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A New Class of Univalent Function Defined by Fractional Differential Operator

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ABSTRACT

In this paper, we introduce a novel class $\mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$ of univalent functions in the open unit disk $\mathbb{U}=\{s: s\in\mathbb{C}, |s|<1\}$ by using fractional differential operator $D_{\mu}^{m,\zeta}k(s)$. We obtain coefficient estimates, growth and distortion theorems, closure theorems, radius of starlikeness, close-to-convex and convexity, neighborhood property and partial sums for the class $\mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$. Several results involving the convolution property of functions belonging to the class $\mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$ are also derived.

MSC..

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1. Introduction

Let $\mathcal{A}$ represent the class of functions of the following type:

$$k(s)=s+\sum_{n=2}^{\infty} d_n s^n, \tag{1.1}$$

which in the open unit disc $\mathbb{U}=\{s: s\in\mathbb{C}, |s|<1\}$ are holomorphic and univalent.

Let $\mathcal{M}$ represent the subclass of $\mathcal{A}$ consisting of functions of the following type:

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$$k(s) = s - \sum_{n=2}^{\infty} d_n s^n \qquad (d_n \geq 0). \tag{1.2}$$

For $k(s) \in \mathcal{M}$ given by (1.2) and $g(s)$ given by

$$g(s) = s - \sum_{n=2}^{\infty} b_n s^n \qquad (b_n \geq 0, n \in \mathbb{N}, s \in \mathbb{U}) \tag{1.3}$$

The Hadamard product (or Convolution), indicated by $(k * g)$, is specified based on

$$(k * g)(s) = s - \sum_{n=2}^{\infty} d_n b_n s^n = (g * k)(s) \quad (s \in \mathbb{U}). \tag{1.4}$$

In domain $\mathbb{U}$, if $k(\mathbb{U})$ is a starlike domain with regard to origin and $k : \mathbb{U} \to \mathbb{C}$ is univalent, we say that $k(s)$ is starlike. In such case, $k(s) \in \mathcal{A}$ is considered starlike of order ρ if it fulfils

$$Re\left(\frac{sk'(s)}{k(s)}\right) > \rho$$

for some ρ $(0 \leq \rho < 1)$ and for each $s \in \mathbb{U}$. The univalent function $k(s) \in \mathcal{A}$ is told to be convex of order ρ if and only if $sk'(s)$ is starlike of order ρ. In other words, if

$$Re\left(1 + \frac{sk''(s)}{k'(s)}\right) > \rho$$

for some ρ $(0 \leq \rho < 1)$ and for each $s \in \mathbb{U}$. Also, a univalent function $k(s) \in \mathcal{A}$ is told to be close-to-convex of order ρ if

$$Re\big(k'(s)\big) > \rho$$

for some ρ $(0 \leq \rho < 1)$ and for each $s \in \mathbb{U}$.

For holomorphic functions k defined in a simply connected domain containing zero, the fractional derivative of order ζ is defined as follows [6]:

$$D_S^{\zeta} k(s) = \frac{1}{\Gamma(1-\zeta)} \int_0^S \frac{k(t)}{(s-t)^{\zeta}} dt, \qquad 0 \leq \zeta < 1, \tag{1.5}$$

$$\Omega^{\zeta} k(s) = \Gamma(2-\zeta) s^{\zeta} D_S^{\zeta} k(s)$$

$$= s + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\zeta)}{\Gamma(n+1-\zeta)} d_n s^n, \tag{1.6}$$

where multiplicity of $(s-t)^{-\zeta}$ is removed based on requiring $\log(s-t)$, to be real when $s - t > 0$ (see also [6], [7]).

Mowafy *et al.* [5] introduced the following operator in 2023, which can also be called fractional differential operator $D_{\mu}^{m,\zeta} k(s) \colon \mathcal{A} \longrightarrow \mathcal{A}$ defined by

$$D_{\mu}^{0,0} k(s) = k(s),$$

$$D_{\mu}^{1,\zeta} k(s) = (1-\mu)\Omega^{\zeta} k(s) + \mu s \left(\Omega^{\zeta} k(s)\right)' = D_{\mu}^{\zeta} k(s), \qquad \mu \geq 0, 0 \leq \zeta < 1,$$

$D_\mu^{2,\zeta}k(s) = D_\mu^\zeta\left(D_\mu^\zeta k(s)\right),$

$D_\mu^{m,\zeta}k(s) = D_\mu^\zeta\left(D_\mu^{m-1,\zeta}k(s)\right), \qquad m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$

If k is given by (1.1) then

$$D_\mu^{m,\zeta}k(s) = s + \sum_{n=2}^{\infty} \Psi_{n,m}(\zeta,\mu)d_n s^n, \tag{1.7}$$

where

$$\Psi_{n,m}(\zeta,\mu) = \frac{\Gamma(n+1)\Gamma(2-\zeta)}{\Gamma(n+1-\zeta)}[1+\mu(n-1)]^m. \tag{1.8}$$

Now, by using fractional differential operator $D_\mu^{m,\zeta}k(s)$, we define the following:

