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Rough Convergence of Nets

Ruqaya Nameer Abd Ali [a]*, Sattar Hameed Hamzah [b]

[a] *Department of Mathematics, College of Education, University of Al-Qadisiyah , Iraq.Email: edu-math18.57@qu.edu.iq*

[b] *Department of Mathematics, College of Education, University of Al-Qadisiyah , Iraq.Email : sattar.hamzah@qu.edu.iq*

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ABSTRACT

This paper investigates fundamental and advanced concepts in rough topology by extending classical topological notions through the lens of rough set theory. We define and analyze rough-open and rough-closed sets, rough convergence of nets and filters, rough cluster and limit points, and separation axioms in rough spaces. The study introduces new forms of continuity such as rough-continuity, rough-irresoluteness, and rough-proper functions, and explores their interactions with compactness and convergence. Several illustrative examples and theorems are provided to demonstrate how rough approximations influence topological structure and behavior. The results generalize classical convergence theory and offer a unified framework for studying indiscernibility-based topology.

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Introduction

Rough set theory, introduced by Pawlak, models uncertainty via approximation spaces. Its integration with topology gives rise to rough topological spaces, where classical notions like openness, closure, and continuity are reinterpreted. However, the concept of convergence especially via nets has not been fully developed in this setting. This paper introduces a new framework for rough convergence of nets, along with related concepts such as rough-limit points, rough-cluster points, and rough-exceptional set. The definitions and results are original and serve to generalize classical convergence within the rough context. As the main contribution is theoretical, most definitions and theorems are developed independently, with minimal reliance on prior convergence literature. This work aims to establish a foundational reference for future studies in rough topology and its applications.

Basic Definitions and Notations

∗Corresponding author: Ruqaya Nameer Abd Ali

Email addresses: edu-math18.57@qu.edu.iq

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Definition1.1:[6] A pair $\mathcal{W} = (\Gamma, \delta)$ where Γ is a non-empty universe of discourse and δ is an equivalence relation (indiscernibility relation) on Γ called An approximation space .The lower and upper approximations of $A \subseteq \Gamma$ with respect to δ are defined as:

$\delta_*(A) = \{x \in \Gamma \mid \delta(x) \subseteq A\}$ (The lower approximation of A)

$\delta^*(A) = \{x \in \Gamma \mid \delta(x) \cap A \neq \phi\}$ (The upper approximation of A)

Here, $\delta(x)$ denotes the equivalence class of the element x under the relation δ.

Definition 1.2:[6] A set $\chi = (\delta^*(x), \delta^*(x))$ is said to be rough in $\mathcal{W}$, if $\delta^*(x) \neq \delta^*(x)$ otherwise, χ is an exact set (definable) in $\mathcal{W}$.

Definition 1.3: Let $\mathcal{W}=(\Gamma, \delta)$ be an approximation space and τ be a topological space, A subset $A \subseteq \Gamma$ is said to be rough-closed if for every open set $O \in \tau$ such that $A \subseteq O$, it holds that $\delta^*(A) \subseteq O$. The complement of rough-closed is called rough open and the triple (Γ, δ, τ) is called rough topological space and dented by RTSpace. And the collection of all rough open set dented by τ_δ .it is clear that A is rough-open $\Leftrightarrow \delta^*(\Gamma \setminus A) \subseteq O$ for every open set $O \in \tau$ such that $\Gamma \setminus A \subseteq O$.

Example 1.4: Let $\Gamma = \{x, y, z\}$, $\delta = \{(x, x), (y, y), (z, z), (y, x), (x, y)\}$,and $\tau = \{\emptyset, \{x\}, \Gamma\}$. Let $A=\{y\}$ .Then $\delta^*(A) = \{y, x\}$ and since $\{y, x\} \subseteq \Gamma$, A is rough-closed Let $A^c = \Gamma \setminus A = \{x,z\}$ and $\delta^*(A^c) = \Gamma$. So $A^c = \{x,z\}$ is rough-closed then A is rough-open in (Γ, δ, τ). When $A=\{x\}$ then $\delta^*(A) = \{y, x\} \not\subset O = \{x\}$ so A is not rough-closed.

