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New Subclass of Multivalent functions Defined by Multiplier Differential Operator

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ABSTRACT

In this paper, we study a novel subclass $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ of multivalent functions with negative coefficients defined based on multiplier differential operator $\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)$ in the open unit disk $\mathbb{U}=\{s : s\in\mathbb{C}\ and\ |s|<1\}$. We get some interesting properties, like, coefficient estimate, radii of starlikeness, convexity and close-to-convexity, extreme points, weighted mean and arithmetic mean, integral operators, integral means inequalities, convex set, neighborhood property for functions being a member of the class $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$.

MSC..

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1. Introduction

Let $\mathcal{A}_p$ be denote the class of all functions of the form

$$k(s)=s^p+\sum_{n=p+1}^{\infty}d_n s^n,\qquad (p\in\mathbb{N}=\{1,2,\dots\}),\tag{1.1}$$

which are holomorphic and multivalent in the open unit disk $\mathbb{U}=\{s : s\in\mathbb{C}\ and\ |s|<1\}$.

Let $\mathcal{M}_p$ be denote the subclass of $\mathcal{A}_p$ consisting of functions of the form

$$k(s)=s^p-\sum_{n=p+1}^{\infty}d_n s^n,\qquad (d_n\geq 0, p\in\mathbb{N}).\tag{1.2}$$

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For the function $k(s) \in \mathcal{M}_p$ given by (1.2) and the function $g(s) \in \mathcal{M}_p$ defined by

$$g(s) = s^p - \sum_{n=p+1}^{\infty} b_n s^n, \qquad (b_n \geq 0, p \in \mathbb{N}) \tag{1.3}$$

we defined Hadamard product (or the convolution) of $k(s)$ and $g(s)$ by

$$(k * g)(s) = s^p - \sum_{n=p+1}^{\infty} d_n b_n s^n = (g * k)(s). \tag{1.4}$$

A function $k(s) \in \mathcal{A}_p$, is told to be p-valently starlike of order ρ if it fulfills the inequality

$$Re\left(\frac{sk'(s)}{k(s)}\right) > \rho, \qquad (s \in \mathbb{U}; 0 \leq \rho < p; p \in \mathbb{N}). \tag{1.5}$$

We represent the class of all p-valently starlike functions of order ρ by $S_n^*(p,\rho)$.

Also, the function $k(s) \in \mathcal{A}_p$, is told to be p-valently convex of order ρ if it fulfills the inequality

$$Re\left(1 + \frac{sk''(s)}{k'(s)}\right) > \rho, \qquad (s \in \mathbb{U}; 0 \leq \rho < p; p \in \mathbb{N}). \tag{1.6}$$

We represent the class of all p-valently convex functions of order ρ by $C_n(p,\rho)$. The class $S_n^*(p,\rho)$ and $C_n(p,\rho)$ are studied by Owa [8].

A function $k(s) \in \mathcal{A}_p$, is told to be p-valently close-to-convex of order ρ if it fulfills the inequality

$$Re\left(\frac{k'(s)}{s^{p-1}}\right) > \rho \qquad (s \in \mathbb{U}; 0 \leq \rho < p; p \in \mathbb{N}). \tag{1.7}$$

Recently, Sambo and Lasode [11] present novel multiplier differential operator as follows:

Definition (1.1)[11]: Let $p \in \mathbb{N}$; $m \in \mathbb{N}_0 = \{0,1,2,\dots\}$; $\xi, \beta, \mu, \lambda \geq 0$; $0 \leq \eta \leq \lambda$; $0 < v \leq 1$ and $\xi + \beta > 0$, then for $k \in \mathcal{A}_p$, we defined the multiplier differential operator $\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\colon \mathcal{A}_p \to \mathcal{A}_p$ by

$$\mathcal{D}_{\lambda,\mu}^{0,p}(\xi,\beta,\eta,v)k(s) = k(s) \tag{1.8}$$

$$(\xi+\beta)\mathcal{D}_{\lambda,\mu}^{1,p}(\xi,\beta,\eta,v)k(s) = [\xi+\beta+\eta-(2v-1)(\lambda+\mu)]k(s) - (p-1)\lambda\eta s^p$$