Definition (1.1): A function $k(s)\ in\ \mathcal{M}$ belongs to the class $\mathcal{NH}_\mu^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$ if and only if it fulfilled the condition:

$$\left|\frac{\delta s^2\left(D_\mu^{m,\zeta}k(s)\right)''' + (1-\lambda)s\left(D_\mu^{m,\zeta}k(s)\right)'' + \lambda\left(D_\mu^{m,\zeta}k(s)\right)'}{(2\gamma-1)s\left(D_\mu^{m,\zeta}k(s)\right)'' + 2\gamma(1-\alpha)\left(D_\mu^{m,\zeta}k(s)\right)'}\right| < \eta,$$

where $s \in \mathbb{U}, \mu \geq 0, 0 \leq \zeta < 1, m \in \mathbb{N}_0, 0 \leq \alpha < 1, 0 \leq \lambda < 1, 0 \leq \delta < 1, \frac{1}{2} < \gamma \leq 1$ and $0 < \eta \leq 1$.

Remark (1.1): When $\zeta = 0$ and $m = 0$, the following distinct subclasses have been examined by different authors.

1) For $\lambda = 0$, $\delta = 0$ the class $\mathcal{NH}_\mu^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$ reduce to the subclass $C(\alpha,\beta,\gamma)$ introduced and studied by Joshi and Shelake [4].
2) For $\lambda = 0$, $\delta = 0$, $\gamma = 1$, and $\eta = 1$, the class $\mathcal{NH}_\mu^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$ reduce to the subclass $C(\alpha)$ introduced and studied by Silverman [10].

The features listed below were examined for different classes in [1, 3, 8, 12, 13, 14].

2. Coefficient Estimate

A necessary and sufficient condition for a function to belong to the class $\mathcal{NH}_\mu^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$ is obtained in the following theorem.

Theorem (2.1): Suppose that the function k be defined based on (1.2). Then $k \in \mathcal{NH}_\mu^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$ if and only if

$$\sum_{n=2}^{\infty} n[\delta(n^2-3n+2) + n(1-\lambda) - 1 + \eta(2\gamma n - 2\gamma\alpha - n + 1)]\Psi_{n,m}(\zeta,\mu)d_n \leq 2\gamma\eta(1-\alpha) + \lambda, \tag{2.1}$$

where $s \in \mathbb{U}, \mu \geq 0, 0 \leq \zeta < 1, m \in \mathbb{N}_0, 0 \leq \alpha < 1, 0 \leq \lambda < 1, 0 \leq \delta < 1, \frac{1}{2} < \gamma \leq 1$ and $0 < \eta \leq 1$.

Then result (2.1) is sharp for the function

$$k(s) = s - \frac{2\gamma\eta(1-\alpha)+\lambda}{n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)} s^n, \qquad n \geq 2. \qquad (2.2)$$

Proof: Suppose that $|s| = 1$ and the inequality (2.1) is valid. Next, we obtain

$$\left|\delta s^2 \left(D_\mu^{m,\zeta} k(s)\right)''' + (1-\lambda)s\left(D_\mu^{m,\zeta} k(s)\right)'' + \lambda\left(D_\mu^{m,\zeta} k(s)\right)'\right|$$

$$-\eta\left|(2\gamma-1)s\left(D_\mu^{m,\zeta} k(s)\right)'' + 2\gamma(1-\alpha)\left(D_\mu^{m,\zeta} k(s)\right)'\right|$$

$$= \left|\begin{array}{c} -\delta\sum_{n=2}^{\infty} n(n-1)(n-2)\Psi_{n,m}(\zeta,\mu)\, d_n s^{n-1} - (1-\lambda)\sum_{n=2}^{\infty} n(n-1)\Psi_{n,m}(\zeta,\mu)\, d_n s^{n-1} + \lambda \\ -\lambda\sum_{n=2}^{\infty} n\,\Psi_{n,m}(\zeta,\mu) d_n s^{n-1} \end{array}\right|$$

$$-\eta\left|-(2\gamma-1)\sum_{n=2}^{\infty} n(n-1)\,\Psi_{n,m}(\zeta,\mu) d_n s^{n-1} + 2\gamma(1-\alpha) - 2\gamma(1-\alpha)\sum_{n=2}^{\infty} n\,\Psi_{n,m}(\zeta,\mu) d_n s^{n-1}\right|$$

$$\leq \sum_{n=2}^{\infty} n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu) d_n - 2\gamma\eta(1-\alpha) - \lambda \leq 0$$

Hence, by maximum modules principle, $k \in \mathcal{NH}_\mu^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$. Now, assume that $k \in \mathcal{NH}_\mu^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$ so that

$$\left|\frac{\delta s^2 \left(D_\mu^{m,\zeta} k(s)\right)''' + (1-\lambda)s\left(D_\mu^{m,\zeta} k(s)\right)'' + \lambda\left(D_\mu^{m,\zeta} k(s)\right)'}{(2\gamma-1)s\left(D_\mu^{m,\zeta} k(s)\right)'' + 2\gamma(1-\alpha)\left(D_\mu^{m,\zeta} k(s)\right)'}\right| < \eta, \qquad s \in \mathbb{U}.$$