Proposition 1.5:

(i) Every closed set is rough-closed.

(ii) Every open set is rough-open.

Remark 1.6: The converses of proposition 1.5 (i),(ii) is not always true .If $\Gamma = \{x, y, z\}$, $\mathscr{R} = \{(x, x), (y, y), (z, z), (y, x), (x, y)\}$, $\tau = \{\emptyset, \{x,y\}, \Gamma\}$ and $A=\{x\}$ then $\delta^*(A) = \{x, y\} \subseteq O$ for every $O \in \tau$ such that $A \subseteq O$ so A is rough-closed but not closed.Also, in example 1.4 when $A = \{y\}$ then $\delta^*(A)=\{x,y\}$ A is not open, $\Gamma \setminus A=\{x,z\}$ and $\delta^*(\Gamma \setminus A) = \Gamma$, since every equivalence class intersects {x,z}So $\Gamma \setminus A$ is rough-closed $\Rightarrow$ A is rough-open but not open.

Proposition 1.7:

(i) $\emptyset$ and $\boldsymbol{\Gamma}$ are **always rough-open** and **rough-closed**

(ii) If $\{A\alpha\}\alpha\in I$ be a family of rough-closed sets then $\cup\alpha\in I\ A\alpha$ is rough-closed.

If $\{A\alpha\}\alpha\in I$ be a family of rough-open sets then $\cap\alpha\in I\ A\alpha$ is rough-open.

Proof : (ii) Let $A=\cup\alpha\in I\ A\alpha$, and assume each $A\alpha$ is rough-closed.

Let $O\in\tau$ such that $A\subseteq O$, i.e., $\cup\alpha\in I\ A\alpha \subseteq O$.

Since $A\alpha \subseteq O$ for all $\alpha\in I$, and each $A\alpha$ is rough-closed, we have: $\boldsymbol{\delta}^*(A\alpha)\subseteq O$ for all $\alpha\in I$.

Taking the union over all α , $\cup\alpha\in I\ \boldsymbol{\delta}^*(A\alpha) \subseteq O$.Hence $\boldsymbol{\delta}^*(A)= \boldsymbol{\delta}^*(\cup\alpha\in I\ A\alpha) \subseteq \cup\alpha\in I\ \boldsymbol{\delta}^*(A\alpha) \subseteq O$. Thus, for every open set O with $A\subseteq O$, it holds that $\boldsymbol{\delta}^*(A)\subseteq O$. Hence, A is rough-closed.

(iii) Let $A=\cap\alpha\in I\ A\alpha$ .We want to show that for every open set $O\in\tau$ such that $\boldsymbol{\Gamma}\setminus A\subseteq O$, it holds that: $\boldsymbol{\delta}^*(\boldsymbol{\Gamma}\setminus A) \subseteq O$.

Note that: $\boldsymbol{\Gamma}\setminus A = \boldsymbol{\Gamma}\setminus \cap\alpha\in I\ A\alpha = \cup\alpha\in I\ (\boldsymbol{\Gamma}\setminus A\alpha) \subseteq O$.

Let $O\in\tau$ be any open set such that $\cup\alpha\in I\ (\boldsymbol{\Gamma}\setminus A\alpha)\subseteq O$.

Then for each $\alpha\in I$, we have: $\boldsymbol{\Gamma}\setminus A\alpha \subseteq O$.

Since each $A\alpha$ is rough-open, by Definition 1.4, this implies: $\boldsymbol{\delta}^*(\boldsymbol{\Gamma}\setminus A\alpha) \subseteq O$.

Hence: $\forall\alpha\in I$, $\boldsymbol{\delta}^*(\boldsymbol{\Gamma}\setminus A\alpha) \subseteq O$.

Taking the union: $\cup\alpha\in I\ \boldsymbol{\delta}^*(\boldsymbol{\Gamma}\setminus A\alpha) \subseteq O$.

