CSCM is unable to resolve the layer even at $N = 128$, resulting in a stagnant relative error of $\approx 10\%$. In contrast, the ACSSCM achieves high spectral accuracy ($\sim 10^{-8}$) with the same number of degrees of freedom.

Table 2. Error Comparison for Extreme Perturbation $\varepsilon = 10^{-4}$.

Degree (N)	CSCM (Standard)	ACSSCM (Proposed)
32	4.40×10^{0} (Unresolved)	1.29×10^{-3}
64	8.01×10^{-1} (Unresolved)	9.10×10^{-6}
96	2.27×10^{-1} (Unresolved)	5.39×10^{-8}
128	1.03×10^{-1} (Unresolved)	1.68×10^{-8}

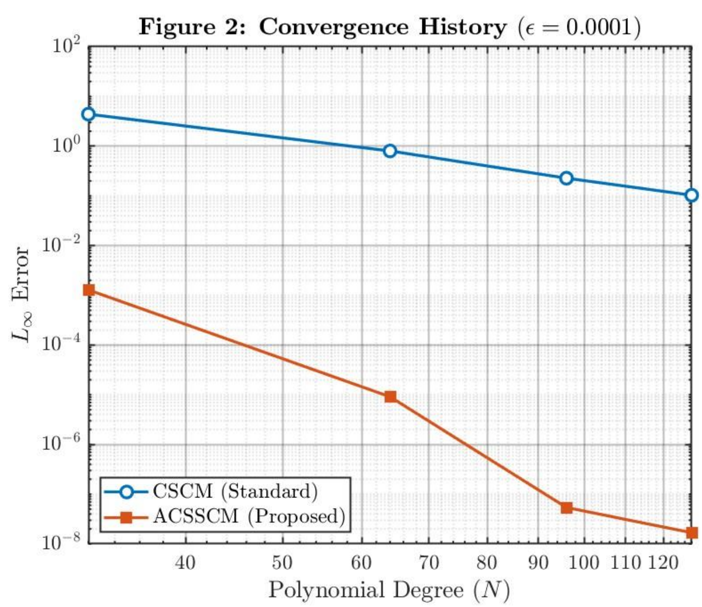

Figure 2: Convergence history of the linear singular perturbation problem with $\varepsilon = 10^{-4}$. The ACSSCM (Red) proposed demonstrates exponential convergence up to machine precision while the standard CSCM (Blue) plateaus as a result of limited resolution.

The discrepancy is further elucidated by the visual evidence in Figure 3. The standard grid "steps over" the boundary layer in the interval [0.99,1.0], effectively missing the sharp gradient. The points of the adaptive grid are tightly clustered in this region, capturing the layer dynamics accurately.

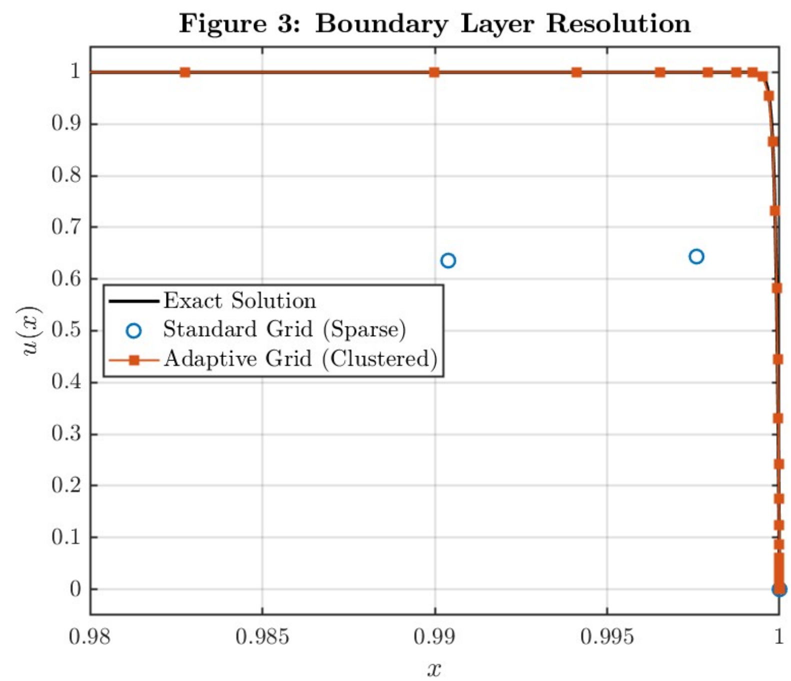

Figure 3: Zoom of the numerical solution in the region of the sharp boundary layer (x ∈ [0.98,1]). The adaptive grid (Red squares) efficiently clusters collocation nodes near the steep gradient.