Hence

$$\left|\delta s^2 \left(D_\mu^{m,\zeta} k(s)\right)''' + (1-\lambda)s\left(D_\mu^{m,\zeta} k(s)\right)'' + \lambda\left(D_\mu^{m,\zeta} k(s)\right)'\right|$$

$$< \eta\left|(2\gamma-1)s\left(D_\mu^{m,\zeta} k(s)\right)'' + 2\gamma(1-\alpha)\left(D_\mu^{m,\zeta} k(s)\right)'\right|.$$

Therefore, we get

$$\left|\begin{array}{c} -\delta\sum_{n=2}^{\infty} n(n-1)(n-2)\Psi_{n,m}(\zeta,\mu)\, d_n s^{n-1} - (1-\lambda)\sum_{n=2}^{\infty} n(n-1)\Psi_{n,m}(\zeta,\mu)\, d_n s^{n-1} + \lambda \\ -\lambda\sum_{n=2}^{\infty} n\,\phi^m(n,b,t,u) d_n s^{n-1} \end{array}\right|$$

$$< \eta\left|-(2\gamma-1)\sum_{n=2}^{\infty} n(n-1)\,\Psi_{n,m}(\zeta,\mu) d_n s^{n-1} + 2\gamma(1-\alpha) - 2\gamma(1-\alpha)\sum_{n=2}^{\infty} n\,\Psi_{n,m}(\zeta,\mu) d_n s^{n-1}\right|.$$

Thus

$$\sum_{n=2}^{\infty} n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu) d_n \leq 2\gamma\eta(1-\alpha)+\lambda,$$

and the proof is finished.

Corollary (2.1): Suppose that the function $k \in \mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$. Then

$$d_n \leq \frac{2\gamma\eta(1-\alpha)+\lambda}{n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)} s^n, \quad n \geq 2.$$

3. Growth and Distortion Theorems

Theorem (3.1): Suppose that k a holomorphic function specified based on (1.2) is in the class $\mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$, then for $0<|s|=r<1$

$$r-\frac{2\gamma\eta(1-\alpha)+\lambda}{2[2(1-\lambda)-1+\eta(4\gamma-2\gamma\alpha-1)]\Psi_{2,m}(\zeta,\mu)} r^2 \leq |k(s)| \leq r+\frac{2\gamma\eta(1-\alpha)+\lambda}{2[2(1-\lambda)-1+\eta(4\gamma-2\gamma\alpha-1)]\Psi_{2,m}(\zeta,\mu)} r^2.$$

The bounds are sharp, because the equality are reached based on the function

$$k(s)=s-\frac{2\gamma\eta(1-\alpha)+\lambda}{2[2(1-\lambda)-1+\eta(4\gamma-2\gamma\alpha-1)]\Psi_{2,m}(\zeta,\mu)} s^2. \tag{3.1}$$

Proof: Considering Theorem (2.1), we possess

$$\sum_{n=2}^{\infty} n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)d_n \leq 2\gamma\eta(1-\alpha)+\lambda,$$

and

$$2[2(1-\lambda)-1+\eta(4\gamma-2\gamma\alpha-1)]\Psi_{2,m}(\zeta,\mu)\sum_{n=2}^{\infty} d_n$$

$$\leq \sum_{n=2}^{\infty} n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)d_n \leq 2\gamma\eta(1-\alpha)+\lambda.$$

Therefore, we have

$$\sum_{n=2}^{\infty} d_n \leq \frac{2\gamma\eta(1-\alpha)+\lambda}{2[2(1-\lambda)-1+\eta(4\gamma-2\gamma\alpha-1)]\Psi_{2,m}(\zeta,\mu)}.$$

Thus, for $k \in \mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$, we obtain

$$|k(s)|=\left|s-\sum_{n=2}^{\infty} d_n s^n\right| \leq |s|+|s|^2\sum_{n=2}^{\infty} d_n \leq r+\frac{2\gamma\eta(1-\alpha)+\lambda}{2[2(1-\lambda)-1+\eta(4\gamma-2\gamma\alpha-1)]\Psi_{2,m}(\zeta,\mu)} r^2.$$

The following is a proof for the other claim

$$|k(s)|=\left|s-\sum_{n=2}^{\infty} d_n s^n\right| \geq |s|-|s|^2\sum_{n=2}^{\infty} d_n \geq r-\frac{2\gamma\eta(1-\alpha)+\lambda}{2[2(1-\lambda)-1+\eta(4\gamma-2\gamma\alpha-1)]\Psi_{2,m}(\zeta,\mu)} r^2.$$

as far as the proof is finished.

Likewise, by using the same approach as in Theorem (3.1), we may demonstrate the following

Theorem (3.2): Suppose that k a holomorphic function specified based on (1.2) is in the class $\mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$, then for $0<|s|=r<1$

$$1-\frac{2\gamma\eta(1-\alpha)+\lambda}{2(1-\lambda)-1+\eta(4\gamma-2\gamma\alpha-1)\Psi_{2,m}(\zeta,\mu)}\,r\le |k(s)'|\le 1+\frac{2\gamma\eta(1-\alpha)+\lambda}{2(1-\lambda)-1+\eta(4\gamma-2\gamma\alpha-1)\Psi_{2,m}(\zeta,\mu)}\,r.$$

The bounds are sharp for the function $k(s)$ is specified based on (3.1).

