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Darboux Integrability of a 3D Symmetric Chaotic System with Non-Isolated Equilibria

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ABSTRACT

In this paper, we study the Darboux-type first integrals of the three-dimensional polynomial dynamical system defined by the equations $\dot{u} = v$, $\dot{v} = -\alpha u + vw$, and $\dot{w} = u^2 - \beta v^2$.

This system exhibits chaotic behavior for suitably selected values of the real parameters α and β. We demonstrate that the system has no polynomial, rational, or Darboux first integrals for any values of α and β. Furthermore, we derive all Darboux polynomials associated with the system, in conjunction with their corresponding exponential factors.

MSC..

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1. Introduction

In the analysis of differential equations on $\mathbb{R}^n$, the existence of a first integral provides a powerful tool for model reduction by decreasing the system's dimension by one. However, establishing the presence of such
an integral for an arbitrary system is a notoriously challenging problem. The Darboux theory of integrability, originating with the work of G. Darboux in 1878, offers a powerful algebraic framework for addressing this question. Subsequently extended by Jouanolou and Llibre and Zhang, the theory provides a systematic approach to constructing first integrals for polynomial differential systems that possess a sufficient number of invariant algebraic hypersurfaces, accounting for their respective multiplicities. This theoretical approach has been successfully applied to the analysis of various physical models [5-8].
This work introduces a novel conservative oscillator that exhibits a continuum of equilibria forming a line,

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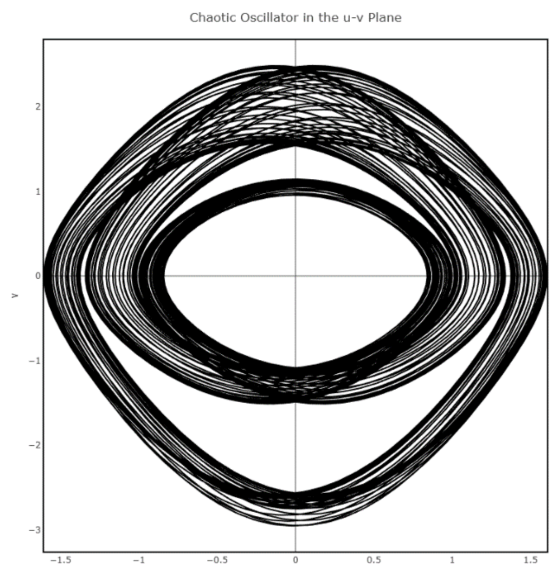
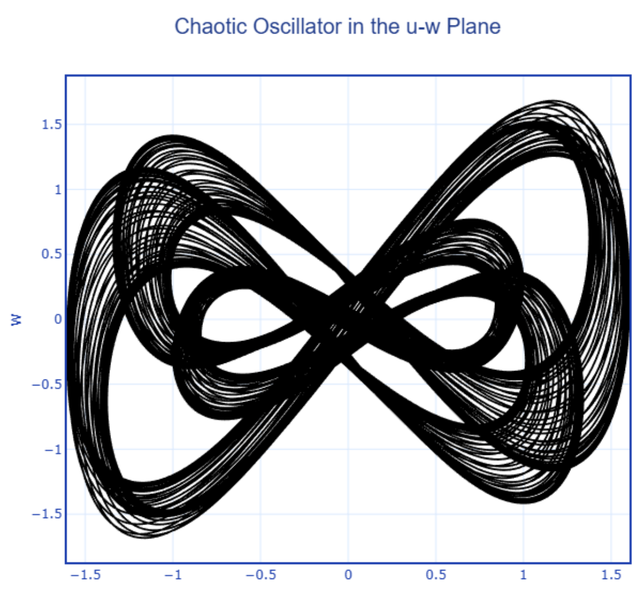
a feature of growing interest in both mathematics and computation [9-11].The system is structurally simple, comprising only five terms, yet its inherent symmetry gives rise to two coexisting symmetric dynamical attractors. The dynamics of this oscillator are described by the following three-dimensional autonomous system of differential equations:

$$\dot{u} = v, \qquad \dot{v} = -\alpha u + vw, \qquad \dot{w} = u^2 - \beta v^2 \tag{1}$$

where u, v, w are the state variables and $\alpha, \beta \in \mathbb{R}$ are real parameters. System (1) represents a quadratic oscillator possessing a line of equilibrium points at $(u, v, w) = (0,0, w)$. For the parameter values $\alpha = 1$, $\beta = 0.68$, and initial conditions $(u_0, v_0, w_0) = (-0.57, -0.99, -0.71)$,