By the monotonicity of $\boldsymbol{\delta}^*$, we have : $\boldsymbol{\delta}^*(\boldsymbol{\Gamma}\setminus A) = \boldsymbol{\delta}^*(\cup\alpha\in I\ (\boldsymbol{\Gamma}\setminus A\alpha)) \subseteq \cup\alpha\in I\ \boldsymbol{\delta}^*(\boldsymbol{\Gamma}\setminus A\alpha)$.

Therefore: $\boldsymbol{\delta}^*(\boldsymbol{\Gamma}\setminus A) \subseteq O$. Hence A is rough-open.

Remark 1.8:

1- Arbitrary intersection of rough-closed sets is not necessarily rough-closed. For example Let $\boldsymbol{\Gamma}=\{1,2,3\}$, $\mathscr{R}=\{(1,1),(2,2),(3,3),(1,2),(2,1)\}$ and $\tau=\{\emptyset,\{1,2\},\{2\},\boldsymbol{\Gamma}\}$ when $\boldsymbol{A}=\{2,3\}$ $\boldsymbol{\delta}^*(A) = U \subseteq O$ for every $O\in\tau$ such that $\boldsymbol{A} \subseteq O$ and $\boldsymbol{B} = \{1,2\}$,

$\boldsymbol{\delta}^*(\boldsymbol{B}) = \{1,2\}\subseteq O$ for every open set O such that $\boldsymbol{B} \subseteq O$ then $\boldsymbol{A}$ and $\boldsymbol{B}$ is rough-closed but $\boldsymbol{A}\cap\boldsymbol{B}=\{2\} \subseteq \boldsymbol{O} = \{\mathbf{2}\}$ but $\boldsymbol{\delta}^*(\boldsymbol{A}\cap\boldsymbol{B}) = \{1,2\} \nsubseteq \{\mathbf{2}\}$ then $\boldsymbol{A}\cap\boldsymbol{B}$ is not rough-closed.

2- Arbitrary union of rough-open sets is not necessarily rough-open. For example Let $\boldsymbol{\Gamma}=\{1,2,3\}$, $\boldsymbol{\delta}=\{(1,1),(2,2),(3,3),(1,2),(2,1)\}$ and $\tau=\{\emptyset,\{1,2\},\{2\},\boldsymbol{\Gamma}\}$ when $\boldsymbol{A}=\{1\}$ then $\boldsymbol{\Gamma}/\boldsymbol{A}=\{2,3\}$, $\boldsymbol{\delta}^*(\boldsymbol{\Gamma}/\boldsymbol{A}) = \boldsymbol{\Gamma} \subseteq O$ for every $O\in\tau$ such that $\boldsymbol{A} \subseteq O$ and $\boldsymbol{B} = \{3\}$ then $\boldsymbol{\Gamma}/\boldsymbol{B}=\{1,2\}$, $\boldsymbol{\delta}^*(\boldsymbol{\Gamma}/\boldsymbol{B}) = \{1,2\} \subseteq O$ for every $O\in\tau$ such that $\boldsymbol{B} \subseteq O$ then $\boldsymbol{A}$ and $\boldsymbol{B}$ is rough-open but $\boldsymbol{A}\cup\boldsymbol{B}=\{1,3\}$ $\boldsymbol{is\ not\ rough-open}$, $\boldsymbol{\Gamma}/(\boldsymbol{A}\cup\boldsymbol{B}) = \{\mathbf{2}\}$ and $\boldsymbol{\delta}^*(\boldsymbol{\Gamma}/(\boldsymbol{A}\cup\boldsymbol{B}))= \{1,2\} \nsubseteq \{\mathbf{2}\}$ is not rough-closed.

From (2) the collection τR of all rough-open subsets of $\boldsymbol{\Gamma}$ does not in general form a topology on $\boldsymbol{\Gamma}$.

Definition 1.9:

A rough topological space $(\boldsymbol{\Gamma}, \boldsymbol{\delta}, \tau)$ is called rough-multiplicative ($\mathscr{R}\tau$-space) if arbitrary intersection of rough-closed sets is rough-closed.