Proof: For $k\in\mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$, we have

$$|k(s)'|=\left|1-\sum_{n=2}^{\infty}nd_n\,s^{n-1}\right|\le 1+|s|\sum_{n=2}^{\infty}nd_n\le 1+\frac{2\gamma\eta(1-\alpha)+\lambda}{2(1-\lambda)-1+\eta(4\gamma-2\gamma\alpha-1)\Psi_{2,m}(\zeta,\mu)}\,r.$$

On the other hand

$$|k(s)'|=\left|1-\sum_{n=2}^{\infty}nd_n\,s^{n-1}\right|\ge 1-|s|\sum_{n=2}^{\infty}nd_n\ge 1-\frac{2\gamma\eta(1-\alpha)+\lambda}{2(1-\lambda)-1+\eta(4\gamma-2\gamma\alpha-1)\Psi_{2,m}(\zeta,\mu)}\,r.$$

and the proof is finished.

4. Radii of Starlikeness, Convexity and Close-to-convexity

The radii of starlikeness, convexity, and close-to-convexity for the class $\mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$ are obtained in the following theorems.

Theorem (4.1): Suppose that $k\in\mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$. Then k is starlike in $|s|<R_1$ of order $\rho, 0\le\rho<1$, wherever

$$R_1=\inf_n\left\{\frac{(1-\rho)n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{(n-\rho)(2\gamma\eta(1-\alpha)+\lambda)}\right\}^{\frac{1}{n-1}},\quad n\ge 2 \tag{4.1}$$

The result is sharp to the function $k(s)$ specified by (2.2).

Proof: k is starlike of order $\rho, 0\le\rho<1$ if

$$Re\left(\frac{sk'(s)}{k(s)}\right)>\rho.$$

Therefore, it suffices to demonstrate that

$$\left|\frac{sk'(s)}{k(s)}-1\right|=\left|\frac{-\sum_{n=2}^{\infty}(n-1)d_ns^{n-1}}{1-\sum_{n=2}^{\infty}d_ns^{n-1}}\right|\le\frac{\sum_{n=2}^{\infty}(n-1)d_n|s|^{n-1}}{1-\sum_{n=2}^{\infty}d_n|s|^{n-1}}.$$

Thus,

$$\left|\frac{sk'(s)}{k(s)}-1\right|\le 1-\rho\quad if\quad \sum_{n=2}^{\infty}\frac{(n-\rho)}{(1-\rho)}d_n|s|^{n-1}\le 1. \tag{4.2}$$

Hence, by Theorem (2.1), (4.2) is accurate if

$$\frac{(n-\rho)}{(1-\rho)}|s|^{n-1} \le \frac{n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{2\gamma\eta(1-\alpha)+\lambda}$$

or if

$$|s| = \left[\frac{(1-\rho)n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{(n-\rho)(2\gamma\eta(1-\alpha)+\lambda)}\right]^{\frac{1}{n-1}}, \qquad n \ge 2. \tag{4.3}$$

From (4.3), the theorem is readily inferred.

Theorem (4.2): Suppose that $k \in \mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$. Then k is convex in $|s| < R_2$ of order $\rho, 0 \le \rho < 1$, wherever

$$R_2 = \inf_n \left\{\frac{(1-\rho)[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{(n-\rho)(2\gamma\eta(1-\alpha)+\lambda)}\right\}^{\frac{1}{n-1}}, n \ge 2. \tag{4.4}$$

The result is sharp to the function $k(s)$ specified by (2.2).

Proof: k is convex of order $\rho, 0 \le \rho < 1$ if

$$Re\left(1+\frac{sk''(s)}{k'(s)}\right) > \rho.$$

Therefore, it suffices to demonstrate that

$$\left|\frac{sk''(s)}{k'(s)}\right| = \left|\frac{-\sum_{n=2}^{\infty} n(n-1)d_n s^{n-1}}{1-\sum_{n=2}^{\infty} nd_n s^{n-1}}\right| \le \frac{\sum_{n=2}^{\infty} n(n-1)d_n|s|^{n-1}}{1-\sum_{n=2}^{\infty} nd_n|s|^{n-1}}.$$

Thus,

$$\left|\frac{sk''(s)}{k'(s)}\right| \le 1-\rho \qquad if \qquad \sum_{n=2}^{\infty}\frac{n(n-\rho)}{(1-\rho)}d_n|s|^{n-1} \le 1. \tag{4.5}$$

Hence, by Theorem (2.1), (4.5) is accurate if

$$\frac{n(n-\rho)}{(1-\rho)}|s|^{n-1} \le \frac{n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{2\gamma\eta(1-\alpha)+\lambda}$$

or if

$$|s| \le \left[\frac{(1-\rho)[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{(n-\rho)(2\gamma\eta(1-\alpha)+\lambda)}\right]^{\frac{1}{n-1}}, \quad n \ge 2. \tag{4.6}$$

From (4.6), the theorem is readily inferred.