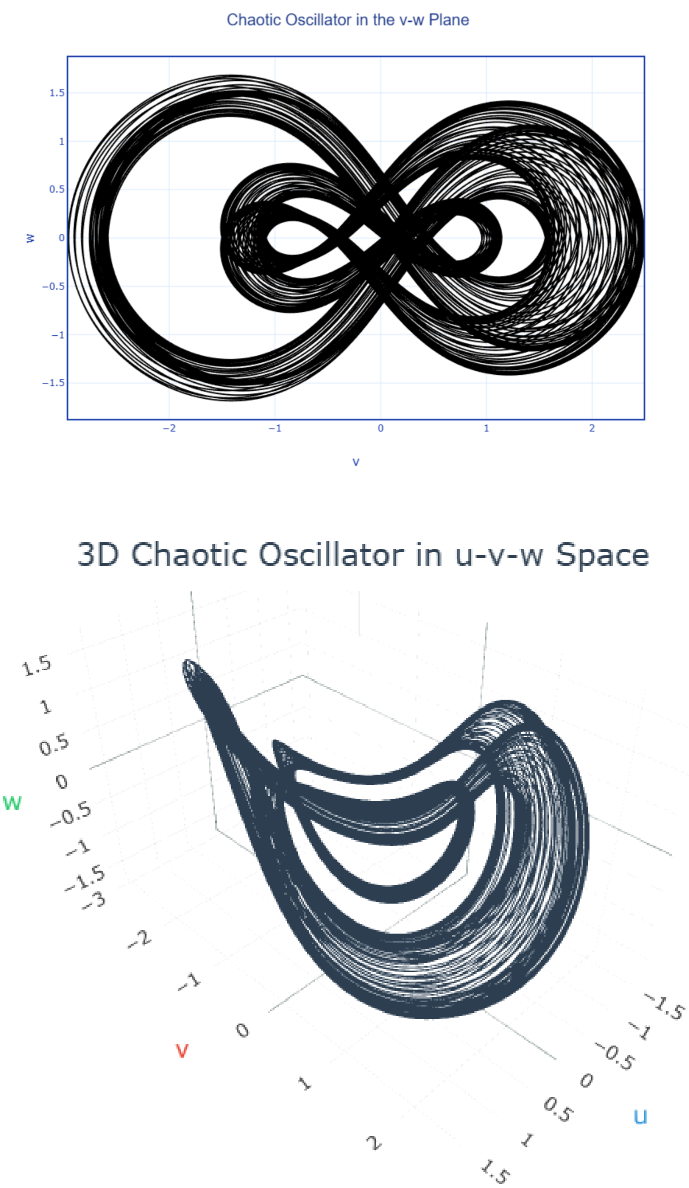


Fig.1 Trajectories corresponding to the initial values $(-0.57, -0.99, -0.71)$, projected onto the UV, UW, and VW planes.

2. Definitions and Preliminary Results

Definition 2.1 Let the polynomial vector field associated with system (1) be given by:

$$v\frac{\partial h}{\partial u} + (vw - \alpha u)\frac{\partial h}{\partial v} + (u^2 - \beta v^2)\frac{\partial h}{\partial w} \qquad (2)$$

defined on an open subset $U \subset \mathbb{R}^3$. A nonconstant C^1 function $H: U \to \mathbb{R}$, is said to be a first integral of system (2.1) on U if $H\big(u(t), v(t), w(t)\big) =$ constant, for all values of t for which $\big(u(t), v(t), w(t)\big)$ is a solution of system (2.1) contained in U. Equivalently, H is a first integral of system (1.4) if and only if

$$XH = v\frac{\partial h}{\partial u} + (vw - \alpha u)\frac{\partial h}{\partial v} + (u^2 - \beta v^2)\frac{\partial h}{\partial w} = 0 \qquad (3)$$

for all $(u, v, w) \in U$.

Definition 2.2 Let $f: U \to \mathbb{C}$ be a non-constant polynomial function defined on an open subset $U \subseteq \mathbb{C}^3$, and let

$$\mathbf{X} = v\frac{\partial f}{\partial u} + (vw - \alpha u)\frac{\partial f}{\partial v} + (u^2 - \beta v^2)\frac{\partial f}{\partial w} \qquad (4)$$

be a polynomial vector field on U, where $L, M, N \in \mathbb{C}[u, v, w]$ are polynomials of degree at most d.
We say that $f = 0$ defines an invariant algebraic surface, or that f is a Darboux polynomial of the vector field $\mathbf{X}$, if there exists a polynomial $K_f \in \mathbb{C}[u, v, w]$, called the cofactor of f, such that

$$\mathbf{X}f = v\frac{\partial f}{\partial u} + (vw - \alpha u)\frac{\partial f}{\partial v} + (u^2 - \beta v^2)\frac{\partial f}{\partial w} = Kf \qquad (5)$$

Definition 2.3 An exponential factor E of the polynomial differential system (2.1) is a non-constant function of the form $E(u, v, w) = \exp\left(\frac{g(u,v,w)}{h(u,v,w)}\right)$, where $g, h \in \mathbb{C}[u, v, w]$ are coprime polynomials (i.e., they have no non-constant common factor in $\mathbb{C}[u, v, w]$) and $h \neq 0$, such that $E \notin \mathbb{C}$ (i.e., E is not identically constant). We say that E is an exponential factor of the associated polynomial vector field $\mathbf{X} = L\,\partial_u + M\,\partial_v + N\,\partial_w$ if it satisfies the partial differential equation

$$\mathbf{X}E = v\frac{\partial E}{\partial u} + (vw - \alpha u)\frac{\partial E}{\partial v} + (u^2 - \beta v^2)\frac{\partial E}{\partial w} = L_E \cdot E \qquad (6)$$

Definition 2.4 A first integral $H(u, v, w)$ of system (2.1) is said to be of Darboux type if it can be expressed in the form $H(u, v, w) = f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_p^{\lambda_p} \cdot E_1^{\mu_1} E_2^{\mu_2} \cdots E_q^{\mu_q}$, where:

a) $f_1, \dots, f_p \in \mathbb{C}[u, v, w]$ are Darboux polynomials of the associated polynomial vector field $\mathbf{X} = L\,\partial_u + M\,\partial_v + N\,\partial_w$.

b) $E_1, \dots, E_q$ are exponential factors of $\mathbf{X}$, each of the form $E_j = \exp(g_j/h_j)$ with $g_j, h_j \in \mathbb{C}[u, v, w]$ coprime.

c) $\lambda_i, \mu_j \in \mathbb{C}$ (complex exponents), not all zero. Furthermore, H satisfies $\mathbf{X}H = 0$ identically on an open subset of $\mathbb{C}^3$, meaning that H is constant along the solutions of system (2.1).

