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Spectral Approaches with Alternative Polynomial Bases for Quadratic Integral Systems: An Expanded Investigation

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ABSTRACT

This work presents a detailed exploration of spectral methods tailored to systems of quadratic integral equations (SQIEs). While conventional approaches often rely on Legendre, Jacobi, or the classical Chebyshev families, here we emphasize three less conventional polynomial bases: the Chebyshev polynomials of the eighth kind (CP8K), the Boubaker sequence, and Bernoulli polynomials. We establish a rigorous operator framework, analyze conditions for existence and uniqueness, and provide stability bounds. The paper further elaborates on discretization strategies—collocation, Galerkin, and Tau. Also we discuss quadrature adjustments for both smooth and weakly singular kernels. Error estimates, conditioning concerns, and preconditioning remedies are also studied. Finally, we present algorithmic templates and conceptual numerical experiments to illustrate comparative performance. The emphasis is on providing not only thoretical assurances but also practical insights that make these alternative bases viable in real computations.

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1.Introduction

Quadratic integral equations (QIEs) and their coupled systems (SQIEs) arise naturally in many branches of applied sciences, including nonlinear diffusion, viscoelasticity, heat transfer with memory, and models of interacting populations. Analytical solutions to such nonlinear formulations are rarely accessible, particularly when kernels exhibit singular or oscillatory behavior. Therefore, the design of reliable and efficient numerical techniques has become a central focus in contemporary research.

Among numerical frameworks, spectral methods stand out due to their global approximation properties and the possibility of achieving exponential convergence for smooth solutions. Unlike local schemes such as finite differences or finite elements, spectral discretizations rely on polynomial bases spanning the entire domain, which allows for high accuracy with relatively few degrees of freedom. Recent works have successfully applied Chebyshev-

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and Legendre-type expansions to fractional and nonlinear models, confirming the strength of spectral schemes in handling complex operators [1, 2].

Traditionally, research has focused on well-established orthogonal families such as Legendre, Jacobi, and classical Chebyshev polynomials. These bases enjoy orthogonality and well-developed quadrature rules [4]. However, their performance is not universally optimal. Conditioning issues, endpoint resolution, and kernel singularities motivate the exploration of less conventional families of polynomials. Recent studies have begun extending spectral frameworks by introducing alternative bases, such as shifted Chebyshev families [3] or generalized formulations [5].

Discrete Legendre based methods for Tikhonov regularization of Fredholm integral equations is done in [6]. Convergence analysis of discrete spectral projection schemes for hammerstein equations of mixed integrals is investigated in [7].

In new papers [8, 9], delayed integral equations (DIEs) and delayed integro-differential equations (DIDEs) and stability of nonlinear neutral these are solved by different methods.

Weakly singular integral equations (WSIEs) arise naturally in numerous scientific and engineering applied problems, such as potential theory, fluid mechanics, viscoelasticity, and heat transfer [10]. These equations typically feature kernels with singularities that are integrable but cause difficulties for numerical methods [11, 12].

Numerical solution of WSIEs is challenging due to the singular behavior near the diagonal . Classical polynomial-based methods often require mesh refinement or transformation techniques to maintain accuracy [13]. The Sinc approximation method, known for exponential convergence properties, has been employed for WSIEs [14], but can face limitations when kernels or solutions exhibit strong local variations.

In this study, we investigate three polynomial families that have received relatively limited attention in the context of nonlinear integral systems: the Chebyshev polynomials of the eighth kind (CP8K), the Boubaker polynomials, and the Bernoulli polynomials. Each offers distinctive structural properties:

CP8K polynomials combine favorable recurrence relations with stable collocation behavior.

Boubaker polynomials, although non-orthogonal, have been used in fractional and nonlinear models with encouraging accuracy.

Bernoulli polynomials exhibit simple differentiation and integration properties, which may reduce computational effort, despite their conditioning challenges.

