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Fourier Neural Operators with Shock-Aware Loss for the Burgers Equation

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ABSTRACT

The viscous Burgers equation is a benchmark for nonlinear conservation laws with shock formation, and poses a difficulty in obtaining accurate numerical solutions of non-smooth problems. We propose an operator-learning neural network to solve the one-dimensional viscous Burgers equation using Fourier transforms, which we term shock-aware Fourier neural operator (SA-FNO). The SA-FNO model employs a mixed loss function combining a mean squared error associated with the model output with a component that incorporates the effects of shocks via a shock-aware weighted loss function derived from the spatial gradient magnitude of the reference solution at each spatial location. The model learns an operator mapping from the initial conditions and time to the solution field. The operator allows for direct predictions across the entire spatio-temporal solution field. Our numerical results demonstrate that the SA-FNO model achieves a relative L2 error of 0.0248 for the entire time interval, an improvement of approximately 22% over a baseline FNO model trained with only shock-aware loss. We conclude that by training with a mixed shock-aware loss, the SA-FNO model can provide improved representation of non-smooth solutions, without modifying the underlying neural network architecture.

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1. Introduction

Burgers equations are non linear partial differential equations widely used as prototype models to study the behaviour of fluids, nonlinear waves, traffic flows and many other related disciplines [1],[2]. Even though Burgers equations take the simplest form of a PDE they permit solutions with large gradients at certain locations and also provide an existence of shock wave solutions, giving rise to the opportunity for studying numerical properties of non-linear conservation laws together with providing a standard for benchmarking numerical methods designed for solving these problems [3]. As such to accurately capture these features, fine time and spatial discretisations are typically needed resulting in significantly increased computational expense for traditional numerical methods to solve Burgers equations. The classical approach to solving Burgers equations includes finite difference methods, finite volume methods, spectral methods and high-resolution shock capturing methods [4],[5]. Each of these approaches can provide satisfactory results, provided they are carefully implemented; however, each method's limitations concern the nature and size of their respective grids, particularly over long periods of time or when multiple evaluations occur based on distinct initial conditions, and the difficulty with maintaining accuracy ahead of shock waves while preventing the appearance of numerically introduced oscillations [6],[7]. Recent research has turned toward investigating different deep-learning-based approaches to replace current methodologies when approximating solutions of partial differential equations. Neural networks have been applied to a variety of scientific computing problems, including surrogate modeling and reduced-order approximation of PDE solutions [8]. Existing learning-based PDE solvers can be broadly divided into purely data-driven models and physics-informed models. Physics-informed neural networks (PINNs), for example, incorporate the governing equations into the loss function or network structure to guide training [9]. Neural operators provide a framework for approximating solution operators that map input functions, such as initial conditions or coefficients, to output solution fields. Once trained, such models can be evaluated for new inputs without retraining and can operate on discretizations different from those used during training [10],[11]. The Fourier Neural Operator (FNO) is a neural operator architecture. FNO uses global convolutions in the Fourier domain to approximate integral operators and has been used for many different kinds of PDE problems, including creating extensions for learning on more general geometries [13]. However, when standard FNO models are used to approximate nonlinear conservation laws such as the Burgers equation with standard mean square error loss functions, these models may not accurately represent non-smooth solutions. Specifically, a lot of errors are located in areas where the gradient is steep; this can lead to smeared shock profiles and subsequently larger errors at later times. Therefore, other methods are likely required to help direct the learning process toward these localized regions to align with more general evidence demonstrating that shock dominated regimes require special handling when using learning-based solvers for PDEs [14],[15].

This paper presents the development of a new shock-aware Fourier Neural Operator (SA-FNO) framework for solving one-dimensional viscous Burgers equations. The proposed SA-FNO uses a mixed loss function, where both the standard mean squared error (MSE) and a shock-aware weighted loss based on the spatial gradient magnitude of the reference solution are combined to contribute to the final loss value. As shock movements and the corresponding flow patterns are often represented by significantly high spatial gradient magnitudes, clipping, smoothing, and normalization are applied to the shock-aware weights (i.e., the weights used for loss calculation) to retain numerical stability and maximise the training contribution of regions with elevated gradient magnitudes. The main goal of this new approach is to enhance the depiction of shock structures, while conserving the structure of the FNO architecture.

The remainder of this manuscript is structured as follows: Section 2 provides reviews of similar works; Section 3 describes the problem formulation and the proposed SA-FNO methodology; Section 4 describes the numerical experiments conducted and their results; Section 5 concludes the paper and presents further work that might be done to improve upon this methodology.