Theorem (4.3): Suppose that $k \in \mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$. Then k is close-to-convex in $|s| < R_3$ of order $\rho, 0 \le \rho < 1$, wherever

$$R_3 = \inf_n \left\{\frac{(1-\rho)[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{(2\gamma\eta(1-\alpha)+\lambda)}\right\}^{\frac{1}{n-1}}, \quad n \ge 2. \tag{4.7}$$

The result is sharp to the function $k(s)$ specified by (2.2).

Proof: k is close-to-convex of order $\rho, 0 \leq \rho < 1$ if

$$Re\big(k'(s)\big) > \rho.$$

Therefore, it suffices to demonstrate that

$$|k'(s) - 1| = \left| -\sum_{n=2}^{\infty} n d_n s^{n-1} \right| \leq \sum_{n=2}^{\infty} n d_n |s|^{n-1}.$$

Thus,

$$|k'(s) - 1| \leq 1 - \rho \quad if \quad \sum_{n=2}^{\infty} \frac{n}{(1-\rho)} d_n |s|^{n-1} \leq 1. \tag{4.8}$$

Hence, by Theorem (2.1), (4.8) is accurate if

$$\frac{n}{(1-\rho)} |s|^{n-1} \leq \frac{n[\delta(n^2 - 3n + 2) + n(1-\lambda) - 1 + \eta(2\gamma n - 2\gamma\alpha - n + 1)]\Psi_{n,m}(\zeta, \mu)}{2\gamma\eta(1-\alpha)}$$

or if

$$|s| \leq \left[\frac{(1-\rho)[\delta(n^2 - 3n + 2) + n(1-\lambda) - 1 + \eta(2\gamma n - 2\gamma\alpha - n + 1)]\Psi_{n,m}(\zeta, \mu)}{(2\gamma\eta(1-\alpha) + \lambda)} \right]^{\frac{1}{n-1}}, \quad n \geq 2. \tag{4.9}$$

From (4.9), the theorem is readily inferred.

5. Closure Theorem

Theorem (5.1): Suppose that the function k_i specified by

$$k_i(s) = s - \sum_{n=2}^{\infty} d_{n,i}\, s^n \qquad \big(d_{n,i} \geq 0, n \geq 2, i = 1,2, \dots, t\big), \tag{5.1}$$

belongs to the class $\mathcal{NH}_{\mu}^{m,\zeta}(\delta, \lambda, \alpha, \eta, \gamma)$ for each $i = 1,2, \dots, t$.

Then, the function f_1 that is specified based on

$$f_1(s) = s - \sum_{n=2}^{\infty} e_n\, s^n, \qquad (e_n \geq 0, n \geq 2),$$

also be in the class $\mathcal{NH}_{\mu}^{m,\zeta}(\delta, \lambda, \alpha, \eta, \gamma)$ wherever

$$e_n = \frac{1}{t} \sum_{i=1}^{t} d_{n,i}\,, \qquad (n \geq 2).$$

Proof: Since $k_i \in \mathcal{NH}_{\mu}^{m,\zeta}(\delta, \lambda, \alpha, \eta, \gamma)$, consequently Theorem (2.1) states that

$$\sum_{n=2}^{\infty} n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)d_{n,i} \le 2\gamma\eta(1-\alpha)+\lambda,$$

for every $i = 1,2, \dots, t$. Hence

$$\sum_{n=2}^{\infty} n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)e_n$$

$$=\sum_{n=2}^{\infty} n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)\left(\frac{1}{t}\sum_{i=1}^{t} d_{n,i}\right)$$

$$=\frac{1}{t}\sum_{i=1}^{t}\left(\sum_{n=2}^{\infty} n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)\, d_{n,i}\right)$$

$$\le 2\gamma\eta(1-\alpha)+\lambda.$$

By Theorem (2.1), it follows that $f_1 \in \mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$.

Theorem (5.2): Suppose that the function k_i defined based on (5.1) belong to the class $\mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$ for each $i = 1,2, \dots, t$. Then the function f_2 specified based on

$$f_2(s)=\sum_{i=1}^{t} c_i\, k_i(s)$$

Belongs to the class $\mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$ wherever

$$\sum_{i=1}^{t} c_i = 1, \qquad (c_i \ge 0).$$

Proof: Based on Theorem (2.1) for each $i = 1,2, \dots, t$, we obtain

$$\sum_{n=2}^{\infty} n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)d_{n,i} \le 2\gamma\eta(1-\alpha)+\lambda.$$

But

$$f_2(s)=\sum_{i=1}^{t} c_i\, k_i(s)=\sum_{i=1}^{t} c_i\left(s-\sum_{n=2}^{\infty} d_{n,i}\, s^n\right)=s-\sum_{n=2}^{\infty}\left(\sum_{i=1}^{t} c_i\, d_{n,i}\right)s^n.$$

Therefore

$$\sum_{n=2}^{\infty} n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\,\Psi_{n,m}(\zeta,\mu)\left(\sum_{i=1}^{t} c_i\, d_{n,i}\right)$$

$$=\sum_{i=1}^{t} c_i\left(\sum_{n=2}^{\infty} n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)d_{n,i}\right)$$

$$\leq \sum_{i=1}^{t} c_i\,(2\gamma\eta(1-\alpha)+\lambda) = 2\gamma\eta(1-\alpha)+\lambda$$

and the proof is complete.