Definition 1.10:

Let ($\boldsymbol{\Gamma}$, $\boldsymbol{\delta}$,τ) be a RTSpace and A⊆ $\boldsymbol{\Gamma}$. Then the rough closure of A, denoted by $\mathscr{R}$cl(A), is defined as:

$\mathscr{R}$cl(A)= ⋂{C⊆ $\boldsymbol{\Gamma}$ | C is rough-closed and A⊆C }

and The rough interior of A, denoted by $\mathscr{R}$int (A), is defined as:

$\mathscr{R}$int(A) = ⋃{O⊆ $\boldsymbol{\Gamma}$ |O⊆A and O is rough-open}.

Remark 1.11 : It is clear note that $\mathscr{R}$cl(A) in general not rough closed and $\mathscr{R}$int(A) is not rough open.

Example 1.12: Let $\boldsymbol{\Gamma}$ = {1, 2, 3,4}, τ = {∅, {3}, {1, 2},{1,2,3}, $\boldsymbol{\Gamma}$ },

$\boldsymbol{\delta}$ ={(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)} let A={1} Then

$\mathscr{R}$cl(A) = {1} and cl(A)={1,2, 4}

And $\mathscr{R}$int(A) = ⋃{O⊆ $\boldsymbol{\Gamma}$ |O⊆A and O is rough-open}= {1}.

int(A) = ⋃{O∈τ:O⊆A}= ∅

Example 1.13: Let $\boldsymbol{\Gamma}$ = {1, 2, 3},τ = {∅, {1},{2}, {1, 2}, $\boldsymbol{\Gamma}$ },and $\boldsymbol{\delta}$ ={(1,1),(3,2),(2,3),(2,2),(3,3)} when A = {2}.Then $\mathscr{R}$cl(A) = cl(A) ={2, 3} and $\mathscr{R}$int(A) = int(A) = {2}.

Remark 1.14 : Let ($\boldsymbol{\Gamma}$, $\mathscr{R}$, τ) be a RTSpace and A⊆ $\boldsymbol{\Gamma}$. Then :

(i) A ⊆ $\mathscr{R}$cl(A).

(ii) $\mathscr{R}$int(A) ⊆ A.

Theorem 1.15: Let ($\boldsymbol{\Gamma}$, $\boldsymbol{\delta}$, τ) be a rough-multiplicative ($\mathscr{R}$ τ-space). Then :

(1) A subset A⊆ $\boldsymbol{\Gamma}$ is rough closed if and only if A= $\mathscr{R}$cl(A)

(2) A subset A⊆ $\boldsymbol{\Gamma}$ is rough open if and only if A= $\mathscr{R}$int(A)

Proof: this is clear.

Definition 1.16: Let ($\boldsymbol{\Gamma}$, $\boldsymbol{\delta}$, τ) be a RTSpace. A set V⊆ $\boldsymbol{\Gamma}$ is called a rough neighborhood of a points x ∈ $\boldsymbol{\Gamma}$ if there exists a rough-open set O∈ $\boldsymbol{\tau_\delta}$ such that: $\boldsymbol{\delta}^*$({x})⊆O⊆V the collection of all such sets is denoted by N$\mathscr{R}$(x),called the rough neighborhood system at x.

Definition 1.17 : A function f :($\boldsymbol{\Gamma}$, $\boldsymbol{\delta}$,τ) → ($\boldsymbol{\Gamma}'$, $\boldsymbol{\delta}'$,σ) between two rough topological spaces is called rough if $\boldsymbol{f}$($\boldsymbol{\delta}$ ∗(A)) ⊆ $\boldsymbol{\delta}'$∗($\boldsymbol{f}$(A)) for every A⊆ $\boldsymbol{\Gamma}$.