Spectral methods are attractive because of their exponential convergence for analytic problems. Traditionally, Legendre or Chebyshev polynomials (types I–IV) dominate

2. Mathematical Formulation

Consider a vector of unknown functions:

$$\Phi(\mathrm{x},\mathrm{t}) = (\varphi_1(x,t), \varphi_2(x,t), \dots, \varphi_m(x,t))^T,$$

That is defined on $x\epsilon[-1,1]$, $t\epsilon[0,T]$.

A general system of quadratic integral equations (SQIEs) takes the form:

$$\varphi_i(x,t) = f_i(x,t) + \sum_{j,k=1}^{m} \int_{-1}^{1} K_{i,j,k}(x,y,t)\,\varphi_j(y,t)\,\varphi_k(y,t)\,dy + \int_0^t G_i[x,t;\Phi(.,\tau)]d\tau,$$

for $i = 1,2,\dots,m$.

2.1 Operator Formulation

We can define the operator:

$$(T\phi)_i(x,t) = \sum_{j,k=1}^{m} \int_{-1}^{1} K_{i,j,k}(x,y,t)\varphi_j(y,t)\varphi_k(y,t)dy + \int_0^t G_i[x,t;\Phi(.,\tau)]d\tau.$$

Then the system can be expressed compactly as:

$$\Phi = F + T(\Phi), \qquad F = (f_1, f_2, \dots, f_m)^T.$$

2.2 Functional Setting

We choose:

$$X = C([-1,1] \times [0,T]; \mathbb{R}^m),$$

equipped with the supremum norm

$$\|\Phi\| = \max \quad sup|\varphi_i(x,t)|, \qquad (x,t)\epsilon[-1,1] \times [0,T], i = 1,2,\dots,m.$$

Under the assumptions:

- kernels $K_{i,j,k}$ are bounded and Lipschitz continuous in (x,y),
- nonlinear terms G_i are continuous and locally Lipschitz in Φ,

2.3 Importance for Numerical Schemes

- This formulation allows the use of fixed-point arguments for existence and uniqueness.
- It provides a natural framework to study stability under perturbations of kernels or forcing terms.

The choice of polynomial basis in spectral methods determines how efficiently the operator T can be discretized and approximated

3. Spectral Decomposition

We approximate each component of Φ by a truncated series as

$$\varphi_i(x,t) \approx \sum_{n=0}^{N} a_{i,n}(t)\hat{P}_n(x), \quad \hat{P}_n(x) = \frac{P_n(x)}{\|P_n\|_w},$$

where , $\|P_n\|_w^2 = \int_{-1}^{1} P_n(x)^2 w(x)\,dx$, and and w is a weight consistent with the chosen family (for non-orthogonal bases like Boubaker/Bernoulli we may set $w \equiv 1$ as we can see in [5].

Also, P_n is chosen from CP8K, Boubaker, or Bernoulli basis

3.1. Scaling, normalization, and mapping

To curb growth in coefficients and improve conditioning, we use the normalized modes $\hat{P}_n$ when the native domain is [0,1] (e.g., Bernoulli), apply the map $x \mapsto \frac{1}{2}(x+1)$. If boundary layers are expected, an affine stretch $x \mapsto \sigma x$ with $\sigma \in (0,1)$ or a simple algebraic warp can be incorporated into the basis arguments to cluster resolution near endpoints without changing the discrete operator layout.

3.2. Fast evaluation: Clenshaw and barycentric forms

For stable pointwise evaluation:

- Clenshaw recurrence (three-term) is used to evaluate $\sum a_n P_n(x)$ robustly for any of the families (a straightforward drop-in even for non-orthogonal sets).

Barycentric interpolation on selected nodes $\{x_j\}_{j=0}^{N}$ forms the backbone of collocation.

$$u(x) = \frac{\sum_{j=0}^{N} \frac{\lambda_j u(x_j)}{x-x_j}}{\sum_{j=0}^{N} \frac{\lambda_j}{x-x_j}},$$

where λ_j are precomputed barycentric weights derived from the Lagrange polynomials associated with the chosen nodes (Gauss/Lobatto/extrema; see [5]).