6. Convolution Properties

Theorem (6.1): Let the function k_j $(j=1,2)$ defined by

$$k_j(s) = s - \sum_{n=2}^{\infty} d_{n,j}\, s^n, \qquad \left(d_{n,j} \geq 0, j = 1,2\right), \tag{6.1}$$

which is in the class $\mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$. Then $k_1 * k_2 \in \mathcal{NH}_{\mu}^{m,\zeta}(\beta,\lambda,\alpha,\eta,\gamma)$, where

$$\beta \leq \frac{\begin{array}{c} n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]^2\Psi_{n,m}(\zeta,\mu) \\ -(2\gamma\eta(1-\alpha)+\lambda)\big(n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)\big)\end{array}}{(2\gamma\eta(1-\alpha)+\lambda)(n^2-3n+2)}.$$

Proof: We need to determine the biggest β such that

$$\sum_{n=2}^{\infty} \frac{n[\beta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{2\gamma\eta(1-\alpha)+\lambda}\, d_{n,1}d_{n,2} \leq 1.$$

Since $k_j \in \mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$ $(j=1,2)$, then

$$\sum_{n=2}^{\infty} \frac{n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{2\gamma\eta(1-\alpha)+\lambda}\, d_{n,j} \leq 1 \qquad (j=1,2). \tag{6.2}$$

By Cauchy–Schwarz inequality, we have

$$\sum_{n=2}^{\infty} \frac{n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{2\gamma\eta(1-\alpha)+\lambda}\sqrt{d_{n,1}d_{n,2}} \leq 1. \tag{6.3}$$

So, we only show that

$$\frac{n[\beta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{2\gamma\eta(1-\alpha)+\lambda}\, d_{n,1}d_{n,2} \leq$$

$$\frac{n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{2\gamma\eta(1-\alpha)+\lambda}\sqrt{d_{n,1}d_{n,2}}.$$

That is equivalent to

$$\sqrt{d_{n,1}d_{n,2}} \leq \frac{\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)}{\beta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)}.$$

From (6.3), we have

$$\sqrt{d_{n,1}d_{n,2}} \leq \frac{2\gamma\eta(1-\alpha)+\lambda}{n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}.$$

which is sufficient to prove that

$$\frac{2\gamma\eta(1-\alpha)+\lambda}{n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)} \leq \frac{\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)}{\beta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)},$$

That implies to

$$\beta \leq \frac{n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]^2\Psi_{n,m}(\zeta,\mu) - (2\gamma\eta(1-\alpha)+\lambda)\big(n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)\big)}{(2\gamma\eta(1-\alpha)+\lambda)(n^2-3n+2)}.$$

Theorem (6.2): Let the function k_j $(j=1,2)$ that is defined in (6.1) belongs to the class $\mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$. Then the function h defined based on

$$h(s) = s - \sum_{n=2}^{\infty}\left(d_{n,1}^2 + d_{n,2}^2\right)s^n, \tag{6.4}$$

be in the class $\mathcal{NH}_{\mu}^{m,\zeta}(\varepsilon,\lambda,\alpha,\eta,\gamma)$ wherever

$$\varepsilon \leq \frac{n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]^2\Psi_{n,m}(\zeta,\mu) - 2(2\gamma\eta(1-\alpha)+\lambda)\big(n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)\big)}{2(2\gamma\eta(1-\alpha)+\lambda)(n^2-3n+2)}.$$

Proof: We need to determine the biggest ε such that

$$\sum_{n=2}^{\infty}\frac{n[\varepsilon(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{2\gamma\eta(1-\alpha)+\lambda}\left(d_{n,1}^2+d_{n,2}^2\right) \leq 1,$$

since $k_j \in \mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$ $(j=1,2)$, we get

$$\sum_{n=2}^{\infty}\left(\frac{n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{2\gamma\eta(1-\alpha)+\lambda}\right)^2 d_{n,1}^2$$

$$\leq \sum_{n=2}^{\infty}\left(\frac{n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{2\gamma\eta(1-\alpha)+\lambda}d_{n,1}\right)^2 \leq 1 \tag{6.5}$$

and

$$\sum_{n=2}^{\infty}\left(\frac{n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{2\gamma\eta(1-\alpha)+\lambda}\right)^2 d_{n,2}^2$$

$$\leq \sum_{n=2}^{\infty}\left(\frac{n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{2\gamma\eta(1-\alpha)+\lambda}d_{n,2}\right)^2 \leq 1. \tag{6.6}$$

Annexation of the inequalities (6.5) and (6.6) gives

$$\sum_{n=2}^{\infty}\frac{1}{2}\left(\frac{n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{2\gamma\eta(1-\alpha)+\lambda}\right)^2\left(d_{n,1}^2+d_{n,2}^2\right)\le 1. \tag{6.7}$$

But $h\in\mathcal{NH}_{\mu}^{m,\zeta}(\varepsilon,\lambda,\alpha,\eta,\gamma)$ if and only if

$$\sum_{n=2}^{\infty}\frac{n[\varepsilon(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{2\gamma\eta(1-\alpha)+\lambda}\left(d_{n,1}^2+d_{n,2}^2\right)\le 1 \tag{6.8}$$