$$+\frac{1}{p}\left[\big((2v-1)(\lambda+\mu)-\eta\big)k(s)' + \lambda\eta s^2 k(s)''\right] \tag{1.9}$$

therefore

$$\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s) = \mathcal{D}_{\lambda,\mu}(\xi,\beta,\eta,v)\left(\mathcal{D}_{\lambda,\mu}^{m-1,p}(\xi,\beta,\eta,v)k(s)\right), \quad m \in \mathbb{N}_0 \tag{1.10}$$

and in general we have that

$$\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s) = s^p + \sum_{n=p+1}^{\infty}\left[\frac{\xi + \left[(2v-1)(\lambda+\mu) + \eta(n\lambda-1)\left(\frac{n}{p}-1\right) + \beta\right]}{\xi+\beta}\right]^m d_n s^n \tag{1.11}$$

or for brevity we have

$$\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s) = s^p + \sum_{n=p+1}^{\infty} \Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)d_n s^n,$$

where

$$\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v) = \left[\frac{\xi + \left[(2v-1)(\lambda+\mu) + \eta(n\lambda-1)\left(\frac{n}{p}-1\right) + \beta\right]}{\xi+\beta}\right]^m.$$

Now, by using multiplier differential operator $\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)$, we define the following:

Definition (1.2): A function of the form (1.2) is said to be in the class $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$; if the following condition is met:

$$Re\left\{\frac{s\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)' + \delta s^2\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)''}{(1-\tau)\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s) + \tau s\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)' + (\delta-\tau)s^2\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)''}\right\}$$

$$> \alpha\left|\frac{s\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)' + \delta s^2\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)''}{(1-\tau)\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s) + \tau s\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)' + (\delta-\tau)s^2\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)''} - p\right| + \gamma, \quad (1.12)$$

where $s \in \mathbb{U}, \lambda, \alpha, \mu, \xi, \beta \geq 0, 0 \leq \gamma < p, 0 \leq \tau \leq 1, \tau \leq \delta, m \in \mathbb{N}_0, n \geq p+1$ and $p \in \mathbb{N}$.

Remark (1.1): When m = 0, the following distinct subclasses have been examined by different authors.

1) For $p = 1$, the subclass $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ shortens to the subclass $TS(\lambda,\mu,\alpha,k,j)$ introduced and studied by Yamini [14].
2) For $\delta = 1$ and $\tau = 1$, the subclass $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ shortens to the subclass $UCV(p,\alpha,\beta)$ introduced and studied by Khairmar and More [7].
3) For $p = 1, \delta = 1$ and $\tau = 1$, the subclass $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ shortens to the subclass $UCT(\alpha,\beta)$ introduced and studied by Bharati et al. [3].
4) For $\alpha = 0$, $\tau = 1$ and $\delta = 1$, the subclass $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ shortens to the subclass $C_n(p,\alpha)$ introduced and studied by Owa [8].
5) For $p = 1, \tau = 1, \delta = 1$ and $\alpha = 0$, the subclass $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ shortens to the subclass $C(\alpha)$ introduced and studied by Silverman [12].
6) For $\tau = 0$ and $\delta = 0$, the subclass $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ shortens to the subclass $UST(\alpha,\beta,p)$ introduced and studied by Khairmar and More [7].
7) For $p = 1, \delta = 0$ and $\tau = 0$, the subclass $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ shortens to the subclass $S_pT(\alpha,\beta)$ introduced and studied by Bharati et al. [3].
8) For $\alpha = 0, \delta = 0$ and $\tau = 0$, the subclass $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ shortens to the subclass $S^*(p,\alpha)$ introduced and studied by Owa [8].
9) For $p = 1, \tau = 0, \delta = 0$ and $\alpha = 0$, the subclass $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ shortens to the subclass $\mathcal{T}^*(\alpha)$ introduced and studied by Silverman [12].

In order to arrive to our primary conclusions, we must remember the following lemmas.