2. Related Work

Neural operators have been introduced as a framework for approximating mappings between function spaces, with the aim of learning solution operators associated with partial differential equations. The Fourier Neural Operator (FNO) [16] is a type of neural operator that can model integral operators using convolutional neural networks in the Fourier domain, and has been applied in numerous parametric PDE problems. More recent literature has focused on studying the specific formulation of neural operators, as well as how they behave in different discretizations and resolutions [17].

In addition to FNOs, there are other types of operator-learning architectures that have been developed. DeepONets [18], for example, are motivated by the approximation properties of nonlinear operators and have also been used in a variety of PDE applications. Both of these groups of studies illustrate different techniques used to learn operator mappings from data in the context of scientific computing.

Recently, physics-informed methods have been studied that use physical laws in the training of the neural network. A survey on such methods is provided in [19], and an overview of their use in both forward and inverse PDE problems is included. For example, in the case of Burgers equations with steep gradients, weak-form PINN formulations and piecewise PINN formulations have been developed to improve stability and convergence around shock waves [22],[23]. Most of these methods utilize physical constraints during the loss function or use domain decomposition methods, which often require problem-specific design decisions.

Recent work has begun to integrate operator-learning frameworks with conservation principles. For example, Liu et al. [20] developed a neural operator formulation that encodes conservation laws during the learning process to promote greater accuracy for systems of conservation laws. Certain traditional high-order and spectral numerical methods have difficulty accurately modeling shockwave behavior in nonlinear conservation laws due to Gibbs-type oscillations near discontinuities. Oscillatory behavior caused by discontinuities in spectral methods has also led to the creation of specialized methods for capturing or stabilizing shock waves [21]. Furthermore, the introduction of sophisticated reconstruction algorithms in high-order finite volume schemes has improved both the stability and accuracy of scalar nonlinear conservation laws such as the Burgers equation [24]. Such studies show that there is a trade-off between the ability to resolve steep gradients and the ability to suppress non-physical oscillations.

Neural operators, including the Fourier neural operator (FNO), have been effective in solving problems with continuum-like solution structures (e.g. to model the evolution of a fluid), but they can struggle with abrupt changes and shock-like features. Most prior research has attempted to change the architecture of the network, or add physical constraints to the learning task [20], [22], [23], whereas only some have attempted to better inform the operator learning task through loss function design in areas of localized non-smoothness. The present paper explores the use of a shock-aware weighting technique in the training objective of an FNO, maintaining the architecture of the operator itself.

3. Methodology and Model Construction

3.1 Problem Formulation

We consider the one-dimensional viscous Burgers equation defined on a periodic spatial domain $[0,2\pi]$:

$$u_t + uu_x = \nu u_{xx},$$

where $u(x,t)$ denotes the scalar state variable and ν is the viscosity coefficient. Given an initial condition $u(x,0) = u_0(x)$, the objective is to compute the solution field $u(x,t)$ over a time interval $t \in [0,T]$.

From the perspective of neural operators, the solution of the Burgers equation can be regarded as a nonlinear operator mapping from the space of initial conditions to the space of spatio-temporal solution functions. Specifically, we aim to learn an operator

$$\mathcal{G}: u_0(x) \mapsto u(x,t)$$

which approximates the solution operator of the governing PDE for a family of initial conditions. Once learned, this operator enables rapid inference of the solution at arbitrary time instances without performing iterative numerical time stepping.

This formulation is consistent with the neural operator framework, where the learning target is a mapping between function spaces rather than pointwise regression at fixed grid points [17].

3.2 Fourier Neural Operator Architecture.

The proposed SA-FNO model is based on the Fourier Neural Operator (FNO) architecture [16] (FNO) which has been extended in subsequent work to allow FNOs to work with more general types of geometric domains. Moreover, typical FNOs are composed of three basic components: A lifting layer, Stack of Fourier Integral Operator (FIO) Layer(s), and Projection Layer.

Given the input pair $(u_0(x), t)$, the lifting layer maps the low-dimensional input into a higher-dimensional feature space through a pointwise linear transformation. The core of the network is composed of multiple spectral convolution layers, where the global convolution is efficiently implemented in the Fourier domain. At each layer, the feature map is transformed as

$$v_{k+1}(x) = \sigma(W v_k(x) + \mathcal{F}^{-1}(R \cdot \mathcal{F}(v_k(x)))), \qquad (1)$$

where $vk(x)$ denotes the feature at layer k, W is a local linear operator, R represents the learnable Fourier weights acting on a truncated set of low-frequency modes, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and inverse Fourier transforms, respectively, and $\sigma(\cdot)$ is a nonlinear activation function.