Proposition 2.5 Let system (2.1) be a polynomial differential system defined by the vector field

$$\mathbf{X} = L\,\partial_u + M\,\partial_v + N\,\partial_w,$$

with $L, M, N \in \mathbb{C}[u, v, w]$. If the system admits a rational first integral, i.e., a non-constant function $H = f/g \in \mathbb{C}(u, v, w)$ such that $\mathbf{X}H = 0$, then one of the following two conditions holds:

1. The system has a polynomial first integral, or
2. There exist two distinct Darboux polynomials $f_1, f_2 \in \mathbb{C}[u, v, w]$, not differing by a constant factor, such that both share the same non-zero cofactor K, i.e.,

$$\mathbf{X}f_1 = Kf_1 \quad \text{and} \quad \mathbf{X}f_2 = Kf_2, \quad \text{with } K \neq 0.$$

Theorem 2.6(Darboux; see also (Dumortier et al., 2006)

Let the polynomial differential system $\dot{u} = L(u, v, w), \quad \dot{v} = M(u, v, w), \quad \dot{w} = N(u, v, w)$ of degree m be associated with the polynomial vector field $\mathbf{X} = L\,\partial_u + M\,\partial_v + N\,\partial_w$. Suppose the system admits:

a) p invariant algebraic surfaces defined by $f_i(u, v, w) = 0$, where $f_i \in \mathbb{C}[u, v, w]$ are Darboux polynomials with corresponding cofactors $K_i \in \mathbb{C}[u, v, w]$, i.e., $\mathbf{X}f_i = K_i f_i, \quad i = 1, \dots, p$;

b) q exponential factors $E_j = \exp(g_j/h_j)$, with $g_j, h_j \in \mathbb{C}[u, v, w]$ coprime and $h_j \not\equiv 0$, satisfying $\mathbf{X}E_j = L_j E_j$, for polynomial cofactors $L_j \in \mathbb{C}[u, v, w]$, $j = 1, \dots, q$. Then, there exist complex constants $\lambda_1, \dots, \lambda_p$ and $\mu_1, \dots, \mu_q$, not all zero, such that

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0, \qquad (7)$$

if and only if the function $H(u, v, w) = f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_p^{\lambda_p} \cdot E_1^{\mu_1} E_2^{\mu_2} \cdots E_q^{\mu_q}$
is a first integral of system (2.1), i.e., $\mathbf{X}H = 0$ on an open subset of $\mathbb{C}^3$.
Moreover, H is called a Darboux-type first integral.

$$\mathbf{X}H = 0 \Leftrightarrow \sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$$

3. Results and their proofs

This section outlines the key outcomes of the research, accompanied by their respective proofs. The findings seek to demonstrate that system (1) exhibits a distinctive irreducible Darboux polynomial when the parameter α is set to zero. Moreover, it is shown that the system does not allow for either a polynomial or a rational first-integral. In addition, it is established that the system includes one exponential factor provided the parameter β is non-zero. Lastly, the investigation verifies that the system lacks Darboux integrability.

Theorem 3.1. System (1) has no polynomial first integrals.
Proof. We proceed by contradiction. Let $H \in \mathbb{R}[u,v,w]$ be a polynomial first integral for System (1) with a zero constant term. This conserved quantity satisfies the partial differential equation (4) and admits a graded decomposition:

$$H(u,v,w) = \sum_{i=0}^{n} h_i\,(u,v)w^i,$$

$h_i \in \mathbb{R}[u,v]$ are polynomial coefficient function. By analyzing the leading-order terms in (3), particularly the coefficient of w^{n+1}, we derive:

$$v\frac{\partial h_n}{\partial v} = 0 \Longrightarrow h_n(u,v) = G_n(u),$$

where $G_n(u)$ is a polynomial in u. For the coefficient of w^n, we obtain:

$$v\frac{\partial h_{n-1}}{\partial v} + v\frac{dG_n}{du} - \alpha u\frac{\partial h_n}{\partial v} = 0.$$

This partial differential equation for h_{n-1} may be solved as follows:

$$h_{n-1}(u,v) = -v\frac{dG_n}{du} + G_{n-1}(u),$$

where $G_{n-1}(u)$ is a polynomial in u. Similarly, for the coefficient of w^{n-1}, we obtain:

$$v\frac{\partial h_{n-2}}{\partial v} - \alpha u\frac{\partial h_{n-1}}{\partial v} + v\frac{\partial h_{n-1}}{\partial u} + (u^2 - \beta v^2)nG_n(u) = 0.$$