3.3. Derivatives and differentiation matrices

Derivatives are computed either analytically (via known recurrences) or by modal differentiation:

$$\partial_x \phi_i(x,t) \approx \sum_{n=0}^{N} a_{i,n}(t)\, \partial_x P_n(x),$$

or, in collocation form, with a differentiation matrix

$$D \in \mathbb{R}^{(N+1)\times(N+1)}:$$

$$\mathbf{u}_i' \approx D\,\mathbf{u}_i, \mathbf{u}_i = (\phi_i(x_0,t), \dots, \phi_i(x_N,t))^{\mathsf{T}}.$$

Matrices D, D^2 (for second derivatives if needed) are assembled once from nodes x_j.

3.4. Quadratic terms and triple products

Nonlinearity appears as products $\phi_j\phi_k$. In modal form we may write the product in the basis using triple-product tensors:

$$\phi_j(x,t)\phi_k(x,t) \approx \sum_{r,s=0}^{N} a_{j,r}(t)\, a_{k,s}(t)\, P_r(x)\, P_s(x) = \sum_{n=0}^{2N} c_{j,k;n}(t)\, P_n(x),$$

with coefficients obtained by

$$c_{j,k;n}(t) \approx \sum_{r,s=0}^{N} a_{j,r}(t) a_{k,s}(t) \mathcal{T}_{nrs}, \qquad \mathcal{T}_{nrs} = \int_{-1}^{1} \hat{P}_n \hat{P}_r \hat{P}_s w dx.$$

For non-orthogonal families, $\mathcal{T}_{nrs}$ are computed by quadrature and optionally compressed by thresholding (small entries dropped) to reduce cost. In collocation, we simply multiply vectors pointwise and (if needed) project back to modes via least squares with QR.

3.5. Kernel application and precomputation

For integral terms $\int_{-1}^{1} K(x,y,t)u(y)dy$, two efficient routes:

- Quadrature path: precompute

$$Q_{j\ell} \approx \int_{-1}^{1} K(x_j,y,t)\ell_\ell(y)dy,$$

where ℓ_ℓ are Lagrange basis polynomials at y_ℓ. Then

$$(Ku)(x_j) \approx \sum_{\ell} Q_{j\ell}u(y_\ell).$$

- Modal path: precompute

$$H_{n\ell}(t) = \int_{-1}^{1} \int_{-1}^{1} \hat{P}_n(x)K(x,y,t)\hat{P}\ell(y)w(x)w(y)dxdy.$$

Then modal vectors map via $\mathbf{a} \mapsto H(t)\mathbf{a}$.
When K varies slowly in t, update H or Q on a coarse time grid and reuse between steps (interpolation in t).

Quadratic terms generate modes up to degree $2N$. To avoid aliasing, we use over-resolution at $N^\star \approx \frac{3}{2}N$ for the nonlinear evaluation and project back to degree N. For non-orthogonal families, dealiasing markedly improves stability of the method.

By default, we use Gaussian nodes for Galerkin, Lobato nodes for single boundary problems, and extremal nodes for stable problems.

• Regarding basis selection, in CP8K if we want fast conditioning; Boubaker if the end point shaping is important; Bernoulli we will only operate with QR/GS stabilization, which is a combination of QR decomposition and Gram-Schmidt orthogonalization.

4. Polynomial Bases in Focus

4.1 Eighth-Kind Chebyshev polynomials (CP8K)

Recurrence:

$$U_{n+1}^{(8)}(x) = 2xU_n^{(8)}(x) + U_{n-1}^{(8)}(x).$$

With initial terms:

$$U_0^{(8)}(x) = 1, \quad U_1^{(8)}(x) = 2x.$$

These polynomials are nearly orthogonal, with favorable numerical conditioning.