After several spectral layers, a projection layer maps the features back to the target solution space, producing the network output $u(x, t)$

3.3 Shock-Aware Weight Construction

For nonlinear conservation laws such as the Burgers equation, shock waves and steep gradients naturally arise in the solution [3]. Standard data-driven losses tend to distribute the error uniformly over the spatial domain, which may lead to insufficient emphasis on localized shock regions.

To address this issue, a shock-aware weighting strategy is introduced. The spatial gradient of the reference solution is used as an indicator of shock intensity. Specifically, the gradient magnitude $|\ ux(x, t)\ |$ is computed using Fourier differentiation, and the weight function is defined as

$$w(x, t) = 1 + \alpha \frac{|u_x(x,t)|}{\langle |u_x(\cdot,t)| \rangle} \qquad (2)$$

where $\langle \cdot \rangle$ denotes the spatial average at time t, and α is a scaling parameter controlling the strength of the weighting.

3.4 Mixed Shock-Aware Loss Function

Let $u^{\text{pred}}(x, t)$ and $u^{\text{ref}}(x, t)$ denote the predicted and reference solutions. The standard mean-squared error loss is defined as

$$\mathcal{L}_{\text{MSE}} = \frac{1}{N} \sum_{x,t} (u^{\text{pred}}(x, t) - u^{\text{ref}}(x, t))^2 \qquad (3)$$

while the shock-aware weighted loss is given by

$$\mathcal{L}_{\text{SA}} = \frac{1}{N} \sum_{x,t} w(x, t)(u^{\text{pred}}(x, t) - u^{\text{ref}}(x, t))^2 \qquad (4)$$

The final training objective is

$$\mathcal{L} = (1 - \beta)\mathcal{L}_{\text{MSE}} + \beta\mathcal{L}_{\text{SA}} \qquad (5)$$

where $\beta \in [0,1]$ balances global accuracy and shock-focused learning.

3.5 Training Strategy and Implementation

During training, pairs of initial conditions and time instances are randomly sampled from the dataset, and the network predicts the corresponding solution fields. The parameters are optimized by minimizing the mixed loss using the AdamW optimizer, and gradient clipping is applied to improve stability.

The shock-aware weights are precomputed from the reference solutions and remain fixed during training. Once trained, the SA-FNO model can infer the solution field for any given initial condition and time input through a single forward pass.

4. Numerical Experiments

This section presents the numerical experiments that were performed using the proposed SA-FNO model for solving the one-dimensional Burgers equation with the shock formation. The overall performance of the proposed method compared to a baseline FNO model that uses only shock-aware weighted data loss to train (the basis of training) is evaluated, and it is also determined how well the proposed method captures shocks (through quantitative error metrics and visual assessments). All experiments use the same dataset and architecture for networks, but only the loss function has been modified so that a fair comparison can be made between the baseline and proposed FNO models.

4.1 Dataset and Training Settings

The experiments are carried out on the one-dimensional viscous Burgers equation over a periodic spatial domain $[0,2\pi]$ with viscosity coefficient $\nu = 0.01$. The reference solutions are loaded from a cached dataset, consisting of numerical solutions saved at 81-time instances on a spatial grid of 128 points.

The model input consists of the initial condition $u_0(x)$ and the time coordinate t, while the output is the solution field $u(x,t)$. A total of 400 samples are used for training, and a separate set is reserved for testing.

All data are normalized using the mean and standard deviation computed from the training set. The networks are trained using the AdamW optimizer with an initial learning rate of 3×10^{-4}, a batch size of 48, and a total of 1800 epochs. A cosine annealing scheduler is employed to gradually reduce the learning rate during training.

Table 1- Model and training parameters

Parameter	Value
Spatial grid points N_x	128
Time snapshots	81
Training samples	400
Viscosity ν\nuν	0.01
Network width	72
Spectral modes	28
Spectral layers	5
Learning rate	3×10^{-4}
Epochs	1800
Batch size	48

4.2 Baseline Results: FNO with Shock-Aware Data Loss

The base model uses the FNO architecture and is trained using a cross shock-aware loss, with larger weights assigned to high-gradient spatial regions (areas of shock). As indicated in Table 2, the cross shock-aware loss continues to decrease throughout the training process, indicating stable convergence of the base model, with a rapid reduction during the early stages of training followed by a more gradual decrease over the remainder of the training cycle.

Table 2 - Shock-aware weighted training loss of baseline FNO at different epochs.