Solving for h_{n-2} yields:

$$h_{n-2}(u,v) = \frac{1}{2}G_n(u)\beta n v^2 - v\frac{d}{du}G_{n-1}(u) + \frac{1}{2}\frac{d^2}{du^2}G_n(u)v^2 - u\left(G_n(u)nu + \frac{d}{du}G_n(u)\alpha\right)\ln(v) + G_{n-2}(u),$$

where $G_{n-2}(u)$ is a polynomial in u. Since h_{n-2} must be a polynomial, the logarithmic term must vanish, which means that:

$$G_n(u)nu + \frac{d}{du}G_n(u)\alpha = 0.$$

For $\alpha \in \mathbb{R}$, we have $n \cdot G_n(u) = 0$. We consider the following two cases:
Case 1. Let $n = 0$. This implies $H = h_0(u,v)$. Substituting this into Equation (3) and analyze the coefficients of w^0 and w^1 yields:
For $i = 1$:

$$v\frac{\partial h_0(u,v)}{\partial v} = 0 \Longrightarrow h_0(u,v) = h_0(u),$$

and for $i = 0$:

$$v\frac{\partial h_0(u)}{\partial u} = 0 \Longrightarrow h_0(u) = c,$$

where c is a constant. This contradicts the assumption that H has a zero constant term (or is a non-trivial polynomial).
Case 2. Suppose $G_n(u) = 0$. It follows that $h_n = 0$, which implies that the polynomial H reduces to a function depending only on lower-degree terms, specifically $H = h_0(u,v)$. The subsequent steps of the proof proceed analogously to those in Case 1. □
Theorem 3.2. The system (1) admits exactly one irreducible Darboux polynomial, given by v, with the corresponding cofactor w, if and only if α = 0.

Proof. Let $H(u,v,w)$ be a polynomial in $R[u,v,w]$. We express H as a polynomial in w with coefficients depending on u and v:

$$H(u,v,w)=\sum_{i=0}^{n} h_i\,(u,v)w^i,$$

where $h_i(u,v)$ are coefficient functions. The definition of a Darboux polynomial requires that H satisfies the condition:

$$\dot{H}=HK,$$

where $K=(k_0+k_1u+k_2v+k_3w)$ is the cofactor. Expanding the total time derivative $\dot{H}=\frac{\partial H}{\partial u}\dot{u}+\frac{\partial H}{\partial v}\dot{v}+\frac{\partial H}{\partial w}\dot{w}$ and substituting the system equations leads to:

$$v\sum_{i=0}^{n}\frac{\partial h_i}{\partial u}w^i+(-\alpha u+v\dot{w})\sum_{i=0}^{n}\frac{\partial h_i}{\partial v}w^i+(u^2-\beta v^2)\sum_{i=0}^{n} i\,h_i w^{i-1}=(k_0+k_1u+k_2v+k_3w)\sum_{i=0}^{n} h_i\,w^i.$$

Analysis of the coefficient of w^{n+1} : By isolating the terms involving w^{n+1}, we obtain:

$$v\frac{\partial h_n}{\partial v}=k_3h_n.$$

Integration yields:

$$h_n(u,v)=G_n(u)v^{k_3},$$

where $G_n(u)$ is an arbitrary function of u, and $k_3\in N\cup\{0\}$.
Analysis of the coefficient of w^n: For the terms involving w^n, the equation simplifies to:

$$v\frac{\partial h_{n-1}}{\partial v}-\alpha u\frac{\partial h_n}{\partial v}+v\frac{\partial h_n}{\partial u}=k_3h_{n-1}+h_n(k_0+k_1u+k_2v).$$

Substituting h_n and solving this partial differential equation for h_{n-1} yields:

$$h_{n-1}(u,v)=\left[G_n(u)k_2v-G_n'(u)v-\frac{G_n(u)\alpha k_2u}{v}+G_n(u)(k_1u+k_0)\ln v+G_{n-1}(u)\right]v^{k_3},$$

where $G_{n-1}(u)$ is an arbitrary function of u. Since $h_{n-1}(u,v)$ must be a polynomial, the term involving $\ln v$ must vanish. This implies:

$$G_n(u)(k_1u+k_0)=0.$$

We consider two cases based on this condition:
Case 1: Suppose $G_n(u)=0$ and $k_0+k_1u\neq 0$. This implies $h_n=0$. Consequently, the degree of the polynomial reduces. Assume $H=h_0(u,v)$, the invariant condition becomes:

$$v\frac{\partial h_0}{\partial u}+(vw-\alpha u)\frac{\partial h_0}{\partial v}=(k_0+k_1u+k_2v+k_3w)h_0(u,v).$$

Comparing coefficients of w:
1. For w^1:

$$v\frac{\partial h_0}{\partial v}=k_3h_0\Longrightarrow h_0(u,v)=G_0(u)v^{k_3}.$$

2. For w^0:

$$v\frac{\partial}{\partial u}\left(G_0(u)v^{k_3}\right)-\alpha u\frac{\partial}{\partial v}\left(G_0(u)v^{k_3}\right)=(k_0+k_1u+k_2v)G_0(u)v^{k_3}.$$

Solving for $G_0(u)$ yields:

$$G_0(u)=c_1\exp\left(\frac{\alpha k_3u^2+2k_3uv+k_1u^2v+2k_2uv^2}{2v^2}\right).$$