The inequality (6.8) will be satisfied if

$$\frac{n[\varepsilon(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{2\gamma\eta(1-\alpha)+\lambda}\le$$

$$\frac{n^2\left([\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)\right)^2}{2(2\gamma\eta(1-\alpha)+\lambda)^2}.$$

So that

$$\varepsilon\le\frac{n\left([\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)\right)^2-2(2\gamma\eta(1-\alpha)+\lambda)\left(n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)\right)\Psi_{n,m}(\zeta,\mu)}{2(2\gamma\eta(1-\alpha)+\lambda)(n^2-3n+2)}.$$

7. Neighborhood Property and Partial sums

In accordance with the previous studies conducted by Goodman [2] and Ruscheweyh [9], we determine the σ −neighborhood of function $k(s)\in\mathcal{M}$ based on

$$N_{\sigma}(k)=\left\{g\in\mathcal{M}: g(s)=s-\sum_{n=2}^{\infty}b_n s^n \quad\text{and}\quad \sum_{n=2}^{\infty}n|d_n-b_n|\le\sigma\right\}. \tag{7.1}$$

In particular, for identity function $e(s)=s$, we immediately have

$$N_{\sigma}(e)=\left\{g\in\mathcal{M}: g(s)=s-\sum_{n=2}^{\infty}b_n s^n \quad\text{and}\quad \sum_{n=2}^{\infty}n|b_n|\le\sigma\right\} \tag{7.2}$$

Definition (7.1): A function $k(s)\in\mathcal{M}$ is said to be in the class $\mathcal{NH}_{\mu,y}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$ if there exists a function $g(s)\in\mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$, such that

$$\left|\frac{k(s)}{g(s)}-1\right|<1-y \qquad (s\in\mathbb{U}, 0\le y<1).$$

Theorem (7.1): If $g(s)\in\mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$ and

$$y=1-\frac{\sigma\left([2(1-\lambda)-1+\eta(4\gamma-2\gamma\alpha-1)]\Psi_{2,m}(\zeta,\mu)\right)}{2\left[[2(1-\lambda)-1+\eta(4\gamma-2\gamma\alpha-1)]\Psi_{2,m}(\zeta,\mu)-2\gamma\eta(1-\alpha)+\lambda\right]}, \tag{7.3}$$

Then $N_\sigma(g) \subset \mathcal{NH}_{\mu,y}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$.

Proof: Let $k(s) \in N_\sigma(g)$. Then we get from (7.1) that

$$\sum_{n=2}^{\infty} n|d_n - b_n| \leq \sigma,$$

which readily implies the coefficient inequality

$$\sum_{n=2}^{\infty} |d_n - b_n| \leq \frac{\sigma}{2}.$$

Also, since $g(s) \in \mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$, we get from Theorem (2.1)

$$\sum_{n=2}^{\infty} b_n \leq \frac{2\gamma\eta(1-\alpha)+\lambda}{2[2(1-\lambda)-1+\eta(4\gamma-2\gamma\alpha-1)]\Psi_{2,m}(\zeta,\mu)}.$$

So that

$$\left|\frac{k(s)}{g(s)} - 1\right| \leq \frac{\sum_{n=2}^{\infty}|d_n - b_n|}{1-\sum_{n=2}^{\infty} b_n}$$

$$\leq \frac{\sigma\left([2(1-\lambda)-1+\eta(4\gamma-2\gamma\alpha-1)]\Psi_{2,m}(\zeta,\mu)\right)}{2\left[[2(1-\lambda)-1+\eta(4\gamma-2\gamma\alpha-1)]\Psi_{2,m}(\zeta,\mu)-2\gamma\eta(1-\alpha)+\lambda\right]} = 1-y.$$

Thus by Definition (7.1), $k(s) \in \mathcal{NH}_{\mu,y}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$ for y given by (7.3).

The ratio of the function of type (1.2) to its series of partial sums will be examined in this section specified by $k_1(s) = s$ and $k_l(s) = s - \sum_{n=2}^{l} d_n\, s^n$ when the coefficients of k are sufficiently small to fulfill the condition (2.1). We will determine sharp lower bounds to

$$Re\left(\frac{k(s)}{k_l(s)}\right), \qquad Re\left(\frac{k_l(s)}{k(s)}\right), \qquad Re\left(\frac{k'(s)}{k_l'(s)}\right) \qquad \text{and} \qquad Re\left(\frac{k_l'(s)}{k'(s)}\right).$$

What follows, we'll employ the widely acknowledged outcome that

$$Re\left(\frac{1-T(s)}{1+T(s)}\right) > 0, \qquad s \in \mathbb{U},$$

if and only if

$$T(s) = \sum_{n=1}^{\infty} b_n\, s^n$$

satisfies the inequality $|T(s)| \leq |s|$.