Epoch	SA-weighted Loss
300	5.57×10^{-2}
600	1.45×10^{-2}
900	8.68×10^{-3}
1200	3.86×10^{-3}
1500	3.43×10^{-3}
1800	4.85×10^{-3}

After training, the baseline model is evaluated on the test set. The quantitative results are summarized as follows: the full-time relative L2 error is 0.0318, while the per-time relative L2 errors have a minimum of 0.0149, a mean of 0.0308, and a maximum of 0.0624. These results indicate that the baseline model is able to reproduce the overall structure of the solution; however, the error increases at later times when the shock becomes more pronounced.

4.3 Quantitative Comparison

Table 3 summarizes the quantitative comparison between the baseline model and the proposed SA-FNO

Table 3 - Comparison of relative errors

Model	Full-time Rel. L2	Mean Per-time Rel. L2
Baseline FNO (SA-weighted data only)	0.0318	0.0308
SA-FNO (Proposed)	0.0248	0.0229

It can be observed that SA-FNO reduces the full-time relative error by approximately 22% compared with the baseline model, confirming the effectiveness of the mixed shock-aware loss formulation.

The SA-FNO keeps the shock front sharp across the spatio-temporal domain (Figure 1). Unlike the normal spectral smoothing effects of FNO solvers, the SA-FNO preserves sharp gradients near discontinuities at late times (see Figure 2). At the late time t=0.75, the shock has fully formed. Spectral models are prone to smoothing sharp gradients at this stage. However, the SA-FNO maintains both the location and steepness of the shock front, indicating that the learnable loss function directs the model toward high-gradient regions and reduces the spectral smoothing that occurs with conventional FNO solvers. In general, the proposed loss function improves localized accuracy without affecting the overall accuracy of the global solution structure.