For h_0 to be a polynomial, the exponent must be constant or logarithmic in a specific way. This requires $k_0=k_1=k_2=0$ and $\alpha k_3=0$. However, this contradicts the assumption that $k_0+k_1u\neq 0$. Therefore, we must pass to the second case where $k_0=k_1=0$.
Case 2: Suppose $G_n(u)\neq 0$. It follows that $k_1u+k_0=0$, which implies $k_0=k_1=0$. We proceed to analyze the term involving w^{n-1}:

$$v\frac{\partial h_{n-1}}{\partial u}-\alpha u\frac{\partial h_{n-1}}{\partial v}+v\frac{\partial h_{n-2}}{\partial v}+(u^2-\beta v^2)nh_n=k_3h_{n-2}+k_2vh_{n-1}.$$

Solving for $h_{n-2}(u,v)$ results in an expression containing a logarithmic term:

$$h_{n-2}(u,v) = \Big[\frac{1}{2}G_n(u)\beta n v^2 + \frac{1}{2}G_n(u)k_2^2v^2 - G_n{}'(u)k_2v^2 + \frac{1}{2}G_n{}''(u)v^2 + G_{n-1}(u)k_2v - G_{n-1}(u)v + \big(G_n(u)\alpha k_2u - G_n(u)nu^2 + G_n(u)\alpha k_3 - \alpha u G_n{}'(u)\big)\ln v - \frac{G_{n-1}(u)\alpha k_3 u}{v} + \frac{G_n(u)\alpha^2k_2u^2(k_3-1)}{2v^2} + G_{n-2}(u)\Big]v^{k_3}.$$

Since h_{n-2} must be a polynomial, the coefficient of $\ln v$ must vanish:

$$G_n(u)\alpha k_2u + G_n(u)nu^2 + G_n(u)\alpha k_3 - \alpha u G_n{}'(u) = 0. \qquad (8)$$

Since we assumed $G_n(u) \neq 0$, we analyze two subcases regarding α:

$$G_n(u)\alpha k_2u - G_n(u)nu^2 + G_n(u)\alpha k_3 - \alpha u G_n{}'(u) = 0.$$

Subcase 2.1: If $\alpha \neq 0$.
Solving equation (8) for $G_\alpha(u)$ yields:

$$G_\alpha(u) = u^{k_1}c_1\exp\left(\frac{2\alpha k_2u - nu^2}{2\alpha}\right).$$

For $G_\alpha(u)$ to be a polynomial, the argument of the exponential must be constant, which implies $n = 0$ and $k_2 = 0$. Thus $H = h_0(u,v)$.
Substituting into the invariant condition:

$$v\frac{\partial h_0}{\partial u} + (vw - \alpha u)\frac{\partial h_0}{\partial v} = k_3wh_0. \qquad (9)$$

Comparing coefficients:
For w^1: $\frac{\partial h_0}{\partial u} = k_3h_0 \Longrightarrow h_0(u,v) = G_0(u)v^{k_3}$.
For w^0: $v\frac{\partial h_0}{\partial u} - \alpha u\frac{\partial h_0}{\partial v} = 0$.
Substituting h_0 leads to $G_0(u) = c_1\exp\left(\frac{\alpha k_2u^2}{2u}\right)$. For this to be a valid solution, we must have $k_3 = 0$, which implies $K = 0$. Thus, no non-trivial Darboux polynomial exists when $\alpha \neq 0$.
Subcase 2.2: If $\alpha = 0$.
Equation (9) reduces to $-nu^2G_\alpha(u) = 0$. Since $G_\alpha(u) \neq 0$, this implies $n = 0$. Thus $H = h_0(u,v)$.
The invariant condition becomes:

$$v\frac{\partial h_0}{\partial u} + vw\frac{\partial h_0}{\partial v} = (k_3w + k_2v)h_0. \qquad (10)$$

Comparing coefficients:
For w^1: $\frac{\partial h_0}{\partial u} = k_3h_0 \Longrightarrow h_0(u,v) = G_0(u)v^{k_3}$.
For w^0: $v\frac{\partial h_0}{\partial u} = k_2vh_0 \Longrightarrow \frac{\partial h_0}{\partial u} = k_2h_0$.
Solving this gives $G_0(u) = c_1e^{k_2u}$. For G_0 to be a polynomial, we must have $k_2 = 0$, and $G_0(u)$ becomes a constant.
Therefore, $H = cv^{k_3}$ with cofactor $K = k_3w$. For the polynomial to be irreducible, we take $k_3 = 1$, yielding $H = v$ and cofactor w. This concludes the proof. □
Theorem 3.3. System (1) does not admit any rational first integrals.
Proof. This conclusion follows directly from Theorems 3.1 and 3.2. System (1) possesses a single irreducible Darboux polynomial and lacks any polynomial first integrals, which together rule out the existence of rational first integrals. □
Theorem 3.4: system has the only e^u exponential factor with the cofactor v.
The following results are necessary to establish the proof of Theorem 3.3. □
Theorem 3.5. For the case when $\alpha \neq 0$, System (1) admits e^u as its unique exponential factor, with the corresponding cofactor v. □
Proof. Let $F = \exp(g/h)$ be an exponential factor of System (1) with cofactor L, where $g, h \in \mathbb{C}[u,v,w]$ are coprime polynomials (i.e., $\gcd(g,h) = 1$). By Theorem 3.1 and Proposition 2.5, the denominator h must be a constant polynomial. Without loss of generality, we may normalize $h = 1$. It follows that $F = \exp(g)$, where the function g satisfies the associated partial differential equation:

$$v\frac{\partial g}{\partial u} + (-\alpha u + vw)\frac{\partial g}{\partial v} + (u^2 - \beta v^2)\frac{\partial g}{\partial w} = Le^g.$$