Lemma (1.1)[1]: Let $y = v + iu$, be a complex number and $\alpha, \varphi \in \mathbb{R}$. Then $Re(y) \geq \varphi$ if and only if $|y - (p + \varphi)| \leq |y + (p - \varphi)|$, where $\varphi \geq 0$.

Lemma (1.2)[1]: Let $y = v + iu$, be a complex number and $\alpha, \varphi \in \mathbb{R}$. Then $Re(y) \geq \alpha|y - p| + \varphi$ if and only if $Re\left(y\left(1 + \alpha e^{i\theta}\right) - p\alpha e^{i\theta}\right) \geq \varphi$.

Lemma (1.3)[4]: If k and g are holomorphic in $\mathbb{U}$, with $k \prec g$, then

$$\int_0^{2\pi} \left|k\left(re^{i\theta}\right)\right|^{\omega} d\theta \leq \int_0^{2\pi} \left|g\left(re^{i\theta}\right)\right|^{\omega} d\theta,$$

where $\omega > 0, s = re^{i\theta}$, $(0 < r < 1)$.

The features listed below were examined for different classes in [2,6,9,13,15,16].

2. Coefficient Estimate

Theorem (2.1): The function $k(s) \in \mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha, \delta, \tau, \gamma)$, if and only if

$$\sum_{n=p+1}^{\infty} \left[(\delta n - \delta + 1)\left(n(1 + \alpha)\right) - (\alpha p + \gamma)(\delta n^2 - n^2\tau - \delta n + 2n\tau - \tau + 1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)d_n$$

$$\leq (1 + \alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1), \quad (2.1)$$

where $s \in \mathbb{U}, \lambda, \alpha, \mu, \xi, \beta \geq 0, 0 \leq \gamma < p, 0 \leq \tau \leq 1, \tau \leq \delta, m \in \mathbb{N}_0, n \geq p + 1$ and $p \in \mathbb{N}$.

The inequality is sharp for the extremal function

$$k(s) = s^p - \frac{(1 + \alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1)}{\left[(\delta n - \delta + 1)\left(n(1 + \alpha)\right) - (\alpha p + \gamma)(\delta n^2 - n^2\tau - \delta n + 2n\tau - \tau + 1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)} s^n. \quad (2.2)$$

Proof: Since $k(s) \in \mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha, \delta, \tau, \gamma)$, when Lemma (1.2) is applied, inequality (1.12) equals

$$Re\left\{\frac{\left(s\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)k(s)\right)' + \delta s^2\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)k(s)\right)''\right)\left(1 + \alpha e^{i\theta}\right)}{(1 - \tau)\mathcal{D}_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)k(s) + \tau s\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)k(s)\right)' + (\delta - \tau)s^2\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)k(s)\right)''} - p\alpha e^{i\theta}\right\} > \gamma$$

$$(s \in \mathbb{U}, \lambda, \alpha, \mu, \xi, \beta \geq 0, 0 \leq \gamma < p, 0 \leq \tau \leq 1, \tau \leq \delta, m \in \mathbb{N}_0, n \geq p + 1 \text{ and } -\pi \leq \theta \leq \pi).$$

So that

$$Re\left\{\begin{array}{c}\dfrac{\left[s\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)k(s)\right)' + \delta s^2\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)k(s)\right)''\right]\left(1 + \alpha e^{i\theta}\right)}{(1 - \tau)\mathcal{D}_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)k(s) + \tau s\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)k(s)\right)' + (\delta - \tau)s^2\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)k(s)\right)''} \\ -\dfrac{p\alpha e^{i\theta}\left[(1 - \tau)\mathcal{D}_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)k(s) + \tau s\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)k(s)\right)' + (\delta - \tau)s^2\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)k(s)\right)''\right]}{(1 - \tau)\mathcal{D}_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)k(s) + \tau s\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)k(s)\right)' + (\delta - \tau)s^2\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)k(s)\right)''}\end{array}\right\} \geq \gamma. \quad (2.3)$$