Definition 1.18: let ($\boldsymbol{\Gamma}$1, $\boldsymbol{\delta}$1, τ1) and ($\boldsymbol{\Gamma}$2, $\boldsymbol{\delta}$2, τ2) be two rough topological spaces. A function $\boldsymbol{f}$:$\boldsymbol{\Gamma}$1→ $\boldsymbol{\Gamma}$2 is called rough-continuous if $\boldsymbol{f}^{-1}$(O) ∈ $\boldsymbol{\tau}_{\delta_1}$, ∀ $\boldsymbol{O}$ ∈ $\boldsymbol{\tau}_{\delta_2}$.

Definition 1.19: A rough topological space ($\boldsymbol{\Gamma}$, $\boldsymbol{\delta}$,τ) is said to be rough-compact if every open cover by rough-open sets has a finite subcover that still covers the upper approximation of $\boldsymbol{\Gamma}$, i.e., $\boldsymbol{\delta}^*$($\boldsymbol{\Gamma}$). Formally For every collection {Oi}i∈I ⊆τ such that $\boldsymbol{\delta}^*$($\boldsymbol{\Gamma}$)⊆ ⋃i∈IOi ,there exists a finite subcollection {$\boldsymbol{O}_{i_1}$,…,$\boldsymbol{O}_{i_n}$} such that $\boldsymbol{\delta}^*$($\boldsymbol{\Gamma}$)⊆ $\bigcup_{k=1}^{n}$Oik

Definition 1.20: Let f :($\boldsymbol{\Gamma}$, $\boldsymbol{\delta}$,τ) → ($\boldsymbol{\Gamma}'$, $\boldsymbol{\delta}'$,σ)be a mapping between rough topological spaces .We say that $\boldsymbol{f}$ is rough-proper if for every rough-compact set K⊆V, the preimage $\boldsymbol{f}^{-1}$(K) is rough-compact in $\boldsymbol{\Gamma}$.

Definition 1.21 :[5] Let D be directed set. A net in a set χ is function X:D⟶χ. The point $\chi(d)$ denoted by χd.

Definition 1.22:[5] A net χd is eventually in a set A if ∃d0∈D such that xd∈A , ∀d ≥ d0.

Definition 1.23 :[5] A net χd is frequently in A if , ∀d∈D, ∃ d' ≥d such that x{d'}∈A.

Rough Convergence of Nets

A new type of convergence in rough topological spaces, namely rough convergence of nets, based on rough approximation operators is introduced in this section. This generalizes classical net convergence by incorporating the indiscernibility relation that underlies rough set theory. Furthermore, we investigate the concepts of rough-limit points and rough-cluster points, supported by examples and theorems.

Definition 2.1: Let (**Γ**, $\boldsymbol{\delta}$,τ) be a rough topological space. A net (xλ)λ∈Λ in **Γ** is said to roughly converge to a point x∈ **Γ**, denoted $x\lambda \xrightarrow{\mathcal{R}} x$, if for every open set O∈τ such that x∈O and $\boldsymbol{\delta}$ *({x}) ⊆ O , there exists λ0∈Λ such that: ∀ λ ≥ λ0 , xλ ∈ O.

Definition 2.2: A point x∈ **Γ** is called a roughlimit point of a net (xλ) if $x\lambda \xrightarrow{\dot{R}} x$

Definition 2.3: A point x∈ **Γ** is said to be a rough-cluster point of a net (xλ) if for every open set O∈τ with x∈O and $\boldsymbol{\delta}$*({x})⊆O, and ∀ λ∈Λ , ∃ λ′ ≥ λ such that: xλ′∈O.

Remark 2.4: If a net roughly converges to a point, then it is frequently within every rough neighborhood of that point. However, the converse is not always true, i.e., a rough-cluster point need not be a rough-limit point.

Example 2.5 : Let **Γ** ={a,b,c}, and let $\boldsymbol{\delta}$ be an equivalence relation such that $\boldsymbol{\delta}$={(x,x),(y,y),(z,z),(x,y),(y,x)}. Let τ ={∅,{x,y},{z}, **Γ** }. Consider the net xn = x for all n. Then the upper approximation $\boldsymbol{\delta}^*$({a})={a,b}, and since xn ∈{x,y} eventually but $\boldsymbol{\delta}^*$({z})={z} and since xn ∉{z} for all n, then is not eventually.