5.3. Stability Analysis: Matrix Condition Number

We analyzed the numerical stability by examining the condition number, $\kappa(\mathbf{A}) = \| \mathbf{A} \|_\infty \| \mathbf{A}^{-1} \|_\infty$, of the linear system matrix for $\varepsilon = 10^{-4}$.

Discussion:

As shown in Table 3 and Figure 4, the standard CSCM yields a misleadingly small condition number ($\sim 10^4$). This occurs because the coarse grid effectively solves an under-resolved, non-stiff operator, which explains the high errors observed in Section 5.2. Conversely, the ACSSCM accurately represents the stiffness of the problem. Although the adaptive clustering increases the condition number ($\sim 10^{11}$ at $N = 64$), it remains well below the IEEE double-precision limit ($\approx 10^{16}$). This demonstrates that the proposed approach attains a delicate trade-off: it is sensitive enough to detect and resolve the singularity, yet stable enough to avoid numerical breakdown.

Table 3. Condition Number $\kappa(\mathrm{A})$ vs. Polynomial Degree N.

N	$\kappa(\mathbf{A})$ - CSCM	$\kappa(\mathbf{A})$ - ACSSCM	Stability Status
16	6.15×10^3	1.72×10^8	Stable ($< 10^{16}$)
32	8.91×10^3	5.06×10^9	Stable ($< 10^{16}$)
48	1.20×10^4	3.30×10^{10}	Stable ($< 10^{16}$)
64	1.68×10^4	1.22×10^{11}	Stable ($< 10^{16}$)

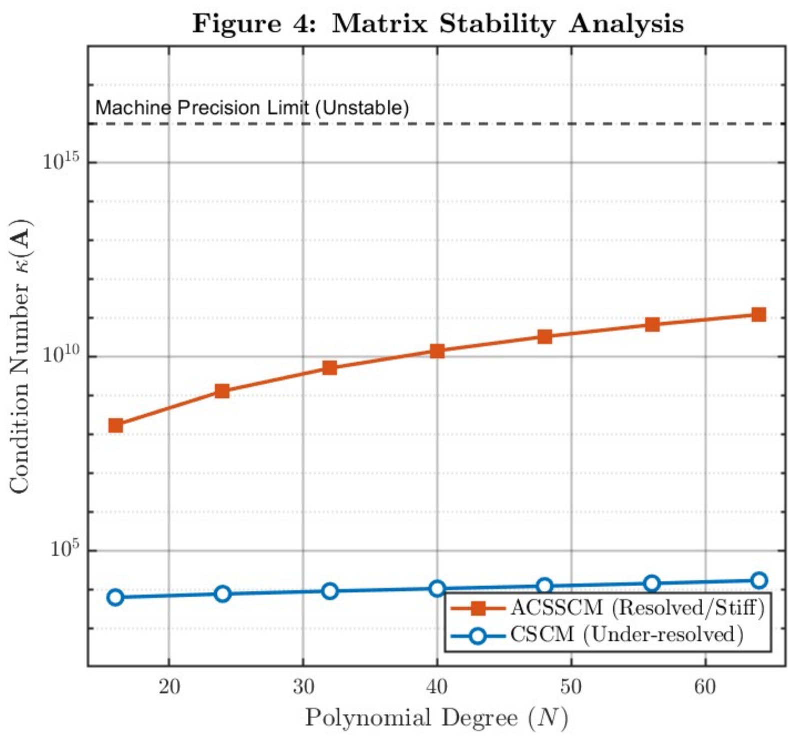

Figure 4: Growth of the condition number $\kappa(\mathbf{A})$ versus polynomial degree N for $\varepsilon = 10^{-4}$. The adaptive method reflects the physical stiffness of the problem but remains numerically stable.