Let

$$N(s) = \left[s\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)k(s)\right)' + \delta s^2\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi, \beta, \eta, v)k(s)\right)''\right]\left(1 + \alpha e^{i\theta}\right)$$

$$-p\alpha e^{i\theta}\left[(1-\tau)\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)+\tau s\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)'+(\delta-\tau)s^2\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)''\right]$$

and

$$N(s)=(1-\tau)\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)+\tau s\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)'+(\delta-\tau)s^2\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)''.$$

Using the Lemma (1.1) in (2.3), we have

$$|N(s)+(p-\gamma)M(s)|\geq|N(s)-(p+\gamma)M(s)|.$$

We have

$|N(s)+(p-\gamma)M(s)|$

$$=\left|\begin{array}{c}\left(ps^p-\sum_{n=1+p}^{\infty}\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,nd_ns^n+(\delta p^2-\delta p)s^p-\sum_{n=1+p}^{\infty}\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,(\delta n^2-\delta n)d_ns^n\right)(1+\alpha e^{i\theta})\\ -p\alpha e^{i\theta}\left(\begin{array}{c}(1-\tau)s^p-\sum_{n=1+p}^{\infty}\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,(1-\tau)d_ns^n+\tau ps^p-\sum_{n=1+p}^{\infty}\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,\tau nd_ns^n+(\delta-\tau)(p^2-p)s^p\\ -\sum_{n=1+p}^{\infty}\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,(\delta-\tau)(n^2-n)d_ns^n\end{array}\right)\\ +(p-\gamma)\left(\begin{array}{c}(1-\tau)s^p-\sum_{n=1+p}^{\infty}\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,(1-\tau)d_ns^n+\tau ps^p-\sum_{n=1+p}^{\infty}\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\tau nd_ns^n+(\delta-\tau)(p^2-p)s^p\\ -\sum_{n=1+p}^{\infty}\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)(\delta-\tau)(n^2-n)d_ns^n\end{array}\right)\end{array}\right|.$$

$$=\left|\begin{array}{c}\left(ps^p-\sum_{n=1+p}^{\infty}\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,nd_ns^n+(\delta p^2-\delta p)s^p-\sum_{n=1+p}^{\infty}\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,(\delta n^2-\delta n)d_ns^n\right)(1+\alpha e^{i\theta})\\ +\\ \left(p-p\alpha e^{i\theta}-\gamma\right)\\ \left(\begin{array}{c}(1-\tau)s^p-\sum_{n=1+p}^{\infty}\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,(1-\tau)d_ns^n+\tau ps^p-\sum_{n=1+p}^{\infty}\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,\tau nd_ns^n+(\delta-\tau)(p^2-p)s^p\\ -\sum_{n=1+p}^{\infty}\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,(\delta-\tau)(n^2-n)d_ns^n\end{array}\right)\end{array}\right|$$

$$=\left|\begin{array}{c}(\delta p^2-\delta p+p)\left(1+\alpha e^{i\theta}\right)s^p+\left(p-\gamma-p\alpha e^{i\theta}\right)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)s^p\\ -\sum_{n=p+1}^{\infty}(\delta n-\delta+1)\left(n\left(1+\alpha e^{i\theta}\right)\right)\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,d_ns^n\\ -\sum_{n=p+1}^{\infty}\left(p-\gamma-p\alpha e^{i\theta}\right)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,d_ns^n\end{array}\right|$$

$$\geq\left((\delta p^2-\delta p+p)(1+\alpha)+(p-\gamma-\alpha p)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)\right)|S|^p$$

$$-\sum_{n=p+1}^{\infty}\left[(\delta n-\delta+1)\left(n(1+\alpha)\right)+(p-\gamma-p\alpha)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)d_n|s|^n.$$