Theorem 2.6: Let (**Γ**, $\boldsymbol{\delta}$,τ) be a RTSpace, and x∈ **Γ** . Then: x ∈ $\mathscr{R}$Cl(A)
if and only if there exists a net (xλ)λ∈Λ⊆A such that $x\lambda \xrightarrow{R} x$

Proof : Suppose s a net (xλ) ⊆ A such that $x\lambda \xrightarrow{\mathcal{R}} x$. Let O∈τ be any open set such that $\boldsymbol{\delta}$*({x})⊆O. So there exists λ0 such that ∀ λ ≥ λ0 , xλ∈O. As xλ∈A, it follows that A∩O ≠ ∅ so x∈ $\mathscr{R}$Cl(A).

Conversely, let x ∈ $\mathscr{R}$Cl (A). The set of all open sets O ∈ τ with

$\boldsymbol{\delta}$ *({x}) ⊆ O is a directed set ordered by inclusion. For each O, since

x ∈ $\mathscr{R}$Cl(A), we have A∩w ≠ ∅ pick xw ∈ A∩O. The net (xw) indexed by the directed system of rough neighborhoods of x satisfies $xw \xrightarrow{\mathcal{R}} x$, with xw∈A.

Corollary 2.7: Let (**Γ**, $\boldsymbol{\delta}$,τ) be a RTSpace and x∈U. Then x∈ $\mathscr{R}$Cl (A) if and only if there exists a net (xλ)λ ∈ Λ ⊆ A such that x is a rough-cluster point of (xλ).

Theorem 2.8: Let (**Γ**, $\boldsymbol{\delta}$,τ) be a RTSpace . Then:

i. A point x∈ **Γ** is a roughlimit point of a set A⊆ **Γ** if and only if there exists a net

(xλ)⊆ A\{x}such that $x\lambda \xrightarrow{R} x$

ii. A set A⊆ **Γ** is rough-closed if and only if no net in A roughly converges to a point in **Γ** \A.

iii. A set A⊆ **Γ** is rough-open if and only if no net in **Γ** \A roughly converges to a point in A.

Proof: (i) By definition of roughlimit point and Theorem 2.6.

(ii) Suppose A is rough-closed and xλ∈A with $x\lambda \xrightarrow{\mathcal{R}} x$. By definition of rough-closure,

x∈ $\mathscr{R}$Cl (A)=A, so x ∉ **Γ** \A. Hence no such net converges to a point outside A.

Conversely, if no net in A roughly converges to a point outside A, then any x∈ $\mathscr{R}$Cl (A) must be in A, thus A is rough-closed.

(iii) Follows from (ii) since A is rough-open ⇔ **Γ** \A is rough-closed.

Theorem 2.9: Let f :(**Γ**, $\boldsymbol{\delta}$,τ) → (**Γ** ′, $\boldsymbol{\delta}$ ′,σ) be a function between two rough topological spaces. Then $\boldsymbol{f}$ is roughly continuous if and only if for every net (xλ)⊆ **Γ** such that $x\lambda \xrightarrow{\mathcal{R}} x$ we have $\boldsymbol{f}(x\lambda) \xrightarrow{\mathcal{R}'} \boldsymbol{f}(x)$ in (**Γ** ′, $\boldsymbol{\delta}$ ′,σ).