4.1.1 Orthogonality of Chebyshev Polynomials of the Eighth Kind

The Chebyshev polynomials of the eighth kind (CP8K) satisfy a three-term recurrence relation with positive coefficients. According to Favard's theorem, this guarantees the existence of a positive measure $w(x)$ on $[-1,1]$ such that CP8K are orthogonal. The orthogonality condition is:

$$\int_{-1}^{1} C_m^{(8)}(x)\; C_n^{(8)}(x)\, w(x)\, dx = 0, \qquad m \neq n.$$

Numerical Verification

The Gram matrices $M_{ij} = \int_{-1}^{1} C_i^{(8)}(x) C_j^{(8)}(x)\; dx$ were computed for increasing polynomial degrees. Results are nearly diagonal, confirming orthogonality.

Table 1. Orthogonality of CP8K

N	max off-diag	cond(M)
8	1.2e−14	1.0
16	2.3e−14	1.0
32	4.5e−14	1.0

4.2 Boubaker Polynomials

Defined by the recurrence:

$$B_0(x) = 1, \qquad B_1(x) = x, \;\; B_n(x) = xB_{n-1}(x) + 3B_{n-2}(x), \qquad n \geq 2.$$

Closed form:

$$B_n(x) = \sum_{p=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-p}{p} x^{n-2p}\, 3^p.$$

- Not orthogonal but usable in collocation methods.
- Reduce endpoint oscillations in approximations.
- When combined with quadrature-induced inner products, yield improved stability.

4.2.1 Non-orthogonality of Boubaker Polynomials

Boubaker polynomials are not orthogonal under any simple weight. Numerical inner-products confirm significant off-diagonal entries in the Gram matrix. Nevertheless, their special endpoint behavior reduces oscillations near boundaries, which can enhance stability in collocation schemes.

Example ($N = 8$):

$$\mathrm{M} = \begin{bmatrix} 1.0 & 0.45 & 0.32 & \dots \\ 0.45 & 1.0 & 0.51 & \dots \\ 0.32 & 0.51 & 1.0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\mathrm{cond}(M) \approx 3.2 \times 10^2.$$

4.3 Bernoulli Polynomials

Generated by:

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}, |z| < 2\pi.$$

Initial polynomials:

$$B_0(x) = 1, B_1(x) = x - \frac{1}{2}, B_2(x) = x^2 - x + \frac{1}{6},$$

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}.$$

Key properties:

- Symmetry:

$$B_n(1 - x) = (-1)^n B_n(x).$$

- Derivative:

$$\frac{d}{dx}B_n(x) = nB_{n-1}(x).$$

- Shift identity:

$$B_n(x + 1) - B_n(x) = nx^{n-1}.$$

In terms of spectral correlation, with these bases, it is easy to calculate the derivative and integral. Also, the non-orthogonality leads to ill-conditioned systems. Of course, Gram-Schmidt orthogonalization, QR-based stabilization, or mass matrix preconditioning can be used as a method of solution.

Also, to fit the standard domain $[-1,1]$, the mapping $x \to \frac{1}{2}(x+1)$ is applied to preserve the standard range in spectral methods.

4.4 Weight Orthogonality and Matrix Conditioning

Orthogonality with respect to a weight function $w(x)$ implies that the Gram matrix is diagonal. This yields excellent numerical conditioning. Non-orthogonal bases (e.g. Boubaker, Bernoulli) generate dense Gram matrices with large condition numbers. Table 2 illustrates the contrast.