Equivalently, $|N(s)-(p+\gamma)M(s)|$

$$=\left|\frac{\left(ps^p-\sum_{n=1+p}^{\infty}\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,nd_ns^n+(\delta p^2-\delta p)s^p-\sum_{n=1+p}^{\infty}\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,(\delta n^2-\delta n)d_ns^n\right)(1+\alpha e^{i\theta})-(p\alpha e^{i\theta}+p+\gamma)}{\left((1-\tau)s^p-\sum_{n=1+p}^{\infty}\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,(1-\tau)d_ns^n+\tau ps^p-\sum_{n=1+p}^{\infty}\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,\tau nd_ns^n+(\delta-\tau)(p^2-p)s^p-\sum_{n=1+p}^{\infty}\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,(\delta-\tau)(n^2-n)d_ns^n\right)}\right|$$

$$\leq\big((p+\gamma+p\alpha)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)-(\delta p^2-\delta p+p)(1+\alpha)\big)|s|^p$$

$$+\sum_{n=p+1}^{\infty}\left[(\delta n-\delta+1)\big(n(\alpha+1)\big)+(p-\gamma-p\alpha)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)d_n|s|^n.$$

Hence

$$|N(s)+(p-\gamma)M(s)|-|N(s)-(\gamma+p)M(s)|\geq(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)$$

$$-\sum_{n=p+1}^{\infty}\left[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)d_n\geq 0.$$

Therefore

$$\sum_{n=p+1}^{\infty}\left[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)d_n$$

$$\leq(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1).$$

Conversely, by inequality (2.1), we need to show that

$$Re\left\{\frac{\left[s\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)'+\delta s^2\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)''\right](1+\alpha e^{i\theta})}{(1-\tau)\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)+\tau s\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)'+(\delta-\tau)s^2\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)''}-\frac{p\alpha e^{i\theta}\left[(1-\tau)\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)+\tau s\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)'+(\delta-\tau)s^2\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)''\right]}{(1-\tau)\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)+\tau s\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)'+(\delta-\tau)s^2\left(\mathcal{D}_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)k(s)\right)''}\right\}\geq 0.\quad(2.4)$$

Let $0\leq s=r<1$, such that $Re(-e^{i\theta})\geq-|e^{i\theta}|=-1$ and $r\to 1^-$, (2.4) is obtained from (2.1).

Corollary (2.1): Suppose the function $k(s)$ defined based on (1.2) be in the class $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$. Then

$$d_n\leq\frac{(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)}{\left[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)},\quad(2.5)$$

where $s\in\mathbb{U},\lambda,\alpha,\mu,\xi,\beta\geq 0, 0\leq\gamma<p, 0\leq\tau\leq 1,\tau\leq\delta, m\in\mathbb{N}_0, n\geq p+1$ *and* $p\in\mathbb{N}$.

3. Radii of Starlikeness, Convexity and Close-to-Convexity

Theorem (3.1): Let $k(s) \in \mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$. Then $k(s)$ is starlike of order ρ $(0 \le \rho < p)$ in the disk $|s| < R_1$, wherever

$$R_1 = \inf_n \left[\frac{(p-\rho)\big[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\big]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{(n-\rho)[(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)]}\right]^{\frac{1}{n-p}},$$

where $s \in \mathbb{U}, \lambda,\alpha,\mu,\xi,\beta \ge 0, 0 \le \gamma < p, 0 \le \tau \le 1, \tau \le \delta, m \in \mathbb{N}_0, n \ge p+1$ and $p \in \mathbb{N}$.

The outcome is sharp for the function $k(s)$ specified based on (2.2).

Proof: It's sufficient to show that

$$\left|\frac{sk'(s)}{k(s)} - p\right| \le p-\rho \quad (0 \le \rho < p),$$

for $|s| < R_1$, we have

$$\left|\frac{sk'(s)}{k(s)} - p\right| \le \frac{\sum_{n=1+p}^{\infty}(n-p)d_n|s|^{n-p}}{1-\sum_{n=1+p}^{\infty}d_n|s|^{n-p}}.$$

Thus

$$\left|\frac{sk'(s)}{k(s)} - p\right| \le p-\rho,$$

if

$$\sum_{n=1+p}^{\infty}\frac{(n-\rho)}{(p-\rho)}d_n|s|^{n-p} \le 1. \tag{3.1}$$

Then by Theorem (2.1), equation (3.1) is equivalent to

$$\frac{(n-\rho)}{(p-\rho)}|s|^{n-p} \le \frac{\big[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\big]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)}.$$

Hence,

$$|s| \le \left[\frac{(p-\rho)\big[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\big]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{(n-\rho)[(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)]}\right]^{\frac{1}{n-p}}.$$

R_1 is obtained by letting $|s| = R_1$ and the proof completes.