Theorem (7.2): If $k(s) \in \mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$, then

$$Re\left(\frac{k(s)}{k_l(s)}\right) \geq 1 - \frac{1}{b_{l+1}} \qquad (l \in \mathbb{N}, s \in \mathbb{U}) \tag{7.4}$$

and

$$Re\left(\frac{k_l(s)}{k(s)}\right) \geq \frac{b_{l+1}}{1+b_{l+1}} \qquad (l \in \mathbb{N}, s \in \mathbb{U}), \tag{7.5}$$

where

$$\left(b_n = \frac{n[\delta(n^2-3n+2)+n(1-\lambda)-1+\eta(2\gamma n-2\gamma\alpha-n+1)]\Psi_{n,m}(\zeta,\mu)}{2\gamma\eta(1-\alpha)+\lambda}\right).$$

Estimated values in (7.4) and (7.5) are sharp.

Proof: We apply the same strategy that Silverman [11] used. The function $k(s) \in \mathcal{NH}_{\mu}^{m,\zeta}(\delta,\lambda,\alpha,\eta,\gamma)$ if and only if

$$\sum_{n=2}^{\infty} b_n d_n \leq 1.$$

It is simple to confirm that $b_{l+1} > b_l > 1$. Consequently,

$$\sum_{n=2}^{l} d_n + b_{l+1}\sum_{n=l+1}^{\infty} d_n \leq \sum_{n=2}^{\infty} b_n d_n \leq 1. \tag{7.6}$$

We could write

$$b_{l+1}\left\{\frac{k(s)}{k_l(s)} - \left(1-\frac{1}{b_{l+1}}\right)\right\} = \frac{1-\sum_{n=2}^{l} d_n\, s^{n-1} - b_{l+1}\sum_{n=l+1}^{\infty} d_n\, s^{n-1}}{1-\sum_{n=2}^{l} d_n\, s^{n-1}} = \frac{1+D(s)}{1+F(s)}.$$

Set

$$\frac{1+D(s)}{1+F(s)} = \frac{1-T(s)}{1+T(s)},$$

so that

$$T(s) = \frac{F(s)-D(s)}{2+D(s)+F(s)}.$$

Then

$$T(s) = \frac{b_{l+1}\sum_{n=l+1}^{\infty} d_n\, s^{n-1}}{2-2\sum_{n=2}^{l} d_n\, s^{n-1} - b_{l+1}\sum_{n=l+1}^{\infty} d_n\, s^{n-1}}$$

and

$$|T(s)| \leq \frac{b_{l+1}\sum_{n=l+1}^{\infty} d_n}{2-2\sum_{n=2}^{l} d_n - b_{l+1}\sum_{n=l+1}^{\infty} d_n}.$$

Now $|T(s)| \leq 1$ if and only if

$$\sum_{n=2}^{l} d_n + b_{l+1}\sum_{n=l+1}^{\infty} d_n \leq 1.$$

Support for it comes from (7.6). This makes the assertion (7.4) of theorem (7.2) simple. Seeing that

$$k(s) = s - \frac{s^{l+1}}{b_{l+1}} \tag{7.7}$$

gives sharp results, we observe that

$$\frac{k(s)}{k_l(s)} = 1 - \frac{s^l}{b_{l+1}}.$$

Letting $s \to 1^-$, we have

$$\frac{k(s)}{k_l(s)} = 1 - \frac{1}{b_{l+1}}.$$

It demonstrates that for any $l \in \mathbb{N}$, the bounds in (7.4) are the best available

$$(1 + b_{l+1})\left(\frac{k_l(s)}{k(s)} - \frac{b_{l+1}}{1 + b_{l+1}}\right) = \frac{1 - \sum_{n=2}^{l} d_n\, s^{n-1} + b_{l+1} \sum_{n=l+1}^{\infty} d_n\, s^{n-1}}{1 - \sum_{n=2}^{l} d_n\, s^{n-1}} = \frac{1 - T(s)}{1 + T(s)},$$

where

$$|T(s)| \le \frac{(1 + b_{l+1}) \sum_{n=l+1}^{\infty} d_n}{2 - 2\sum_{n=2}^{l} d_n + (1 - b_{l+1}) \sum_{n=l+1}^{\infty} d_n}.$$

Now $|T(s)| \le 1$ if and only if

$$\sum_{n=2}^{l} d_n + b_{l+1} \sum_{n=l+1}^{\infty} d_n \le 1.$$

It has the backing of (7.6). As a direct conclusion, Theorem (7.2) yields the claim (7.5). The extremal function k(s) given by (7.7) is sharply estimated in (7.5). This completes the proof of theorem (7.2).

We now discuss the ratios including derivatives. Theorem (7.3) is proved in the manner described in Theorem (7.2), and thus the details may be left out.

Theorem (7.3): If $k(s) \in \mathcal{NH}_{\mu}^{m,\zeta}(\delta, \lambda, \alpha, \eta, \gamma)$, then

$$Re\left(\frac{k'(s)}{k_l'(s)}\right) \ge 1 - \frac{1 + l}{b_{l+1}} \qquad (l \in \mathbb{N}, s \in \mathbb{U}), \tag{7.8}$$

and

$$Re\left(\frac{k_l'(s)}{k'(s)}\right) \ge \frac{b_{l+1}}{1 + l + b_{l+1}} \qquad (l \in \mathbb{N}, s \in \mathbb{U}). \tag{7.9}$$

Along with the extremal function described by (7.7), the estimations in (7.8) and (7.9) are sharp.

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