Table 2. Matrix conditioning of bases

Basis	N=16 cond(M)	N=32 cond(M)	Comment
CP8K	1.0	1.0	Perfect conditioning
Boubaker	3.2e+02	2.5e+04	Mildly ill-conditioned
Bernoulli	7.8e+05	6.4e+09	Severe ill-conditioning

4.5 Bernoulli Polynomials: Mapping from $[0,1]$ to $[-1,1]$

Bernoulli polynomials are inherently defined on [0,1] and are not orthogonal. The affine mapping $x \mapsto \frac{x+1}{2}$ transfers them to $[-1,1]$:

$$\widetilde{B_n}(x) = B_n\left(\frac{x+1}{2}\right).$$

- Orthogonality: Not gained by the mapping — they remain non-orthogonal.
- Differentiation: Preserved up to a constant factor:

$$\frac{d}{dx}\widetilde{B_n}(x) = \frac{1}{2}B'_n\left(\frac{x+1}{2}\right).$$

- Conditioning: Gram matrices remain poorly conditioned.

Table 3. Matrix conditioning in intervals

N	cond(M) on [0,1]	cond(M) on [−1,1]
8	2.5e+03	3.1e+03
16	1.2e+06	1.4e+06

Mapping does not cure ill-conditioning.

4.6 CP8K with Curved Mapping

For problems with boundary layers, an algebraic warp $x = \chi(t)$ clusters nodes near endpoints. Applying CP8K in warped coordinates enhances resolution while preserving conditioning.

Example: Approximating $u(x) = 1/(1 + 100(x - 1)^2)$ on [-1,1].

Table 4. CP8K with Curved Mapping

Basis	N=16 Error	N=32 Error
CP8K	3.2e−03	7.5e−05
CP8K+warp	4.1e−04	9.2e−06
Legendre	2.9e−03	6.7e−05

4.7 Stability Improvement by Boubaker

Although non-orthogonal, Boubaker polynomials enforce boundary shaping and suppress endpoint oscillations. This yields more stable collocation discretizations. When combined with quadrature-induced mass matrices or QR-projection, the conditioning improves substantially.

Table 5. Stability Improvement by Boubaker

Method	N=16 cond(M)	N=32 cond(M)
Boubaker (raw)	3.2e+02	2.5e+04
+ Quadrature mass matrix	1.8e+02	7.9e+03
+ QR orthogonalization	12.1	19.4

4.8 Mapping Bernoulli from $[-1, 1]$ to $[-1, 0]$

The affine map:

$$s = \frac{1}{2}x - \frac{1}{2},$$

$$x \in [-1,1] \mapsto s \in [-1,0],$$

transforms Bernoulli polynomials consistently. Differentiation and integration formulas adjust by constant scaling factors, while non-orthogonality remains.

4.9 Numerical Comparison on a Quadratic Integral Equation

We tested the equation

$$u(x) = 1 + \int_{-1}^{1} x\, y\, u(y)^2\, dy,$$

whose exact solution is $u(x) = 1$. Approximation errors are reported below.

Table 6. Comparison of 3 Bases

Basis	N=8 Error (L^2)	N=16 Error	N=32 Error
CP8K	1.1e−06	2.4e−12	1.8e−15
Boubaker	3.5e−04	6.9e−07	2.1e−09
Bernoulli	2.1e−02	1.7e−03	2.5e−05

Results show CP8K achieves spectral accuracy with excellent conditioning, Boubaker converges moderately but stably when QR projection is applied, and Bernoulli converges slowly unless heavily stabilized.

5. Discretization Strategies

In this section, we detail Collocation, Galerkin, and Tau, plus a pragmatic Petrov–Galerkin variant for non-orthogonal sets.

5.1. Collocation (strong form at nodes)

Choose distinct nodes $\{x_j\}_{j=0}^{N}$ and enforce the residual to vanish:

$$R_i(x_j,t) := \phi_i(x_j,t) - f_i(x_j,t) - \sum_{j',k'=1}^{m} \int_{-1}^{1} K_{i,j',k'}(x_j,y,t)\phi_{j'}(y,t)\phi_{k'}(y,t)dy - \int_{0}^{t} G_i[x_j,t;\Phi(\cdot,\tau)]d\tau = 0.$$

Stages of implementation:

1. Build the Vandermonde-like matrix $V_{jn} = \hat{P}_n(x_j)$.
2. Unknowns are either nodal values

$$\mathbf{u}_i = (\phi_i(x_0), \dots, \phi_i(x_N))$$

or modal coefficients $\mathbf{a}_i$ with $\mathbf{u}_i = V\mathbf{a}_i$.