Theorem (3.2): Let $k(s) \in \mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$. Then $k(s)$ is convex of order ρ $(0 \le \rho < p)$ in the disk $|s| < R_2$, wherever

$$R_2 = \inf_n \left[\frac{p(p-\rho)\big[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\big]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{n(n-\rho)[(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)]}\right]^{\frac{1}{n-p}},$$

where $s \in \mathbb{U}, \lambda,\alpha,\mu,\xi,\beta \ge 0, 0 \le \gamma < p, 0 \le \tau \le 1, \tau \le \delta, m \in \mathbb{N}_0, n \ge p+1$ and $p \in \mathbb{N}$.

The outcome is sharp for the function $k(s)$ specified based on (2.2).

Proof: It's sufficient to show that

$$\left|1+\frac{sk''(s)}{k'(s)}-p\right| \leq p-\rho \quad (0 \leq \rho < p),$$

for $|s| < R_2$, we have

$$\left|1+\frac{sk''(s)}{k'(s)}-p\right| \leq \frac{\sum_{n=1+p}^{\infty} n(n-p)d_n|s|^{n-p}}{p-\sum_{n=1+p}^{\infty} nd_n|s|^{n-p}}.$$

Thus

$$\left|1+\frac{sk''(s)}{k'(s)}-p\right| \leq p-\rho,$$

if

$$\sum_{n=1+p}^{\infty} \frac{n(n-\rho)}{p(p-\rho)} d_n|s|^{n-p} \leq 1. \tag{3.2}$$

Then by Theorem (2.1), equation (3.2) is equivalent to

$$\frac{n(n-\rho)}{p(p-\rho)}|s|^{n-p} \leq \frac{[(\delta n-\delta+1)(n(1+\alpha))-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)}$$

Hence,

$$|s| \leq \left[\frac{p(p-\rho)[(\delta n-\delta+1)(n(1+\alpha))-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{n(n-\rho)[(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)]}\right]^{\frac{1}{n-p}}.$$

R_2 is obtained by letting $|s| = R_2$ and the proof completes.

Theorem (3.3): Let $k(s) \in \mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$. Then $k(s)$ is close to convex of order ρ $(0 \leq \rho < p)$ in the disk $|s| < R_3$, wherever

$$R_3 = \inf_n \left[\frac{(p-\rho)[(\delta n-\delta+1)(n(1+\alpha))-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{n[(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)]}\right]^{\frac{1}{n-p}},$$

where $s \in \mathbb{U}, \lambda,\alpha,\mu,\xi,\beta \geq 0, 0 \leq \gamma < p, 0 \leq \tau \leq 1, \tau \leq \delta, m \in \mathbb{N}_0, n \geq p+1$ and $p \in \mathbb{N}$.

The outcome is sharp for the function $k(s)$ specified based on (2.2).

Proof: It's sufficient to show that

$$\left|\frac{k'(s)}{s^{p-1}}-p\right| \leq p-\rho \quad (0 \leq \rho < p),$$

for $|s| < R_3$, we have

$$\left|\frac{k'(s)}{s^{p-1}}-p\right| \leq \sum_{n=p+1}^{\infty} n d_n |s|^{n-p}.$$

Thus

$$\left|\frac{k'(s)}{s^{p-1}}-p\right| \leq p-\rho,$$

if

$$\sum_{n=p+1}^{\infty} \frac{n d_n |s|^{n-p}}{p-\rho} \leq 1. \tag{3.3}$$

Then by Theorem (2.1), equation (3.3) is equivalent to

$$\frac{n}{p-\rho}|s|^{n-p} \leq \frac{\left[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)},$$

Hence,

$$|s| \leq \left[\frac{(p-\rho)\left[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{n[(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)]}\right]^{\frac{1}{n-p}},$$

R_3 is obtained by letting $|s| = R_3$ and the proof completes.