This equation simplifies to:

$$v\frac{\partial g}{\partial u}+(-\alpha u+vw)\frac{\partial g}{\partial v}+(u^2-\beta v^2)\frac{\partial g}{\partial w}=L, \qquad (11)$$

where the cofactor L takes the form:

$$L=d_0+d_1u+d_2v+d_3w. (11)$$

The polynomial g admits a power series representation in the variable w:

$$g(u,v,w)=\sum_{i=0}^{n} g_i\,(u,v)w^i,$$

where each $g_i\in\mathbb{C}[u,v]$ represents a bivariate polynomial coefficient function. For our initial analysis, we consider the case where the degree satisfies $n\geq 2$.

Through careful examination of the terms involving w^{n+1} in Equation (11), we establish the fundamental relation:

$$v\frac{\partial g_n}{\partial v}=0.$$

This differential constraint immediately implies that $g_n(u,v)$ reduces to a univariate polynomial in u, which we denote as $G_n(u)\in\mathbb{C}[u]$. Proceeding to the next order of analysis, we investigate the coefficients of w^n in Equation (11), which yields:

$$\frac{\partial g_n}{\partial u}+v\frac{\partial g_{n-1}}{\partial v}=0.$$

The solution to this equation is:

$$g_{n-1}(u,v)=-vG_n{}'(u)+G_{n-1}(u),$$

where $G_{n-1}(u)$ is a polynomial in the variable u. By evaluating the coefficients of w^{n-1} in Equation (11), we obtain:

$$v\frac{\partial g_{n-1}}{\partial u}+v\frac{\partial g_{n-2}}{\partial v}-\alpha u\frac{\partial g_{n-1}}{\partial v}+nG_n(u)(u^2-\beta v^2)=0.$$

Solving for g_{n-2} yields:

$$g_{n-2}(u,v)=\frac{1}{2}G_n(u)\beta nv^2+\frac{1}{2}G_n{}''(u)v^2-G_{n-1}{}'(u)v+\left(-G_n(u)nu^2-\alpha uG_n{}'(u)\right)\ln(v)+G_{n-2}(u),$$

where $G_{n-2}(u)$ is required to be a polynomial. Since $g_{n-2}(u,v)$ must be a polynomial, the logarithmic term must vanish, implying:

$$-G_n(u)nu^2-\alpha uG_n{}'(u)=0.$$

Assuming $\alpha\neq 0$, the solution to this differential equation is:

$$G_n(u)=c_1e^{-\frac{nu^2}{2\alpha}}.$$

Since $G_n(u)$ must be a polynomial, we must have $n=0$ or $G_n(u)=0$. If $n=0$, it contradicts the assumption that $n\geq 2$. Therefore, $G_n=0$, which implies $g_n=0$ for all $n\geq 2$. Consequently, g is linear in w:

$$g(u,v,w)=g_0(u,v)+g_1(u,v)w.$$

Substituting this form into Equation (10):

$$v\frac{\partial(g_0+g_1w)}{\partial u}+(vw-\alpha u)\frac{\partial(g_0+g_1w)}{\partial v}+(u^2-\beta v^2)g_1=d_0+d_1u+d_2v+d_3w.$$

By evaluating the coefficients of w^i for $i=2,1,0$, we derive the following:

For $i=2$:

$$v\frac{\partial g_1}{\partial v}=0.$$

This implies $g_1(u,v)=g_1(u)$, where $g_1(u)$ is a polynomial.

For $i=1$:

$$v\frac{\partial g_1}{\partial u}+v\frac{\partial g_0}{\partial v}=d_3.$$

Solving for g_0 yields:

$$g_0(u,v)=-v\frac{dg_1}{du}+d_3\ln v+Q(u),$$

where $Q(u)$ is a polynomial. Given that $g_0(u,v)$ must be a polynomial, the logarithmic term must vanish, which implies $d_3=0$. For $i=0$:

$$v\frac{\partial g_0}{\partial u} - \alpha u\frac{\partial g_0}{\partial v} + (u^2 - \beta v^2)g_1 = d_0 + d_1 u + d_2 v.$$

Substituting $g_0 = -vg_1'(u) + Q(u)$, and analyzing the degrees of the terms (specifically, the $u^2 g_1$ term on the left-hand side versus the linear terms on the right), implies that $g_1(u)$ must be zero. Consequently, the equation simplifies, leading to $Q(u) = d_2 u$.

Thus, we obtain $g(u,v,w) = d_2 u$. This yields the exponential factor $F = e^{d_2 u}$ with the cofactor $L = d_2 v$. Setting $d_2 = 1$ gives the unique factor e^u with cofactor v. □

Proposition 3.6. When α = 0, the system admits e^u as its unique exponential factor, with the corresponding cofactor v.