***Proof*:** (⇒) Assume f is roughly continuous. Let $x\lambda \xrightarrow{\mathcal{R}} x$. Let O′∈ σ be any open set such that $\boldsymbol{\delta}'^{*}$({f(x)})⊆O. Then by rough continuity, $\boldsymbol{f}^{-1}$(O′) is open in τ and contains x, with $\boldsymbol{\delta}^{*}$({x})⊆$\boldsymbol{f}^{-1}$(O′). Hence, eventually xλ∈$\boldsymbol{f}^{-1}$(O′). so $\boldsymbol{f}$(xλ)∈ O′. Thus, $\boldsymbol{f}(x\lambda) \xrightarrow{R'} \boldsymbol{f}(x)$

(⇐) Suppose $\boldsymbol{f}$ preserves rough convergence. Let O′∈σ and $\boldsymbol{\delta}'*$({$\boldsymbol{f}$(x)}) ⊆ O′ . Suppose for contradiction that $\boldsymbol{f}^{-1}$(O′) is not a rough neighborhood of x; then we can construct a net converging to x which does not eventually map into O′, contradicting the assumption.

1. The Rough Exceptional Set of Function

Definition 3.1 : Let f :(**Γ**, $\boldsymbol{\delta}$,τ) → (**Γ** ′, $\boldsymbol{\delta}$ ′,σ)be a function between two rough topological spaces. The rough exceptional set of f , denoted by $\boldsymbol{\varepsilon}$R (f) is defined as:

$\boldsymbol{\varepsilon}$R (f) = {x∈ **Γ** | ∃ a net (xλ)⊆ **Γ** such that $x\lambda \xrightarrow{\mathcal{R}} x$ and $\mathcal{R}'$: f (xλ) ↛ f (x) }

Theorem 3.3: Let $\boldsymbol{f}$: (**Γ**, $\boldsymbol{\delta}$, **τ**) → (**Γ** ′, $\boldsymbol{\delta}$ ′, **σ**) be a function between rough topological spaces. Then $\boldsymbol{f}$ is roughly continuous if and only if $\boldsymbol{\varepsilon}$R($\boldsymbol{f}$) = ∅.

Proof : (⇒) If $\boldsymbol{f}$ is roughly continuous, then by Theorem 2.9, for every net $x\lambda \xrightarrow{\mathcal{R}} x$, we have $\boldsymbol{f}(x\lambda) \xrightarrow{\mathcal{R}'} \boldsymbol{f}(x)$. Thus, x∉ ƐR($\boldsymbol{f}$) for any x , so $\boldsymbol{\varepsilon}$R($\boldsymbol{f}$) = ∅

(⇐) Suppose that $\boldsymbol{\varepsilon}$R($\boldsymbol{f}$)=∅ by definition 3.1 and since $\boldsymbol{\varepsilon}$R($\boldsymbol{f}$) = ∅ there does not exist any point $\boldsymbol{x}$ ∈ **Γ** and any net (xλ) in **Γ** such that $x\lambda \xrightarrow{R} \boldsymbol{x}$ and $\mathcal{R}'$: f (xλ) ↛ f (x) this means $\mathcal{R}'$: f (xλ) → f (x) therefore, $\boldsymbol{f}$ is roughly continuous.

Theorem 3.4 : Let $\boldsymbol{f}$: (**Γ**, $\boldsymbol{\delta}$, **τ**) → (**Γ** ′, $\boldsymbol{\delta}$ ′, **σ**) be a rough continuous function Then: If K⊆U is rough-compact, and $\boldsymbol{f}$ is rough, then $\boldsymbol{f}$(K) is rough-compact in (**Γ** ′, $\boldsymbol{\delta}$ ′, **σ**).

Proof: Let {$\grave{\boldsymbol{O}}_{\iota}$}i∈I be a rough-open cover of $\boldsymbol{f}$(K), i.e. $\grave{\boldsymbol{\delta}}^{*}$($\boldsymbol{f}$(K)) ⊆ ∪i∈I $\grave{\boldsymbol{O}}_{\iota}$.

Since $\boldsymbol{f}$ is rough: $\boldsymbol{f}$($\boldsymbol{\delta}^{*}$(K)) ⊆ $\grave{\boldsymbol{\delta}}^{*}$ ($\boldsymbol{f}$(K))⊆ ∪i∈I $\grave{\boldsymbol{O}}_{\iota}$

So $\boldsymbol{\delta}^{*}$(K) ⊆ $\boldsymbol{f}^{-1}$(∪i∈I $\grave{\boldsymbol{O}}_{\iota}$) = ∪i∈I $\boldsymbol{f}^{-1}$($\grave{\boldsymbol{O}}_{\iota}$).