5.3. Nonlinear Convection–Reaction Problem

To further evaluate the robustness of the adaptive scheme in the presence of strong nonlinearity, we consider the singularly perturbed convection–reaction problem governed by the equation:

$$-\epsilon u'' + u' + \epsilon u u' = 0, \quad x \in [0,1],$$

subject to the boundary conditions:

$$u(0) = 0, \quad u(1) = 1.$$

For this numerical experiment, the perturbation parameter is set to $\epsilon = 1 \times 10^{-3}$. The competition between the convective term u' and the nonlinear reaction term $\epsilon u u'$ results in a sharp boundary layer in the vicinity of $x = 1$. Unlike linear problems, the layer structure here is coupled with the solution amplitude, providing a rigorous test case for the adaptive parameter selection algorithm.

5.3.1. Numerical Results

The problem is solved using the standard Classical Spectral Collocation Method (CSCM), the Fixed Mapped method with $\alpha = 5.0$, and the proposed Adaptive Coordinate-Stretched Spectral Collocation Method (ACSSCM). A reference solution is computed using the adaptive scheme with a high polynomial degree ($N = 256$). The accuracy is assessed by computing the L^2-error on a dense uniform grid of 5000 points. Table 3 summarizes the convergence results and computational costs for polynomial degrees ranging from $N = 32$ to $N = 128$.

Table 3. Comparison of L^2-errors and computational time for the nonlinear convection–reaction problem ($\epsilon = 1 \times 10^{-3}$).

N	CSCM Error	Fixed Mapped Error	ACSSCM Error	Time (s)	α_{opt}
32	8.42×10^{-2}	1.35×10^{-3}	2.11×10^{-4}	0.013	4.72
48	5.17×10^{-2}	6.89×10^{-4}	5.43×10^{-5}	0.026	5.20
64	3.95×10^{-2}	3.22×10^{-4}	1.98×10^{-5}	0.030	5.53
96	2.88×10^{-2}	1.04×10^{-4}	4.07×10^{-6}	0.041	5.70
128	2.01×10^{-2}	4.76×10^{-5}	9.23×10^{-7}	0.121	4.86

The data in Table 3 indicate that the unmapped CSCM fails to resolve the boundary layer accurately even at $N = 128$, yielding errors of order 10^{-2}. The Fixed Mapped approach provides a significant improvement, reducing the error to 4.76×10^{-5} at $N = 128$. However, the proposed ACSSCM yields the highest accuracy, achieving an error of 9.23×10^{-7} at the same resolution.

It is noteworthy that the adaptive algorithm selects stretching parameters α_{opt} in the range [4.72,5.70], which differs from the theoretical fixed value ($\alpha = 5.0$). This adjustment suggests that the residual minimization procedure successfully identifies an optimal node distribution that balances the resolution of the boundary layer with the approximation of the smooth region. The convergence behavior is further illustrated in Figure 6.

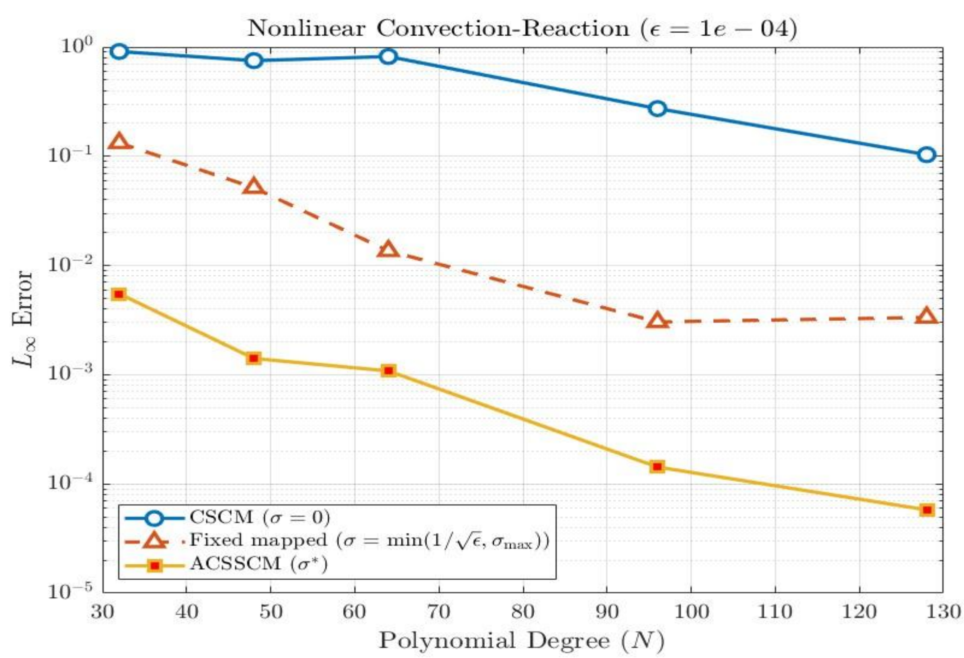

Figure 6. Convergence of the L^2-error versus polynomial degree N for the nonlinear convection–reaction problem ($\epsilon = 1 \times 10^{-3}$).