Proof. Following Proposition 2 and Theorem 5, under the condition α = 0, the exponential factors of System (1) must assume the form:

$$E = \exp\left(\frac{g}{v^s}\right),$$

where s ∈ Z≥0 is a non-negative integer, $g \in C[u,v,w]$ is a multivariate polynomial, and g and v^s are coprime. As established by Theorem 6, the exponential factor E must satisfy the governing partial differential equation:

$$v\frac{\partial}{\partial u}\left(e^{g/v^s}\right) + (vw - \alpha u)\frac{\partial}{\partial v}\left(e^{g/v^s}\right) + (u^2 - \beta v^2)\frac{\partial}{\partial w}\left(e^{g/v^s}\right) = Le^{g/v^s},$$

where L is given by Equation (11). Upon simplification, the equation can be rewritten in the following equivalent form:

$$\frac{v}{\partial g}g + vw\frac{\partial}{\partial v}g + (u^2 - \beta v^2)\frac{\partial}{\partial w}g - swg = Lv^s. \quad (12)$$

Case 1: Analysis for $s \geq 1$. Let $\hat{g}$ denote the restriction of g to the hyperplane $v = 0$. We first observe that $\hat{g} \neq 0$; otherwise, v would divide g, violating the coprimality condition. By restricting Equation (12) to $v = 0$, the function $\hat{g}$ must satisfy the reduced partial differential equation:

$$u^2\frac{\partial \hat{g}}{\partial w} - sw\hat{g} = 0. \quad (13)$$

The general solution to this equation takes the form:

$$\hat{g}(u,w) = f(u)\exp\left(\frac{sw^2}{u^2}\right).$$

However, for $s \neq 0$, the essential singularity at $u = 0$ forces $f(u) = 0$, which results in the contradiction $\hat{g} = 0$. Consequently, no valid solutions exist in this parameter regime.

Case 2: Analysis for $s = 0$. In this case, $E = e^g$, where $g \in \mathbb{C}[u,v,w]$ is a polynomial. Setting $\alpha = 0$ in Theorem 7 yields the solution $g(u,v,w) = d_2 u$, with the corresponding cofactor $L = d_2 v$. This concludes the proof. □

Theorem 3.7. System (1) does not possess any Darboux-type first integrals for any value of the parameter α.

Proof. The proof relies on Theorem 3, which provides a necessary and sufficient condition for the existence of Darboux-type first integrals. Specifically, such a first integral exists if and only if there exist constants $\lambda i, \mu j \in \mathbb{C}$ (not all zero) satisfying the equation:

$$\sum_i \lambda_i K_i + \sum_j \mu_j L_j = 0, \quad (14)$$

where K_i are the cofactors of the invariant polynomials and L_j are the cofactors of the exponential factors. We examine the following cases depending on the parameter values:

Case 1: Suppose α ≠ 0. According to Theorem 3.2, System (1) admits no irreducible Darboux polynomials; thus, there are no cofactors K_i. From Theorem 3.5 (and Proposition 3.6), the system admits a unique exponential factor with the cofactor $L_1 = d_2 v$. Substituting these into Equation (14) yields:

$$\mu_1 v = 0.$$

This equation implies $\mu_1 = 0$. Since no non-trivial solution for the constants exists, the system possesses no Darboux-type first integrals in this case.

Case 2: Suppose $\alpha = 0$ (and $\beta \neq 0$).
In this scenario, Theorem 5 states that the system possesses a single irreducible Darboux polynomial with the cofactor $K_1 = k_3 w$ (where k_3 is a constant). Additionally, according to Proposition 3.6, the system admits an exponential factor with the cofactor $L_1 = d_2 v$. Consequently, Equation (7) takes the form:
$\lambda_1(k_3 w) + \mu_1(d_2 v) = 0$.
Due to the linear independence of v and w, we must have $\lambda_1 = 0$ and $\mu_1 = 0$. As there are no non-zero constants satisfying the condition, the system admits no Darboux-type first integrals.
This concludes the proof. ▫

Results

The investigation into the integrability of System (1) using the Darboux theory of integrability yields the following established facts:

1. Absence of Polynomial First Integrals: Theorem 4 demonstrates that System (1) possesses no polynomial first integrals. The analysis of the partial differential equation governing the conserved quantities proves that no non-trivial polynomial solution exists.
2. Classification of Darboux Polynomials: Theorem 5 characterizes the invariant algebraic surfaces of the system. It is established that the system admits an irreducible Darboux polynomial if and only if the parameter $\alpha = 0$. In this specific case, the unique Darboux polynomial is $H = v$, with the associated cofactor $K = w$. For $\alpha \neq 0$, the system admits no Darboux polynomials.
3. Non-existence of Rational First Integrals: As a direct consequence of the scarcity of Darboux polynomials and the absence of polynomial first integrals, Theorem 6 confirms that System (1) admits no rational first integrals.
4. Exponential Factors: Theorem 8 and Proposition 9 provide a complete classification of the exponential factors.
   - For $\alpha \neq 0$, the unique exponential factor is $E = e^u$ with the cofactor $L = v$.
   - For $\alpha = 0$, the result remains consistent; the system admits e^u as the unique exponential factor with the cofactor v.
5. Non-integrability in the Darboux Sense: Theorem 10 serves as the culminating result. By examining the linear dependence of the cofactors derived in the previous theorems, it is proven that no linear combination of cofactors vanishes. Specifically:
   - If $\alpha \neq 0$, the only cofactor is v, which is non-zero.
   - If $\alpha = 0$, the cofactors are w and v, which are linearly independent.