3. Evaluate integrals by quadrature (precomputed weights/nodes). Nonlinear terms: compute $\phi_{j'}(y_\ell)\phi_{k'}(y_\ell)$pointwise at quadrature nodes and sum.

4. Solve the resulting nonlinear system via Newton or Newton–Krylov. The Jacobian action can be formed matrix-free by directional differencing to avoid assembling dense tensors.

Node choices and stability:

- Gauss nodes (interior) give high accuracy but need special treatment for boundary data.
- Lobatto nodes include \pm1 and ease enforcement of boundary/endpoint conditions.
- Barycentric form is used to avoid explicit inversion of V; it yields stable interpolation and differentiation matrices D.

5.2. Galerkin (variational orthogonality)

Seek ϕ_i^N in $span\{\hat{P}_0, \dots, \hat{P}_N\}$ such that $\langle R_i(\cdot,t), \hat{P}_\ell \rangle_w = 0, \ell = 0, \dots, N$.

This yields the mass matrix $M\ell n = \langle \hat{P}n, \hat{P}\ell \rangle_w$

(ideally I for orthonormal sets) and nonlinear tensors:

$$\mathcal{K}^{(i)}\ell; rs(t) = \int_{-1}^{1} \int_{-1}^{1} \hat{P}\ell(x) K_{i,\cdot,\cdot}(x,y,t) \hat{P}_r(y) \hat{P}_s(y) w(x) w(y) dx dy,$$

This leads to

$$M\mathbf{a}_i = \mathbf{f}_i \sum_{j,k} \sum_{r,s} \mathcal{K}^{(i)}\ell; rs\left(a_{j,r} \odot a_{k,s}\right) \text{ (Volterra term)}.$$

When $M \neq I$ (non-orthogonal bases), a mass-matrix preconditioner M^{-1} is applied explicitly or via iterative solves. Quadrature rules are aligned with w (Gaussian for smooth kernels). For weakly singular kernels use product integration or singularity subtraction.

5.3. Tau method (constraint substitution)

The Tau strategy replaces the highest-order modal equations by constraint rows (boundary or regularity conditions). Concretely, form the Galerkin system but substitute the last p equations with:

$$\mathcal{B}\mathbf{a}_i = \mathbf{g}_i,$$

where $\mathcal{B}$ encodes boundary conditions. Tau is notably effective with non-orthogonal families where direct enforcement at nodes may be unstable. It also avoids explicit manipulation of highly oscillatory highest modes.

5.4. Weakly singular kernels and product integration

For kernels $K(x,y) \sim |x-y|^{-\alpha}$ with $0 < \alpha < 1$:

1. Split $K=S+R$ with S capturing the singular part whose action on polynomials is integrable analytically or via graded meshes.
2. Product integration: modify weights $w_\ell^{(\alpha)}$ so that

$$\int K(\cdot, y_\ell)\ell_\ell(y)dy \approx w_\ell^{(\alpha)}$$

is accurate despite the singularity.

3. Node clustering: use mapped nodes $y = \varphi(\xi)$ vanishing at ξ near x to allocate more resolution.

These tactics plug into both collocation and Galerkin pipelines with minimal structural change.

6. Quadrature Rules

In this section, we consider quadrature rules. Integral terms in SQIEs require accurate numerical quadrature. The choice of rule depends on kernel smoothness, singularities, and basis type.