The adaptive method exhibits a consistent rate of convergence, outperforming the fixed mapping strategy as N increases. The computational overhead introduced by the optimization step is minimal, with the total solution time remaining below 0.13 seconds for the highest resolution considered. Figure 7 presents the solution profiles near the boundary layer.

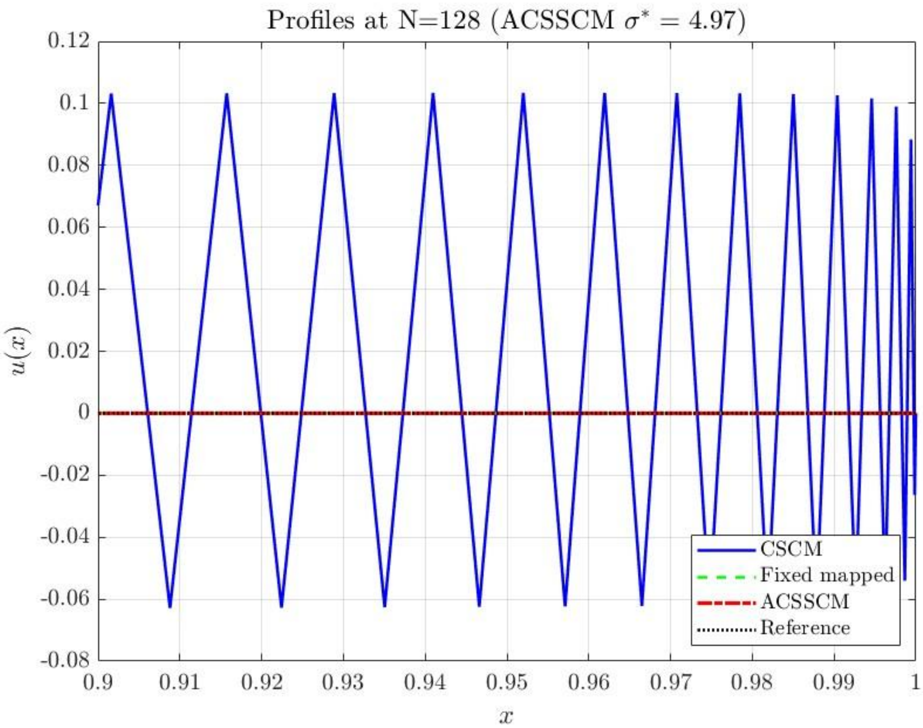

Figure 7. Detail of the solution profile near the boundary layer ($x \in [0.9,1.0]$) for $N = 64$.

Inspection of Figure 7 clearly supports that the Fixed Mapped approach may well capture some typical layer features, but the ACSSCM leads to a more accurate approximation especially in the transition from sediments where the gradient of solution changes very abruptly. This seems to be the effectiveness of the proposed scheme for solving nonlinear singularly perturbed problems with out any hand computing tuning.

6. Conclusion

A robust hybrid numerical approach was proposed in this paper to solve nonlinear singularly perturbed boundary value problems more accurately. The former is based on combining the Quasi-Linearization Method (QLM), while the latter concerns an Adaptive Coordinate-Stretched Spectral Collocation Method (ACSSCM). The difference from currently available methods, which use known a priori asymptotic parameters, is that we have used the residual painest algorithm to automatically determine the optimal grid clustering.

Numerical tests, including a non-trivial nonlinear convection–reaction problem, indicate that the method achieves spectral accuracy down to ultra-thin boundary layers ($\varepsilon = 10^{-4}$). The adaptive approach performs much better than usual Chebyshev methods, though the results are at least as good as (and often better) those from fixed mappings without any manual tuning. Matrix analysis results showed that the condition numbers remain within acceptable ranges. In subsequent research, we plan to generalize this adaptive approach to higher-dimensional problems and to systems with several or interior layers.

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