Consequently, System (1) does not possess a first integral of the Darboux type for any value of the parameters.

Discussion

The results presented in this study provide a comprehensive algebraic characterization of System (1). The primary conclusion is that the system is not integrable within the class of Darboux functions, suggesting that its dynamics are not confined to algebraic or generalized algebraic foliations of the phase space.

Conclusion

We have proven that System (1) is not Darboux integrable. This implies that any first integral, if one exists, must belong to a more complex functional class or the system is strictly non-integrable. Given the absence of even rational first integrals, System (1) is a strong candidate for exhibiting chaotic behavior, subject to further numerical or analytical investigation of its global dynamics.

References

[1] G. Darboux, “De l’emploi des solutions particulières algébriques dans l’intégration des systèmes d’équations différentielles algébriques,” **CR Math. Acad. Sci. Paris**, vol. 86, pp. 1012–1014, 1878.

[2] G. Darboux, “Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré,” **Bull. Sci. Math. Astron.**, vol. 2, no. 1, pp. 151–200, 1878.

[3] J.-P. Jouanolou, *Equations de Pfaff algébriques*. Berlin, Germany: Springer, 1979.

[4] J. Llibre and X. Zhang, “Darboux theory of integrability in Cn taking into account the multiplicity,” *J. *Differ. Equ.**, vol. 246, no. 2, pp. 541–551, 2009.

[5] J. Llibre and C. Valls, “Integrability of the Bianchi IX system,” *J. *Math. Phys.**, vol. 46, no. 7, p. 072901, 2005.

[6] J. Llibre and C. Valls, “On the integrability of the Einstein–Yang–Mills equations,” *J. *Math. Anal. Appl.**, vol. 336, no. 2, pp. 1203–1230, 2007.

[7] C. Valls, “Rikitake system: analytic and Darbouxian integrals,” **Proc. R. Soc. Edinb. A**, vol. 135, no. 6, pp. 1309–1326, 2005.

[8] X. Zhang, “Exponential factors and Darbouxian first integrals of the Lorenz system,” *J. *Math. Phys.**, vol. 43, no. 10, pp. 4987–5001, 2002.

[9] G. Chen and T. Ueta, “Yet another chaotic attractor,” *Int. J. *Bifurcation Chaos**, vol. 9, no. 07, pp. 1465–1466, 1999.

[10] E. N. Lorenz, “Deterministic nonperiodic flow,” *J. *Atmos. Sci.**, vol. 20, no. 2, pp. 130–141, 1963.

[11] O. E. Rössler, “An equation for continuous chaos,” **Phys. Lett. A**, vol. 57, no. 5, pp. 397–398, 1976.

[12] K. Rajagopal, A. Karthikeyan, and A. Srinivasan, “FPGA implementation of novel fractional-order chaotic systems with two equilibriums and no equilibrium and its adaptive sliding mode synchronization,” **Nonlinear Dyn.**, vol. 87, pp. 2281–2304, 2017.

[13] X. Wang and G. Chen, “A chaotic system with only one stable equilibrium,” **Commun. Nonlinear Sci. Numer. Simul.**, vol. 17, no. 3, pp. 1264–1272, 2012.

[14] Z. Wei, “Dynamical behaviors of a chaotic system with no equilibria,” **Phys. Lett. A**, vol. 376, no. 2, pp. 102–108, 2011.

[15] F. Nazarimehr, S. Jafari, J. C. Sprott, and S. M. R. H. Golpayegani, “A new four-dimensional system containing chaotic or hyper-chaotic attractors with no equilibrium, a line of equilibria and unstable equilibria,” **Chaos, Solitons Fractals**, vol. 111, pp. 108–118, 2018.

[16] F. Nazarimehr and J. C. Sprott, “Investigating chaotic attractor of the simplest chaotic system with a line of equilibria,” **Eur. Phys. J. Spec. Top.**, vol. 229, pp. 1289–1297, 2020.

[17] J. P. Singh and B. Roy, “The simplest 4-D chaotic system with line of equilibria, chaotic 2-torus and 3-torus behaviour,” **Nonlinear Dyn.**, vol. 89, pp. 1845–1862, 2017.

[18] A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, “Determining Lyapunov exponents from a time series,” **Physica D**, vol. 16, no. 3, pp. 285–317, 1985.

[19] C. Christopher, J. Llibre, C. Pantazi, and S. Walcher, “On planar polynomial vector fields with elementary first integrals,” *J. *Differ. Equ.**, vol. 267, no. 8, pp. 4572–4588, 2019.

[20] C. Christopher, J. Llibre, and J. V. Pereira, “Multiplicity of invariant algebraic curves in polynomial vector fields,” **Pac. J. Math.**, vol. 229, no. 1, pp. 63–117, 2007.

[21] F. Dumortier, J. Llibre, and J. C. Artés, *Qualitative Theory of Planar Differential Systems*. *Berlin, Germany: Springer*, 2006.

[22] J. Llibre and X. Zhang, “Darboux theory of integrability for polynomial vector fields in Rn taking into account the multiplicity at infinity,” **Bull. Sci. Math.**, vol. 133, no. 7, pp. 765–778, 2009.