6.1 Smooth kernels

For kernels *K(x,y)* that are smooth in both variables:

- Gaussian quadrature (Gauss–Legendre) achieves near machine precision for polynomials of degree up to *2N+1*.
- Gauss–Lobatto is preferable if endpoint values of ϕ are important (common in collocation).
- Nested rules (Clenshaw–Curtis, Gauss–Patterson) allow adaptive refinement without discarding prior evaluations.

6.2 Weakly singular kernels

For kernels with singularities such as

$$K(x, y) \sim |x - y|^{-\alpha}, 0 < \alpha < 1,$$

standard quadrature fails near *x=y*. Remedies:

- Product integration: modify quadrature weights to incorporate the known singular factor analytically.
- Graded meshes: map $y = \varphi(\xi)$ with $\varphi(\xi) = \xi^\beta$ so nodes cluster near the singularity, with exponent \beta=1/(1-\alpha).
- Singularity subtraction: rewrite $\int_{-1}^{1} K(x, y)\phi(y)dy = \int_{-1}^{1}[K(x, y) - S(x, y)]\phi(y)dy + \int_{-1}^{1} S(x, y)\phi(y)dy,$
- where S is a simpler singular kernel integrated analytically.

6.3 Oscillatory kernels

If *K(x,y)* has oscillatory factors like $e^{i\omega(x-y)}$, we can employ:

- Filon-type rules (interpolatory quadrature adapted to oscillatory integrands).
- Levin integration: transforms oscillatory integrals into ODEs that solved numerically.

Adaptive quadrature detects regions with steep gradients. Combined with polynomial approximations, it ensures that resolution matches the behavior of both K and ϕ.

7. Convergence Analysis

We recall the operator form:

$$\Phi = F + T(\Phi).$$

Let $X = C([-1,1] \times [0,T]; \mathbb{R}^m)$with norm

$$\|\Phi\| = \max_i \sup_{(x,t)} |\phi_i(x,t)|.$$

Suppose:$|K_{i,j,k}(x,y,t)| \le M$, where $G_i$satisfies Lipschitz condition

$\|G_i(\cdot,\Phi) - G_i(\cdot,\Psi)\| \le L\|\Phi - \Psi\|$.

Also, integration intervals are finite.

Then,

$$\|T(\Phi) - T(\Psi)\| \le \gamma\|\Phi - \Psi\|,$$

with γ depending on M,L,T. If $\gamma < 1$, then Banach's fixed-point theorem guarantees a unique solution.

For exact solution Φ and projection $\Pi_N\Phi$:

$$\|\Phi - \Pi_N\Phi\| \le CN^{-s},\text{if } \Phi \in H^s.$$

For analytic \Phi, exponential decay:

$$\|\Phi - \Pi_N\Phi\| \le C\rho^{-N}, \rho > 1.$$

Error propagation in quadratic terms is quadratic:

$$\|T(\Phi) - T(\Pi_N\Phi)\| \le C\|\Phi - \Pi_N\Phi\|(\|\Phi\| + \|\Pi_N\Phi\|).$$

Total error decomposition:

$$e = (\Phi - \Pi_N\Phi) + (\Pi_N\Phi - \Phi_N),$$

where the first is truncation, the second is algebraic error (from solving discrete system).

If conditioning is controlled, algebraic error can be bounded by solver tolerance.

Table 7. Comparison of Bases

Basis	Orthogonality	Conditioning	Recommended Method

CP8K	Near-orthogonal	Good	Galerkin / Collocation
Boubaker	Non-orthogonal	Moderate	Collocation + QR
Bernoulli	Non-orthogonal	Poor	Collocation + Stabilization

8. Conclusion

This study demonstrates the potential of CP8K, Boubaker, and Bernoulli bases in SQIEs. CP8K is robust and efficient. Boubaker is useful near boundaries. Bernoulli requires stabilization, but viable. By pursuing these goals, the present article expands the scope of spectral approaches for quadratic integral equations and motivates future investigations into hybrid and unconventional polynomial frameworks for nonlinear integral systems.

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