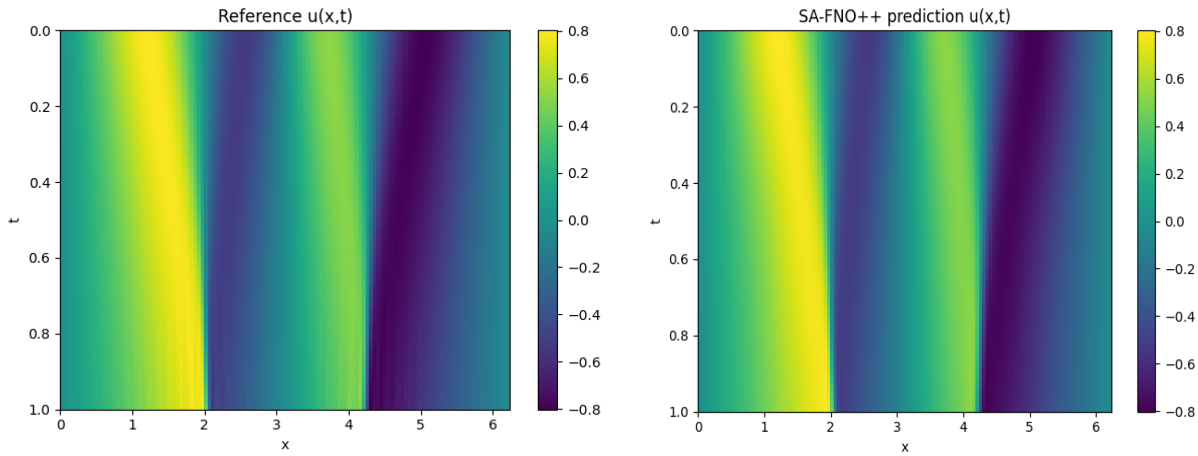

Fig 1- Reference and SA-FNO predictions of $u(x,t)$ for the 1D Burgers equation

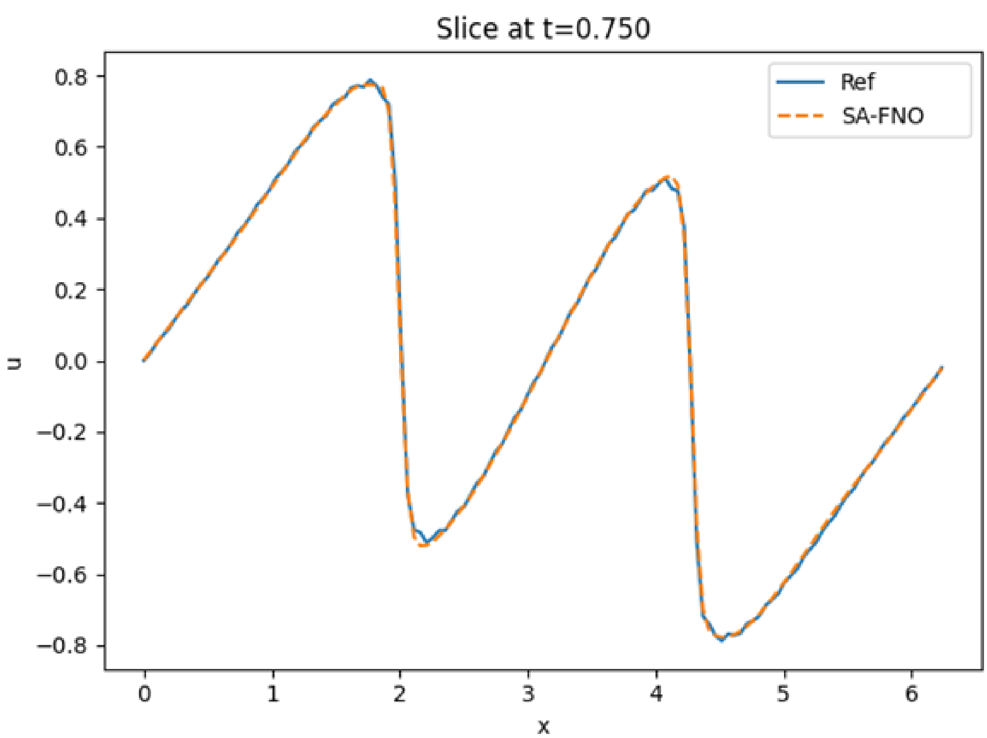

Fig 2- Spatial slice at $t = 0.75$ comparing reference solution and SA-FNO prediction.

4.4 Discussion

The results of the evaluation show how the proposed SA-FNO model compares to the traditional FNO model when using shock-aware data loss. The quantitative evaluation shows that the addition of the mean-squared error objective leads to a significant decrease in the overall (relative) L2 error, indicating that the standard mean-squared error term helps provide a good balance between global accuracy and greater emphasis on localized error around shock regions.

The temporal variance of the (relative) error shows that both models produce low errors for early time instances, when the solution exhibits few features of being smooth. With time, the non-smooth solution features begin to develop sharper, and both methods begin exhibiting increased errors related to the inherent difficulty of modeling these features using spectral-based neural operators. The proposed SA-FNO method produces lower errors on average than the baseline FNO model; however, the continued growth of error with time shows that the proposed reweighting of the loss does not prevent the continued accumulation of inaccuracies in the presence of shock-dominated regimes.

The spatial slice comparison at a time of t=0.75 demonstrates how well the SA-FNO can reproduce the profile of the reference solution when shock structures become well-developed. In this instance, while both models can capture the primary location and shape of the steep gradients near the shock fronts, there are still some differences in the areas surrounding the shock front where both methods are performing poorly (hence the elevated error in the late-time regime). Therefore, even though the shock-aware weighting provides increased resolution locally, the global representation limitations and fixed capacity of the model continue to affect accuracy in areas with rapid changes.

In terms of methodology, the results of this study demonstrate that the design of the loss function can significantly influence the ability of a neural operator to identify important regions for maintaining the fidelity of a solution without necessitating any changes to the underlying architecture of the network. Conversely, the remaining errors indicate that simply reweighting may not be sufficient to solve all of the challenges of modeling discontinuities and steep gradients present in nonlinear conservation laws.

5. Conclusions

A shock-aware Fourier Neural Operator (SA-FNO) method has been developed for solving the one-dimensional viscous Burgers equation with shock (discontinuity) formation. The SA-FNO utilizes a mixed loss approach to encode the shock information into the training process, increasing the ability of the neural operator to reconstruct a non-smooth solution, while retaining overall accuracy. The numerical experiments have shown that the SA-FNO has a full time-referenced L2 error of 0.0248; about 22% better than the baseline FNO model using a shock-aware data loss only. The results demonstrate that the proposed SA-FNO framework provides enhanced temporal stability and superior shock resolution compared with conventional FNO models, providing good (quantitative) agreement between predicted and reference solutions and visually. Unlike conventional (data-driven) FNO frameworks, the proposed approach allows for a straightforward implementation, as it does not require any adjustment of the neural network architecture and only uses a lightweight loss reweighting scheme. This demonstrates the importance of specifically training on localized high-gradient regions for the accurate learning of solution operators for nonlinear conservation laws.

Future efforts will focus on expanding SA-FNO to higher dimensional problems as well as combining a physics based or adaptive restriction along with its application to more difficult and complicated real world applications that involve complex parameter and geometry variations.

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