Each $\boldsymbol{f}^{-1}$($\grave{\boldsymbol{O}}_{\iota}$) is rough-open in (**Γ**, $\boldsymbol{\delta}$, **τ**), since $\boldsymbol{f}$ is rough continuous.

Because K is rough-compact, there exists a finite subcover {$\grave{\boldsymbol{O}}_{i1}$,…,$\grave{\boldsymbol{O}}_{\iota n}$} such that:

$\delta*(K) \subseteq \bigcup_{k=1}^{n} f^{-1}(O'_{ik}) \Rightarrow f(\delta^*(K)) \subseteq \bigcup_{k=1}^{n} O'_{ik}$. So $f(K)$ is rough-compact.

Theorem 3.5 : Let $f:(\Gamma,\delta,\tau) \rightarrow (\Gamma',\delta',\sigma)$, g: $(\Gamma',\delta',\sigma) \rightarrow (\Gamma'',\delta'',\theta)$ be two functions.

If both f and g are rough-proper, then $g\circ f:(\Gamma,\delta,\tau) \rightarrow (\Gamma'',\delta'',\theta)$ is rough-proper.

Proof : Let K⊆W be rough-compact. Since g is rough-proper, g−1(K)⊆V is rough-compact.

Since f is also rough-proper, f −1(g−1(K))= $(g \circ f)$-1(K) is rough-compact.

Theorem 3.6 : Let (Γ,δ,τ) be a rough multiplication topological space. Then the following statement are equivalent:

(i) Γ is rough-compact.

(ii) Every net in Γ has a rough-cluster point.

Proof: (i ⇒ ii):Let (xλ) be a net in Γ. Construct the tails: A_{λ_0}={xλ| λ≥λ0}.

By rough-compactness, the collection $\{\mathscr{R}Cl(A)\}_{\lambda_0}$ has a nonempty intersection.

Any point in this intersection is a rough-cluster point.

(ii ⇒ i): Assume every net has a rough-cluster point. Suppose Γ is not rough-compact.

Then there exists a cover {Oi} such that $\delta^*(\Gamma)$ is not contained in any finite union.Construct a net that escapes every finite subcover. It will have no cluster point — contradiction.

Theorem 3.7 : Let$f:(\Gamma,\delta,\tau) \rightarrow (\Gamma',\delta',\sigma)$ be rough continuous and rough. If K⊆ Γ is rough-compact, then $f(K)$ is rough-compact in V.

Proof: Clear

Remark 3.8 : In rough topological spaces:

Every rough-proper map is rough.

Every rough map need not be rough-proper.

Every rough-continuous function preserves rough convergence, but not necessarily compactness.

This highlights a natural hierarchy: Rough-Proper ⇒ Rough ⇒ Rough-Continuous.

Theorem 3.9: Let$f:(\Gamma,\delta,\tau) \rightarrow (\Gamma',\delta',\sigma)$ be rough and rough continuous. Then f is rough-proper if and only if for every rough-compact set K⊆U, the image $f(K)$ is rough-closed and rough-compact in (Γ',δ',σ).

***Proof*:** (⇒) If f is rough-proper, then $f(K)$ is rough-compact and rough continuity ensures images of closed sets are closed.

(⇐) If $f(K)$ is always rough-compact and rough-closed for rough-compact K, then for any rough-compact L⊆V, $f^{-1}(L)$ must also be rough-compact by roughness of f.

Theorem 3.10: Every infinite net in a rough-compact space (Γ, δ, τ) has at least one rough-cluster point.

***Proof*:** Let (xλ) be a net in **Γ**. Since **Γ** is rough-compact, the tails {xλ|λ≥λ0} have closures whose intersection is nonempty (by Theorem 3.6).Any point in that intersection is a rough-cluster point.

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