4. Extreme Points

The extreme points of the class $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ are discussed in the following theorem.

Theorem (4.1): Let $k_p(s) = s^p$ and

$$k_n(s) = s^p - \frac{(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)}{\left[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)} s^n,$$

where $s \in \mathbb{U}, \lambda, \alpha, \mu, \xi, \beta \geq 0, 0 \leq \gamma < p, 0 \leq \tau \leq 1, \tau \leq \delta, m \in \mathbb{N}_0, n \geq p+1 \text{ and } p \in \mathbb{N}$.

Then the function $k(s)$ belongs to the class $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ if and only if it can be written as:

$$k(s) = \mathcal{L}_p s^p + \sum_{n=p+1}^{\infty} \mathcal{L}_n k_n(s), \tag{4.1}$$

such that

$$\left(\mathcal{L}_p \geq 0, \mathcal{L}_n \geq 0, n \geq p+1 \text{ and } \mathcal{L}_p + \textstyle\sum_{n=p+1}^{\infty} \mathcal{L}_n = 1\right).$$

Proof: Suppose that $k(s)$ that defined in (4.1). Then

$$k(s) = \mathcal{L}_p s^p + \sum_{n=p+1}^{\infty} \mathcal{L}_n$$

$$\left(s^p-\frac{(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)}{\left[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}s^n\right)$$

$$=s^p-\sum_{n=p+1}^{\infty}\frac{(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)}{\left[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}\mathcal{L}_n s^n.$$

Hence

$$\sum_{n=p+1}^{\infty}\frac{\left[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)}$$

$$\times\frac{\left[(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)\right]\mathcal{L}_n}{\left[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}$$

$$=\sum_{n=p+1}^{\infty}\mathcal{L}_n=1-\mathcal{L}_p\le 1.$$

Thus $k(s)\in\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$.

Conversely, suppose that $k(s)\in\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$, we may set

$$\mathcal{L}_n=\frac{\left[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)}d_n,$$

where d_n is defined in (2.5). Then

$$k(s)=s^p-\sum_{n=p+1}^{\infty}d_n s^n$$

$$=s^p-\sum_{n=p+1}^{\infty}\frac{(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)}{\left[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}\mathcal{L}_n s^n$$

$$=s^p-\sum_{n=1+p}^{\infty}\big(s^p-k_n(s)\big)\mathcal{L}_n=\left(1-\sum_{n=1+p}^{\infty}\mathcal{L}_n\right)s^p+\sum_{n=1+p}^{\infty}\mathcal{L}_n k_n(s)=\mathcal{L}_p s^p+\sum_{n=1+p}^{\infty}\mathcal{L}_n k_n(s).$$

This completes the proof of Theorem (4.1).

5. Arithmetic Mean and Weighted Mean

We shall elaborate that the class $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ is closed under arithmetic mean in the next theorem.

Theorem (5.2): Let $k_1(s), k_2(s), k_3(s), \dots, k_\vartheta(s)$ that defined by

$$k_\ell(s)=s^p-\sum_{n=p+1}^{\infty}d_{n,\ell}\,s^n\quad\big(d_{n,\ell}\ge 0,\ell=1,2,\dots,\vartheta,n\ge 1+p\big),\tag{5.1}$$

are in the class $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$, then the arithmetic mean of $k_\ell(s)$ $(\ell=1,2,\dots,\vartheta)$ which clarified based on

$$h(s) = \frac{1}{\vartheta}\sum_{\ell=1}^{\vartheta} k_\ell(s) \tag{5.2}$$

is also a member of the class $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$.

Proof: Based on (5.1 and 5.2), we are able to write

$$h(s) = \frac{1}{\vartheta}\sum_{\ell=1}^{\vartheta}\left(s^p - \sum_{n=1+p}^{\infty} d_{n,\ell}\, s^n\right) = s^p - \sum_{n=1+p}^{\infty}\left(\frac{1}{\vartheta}\sum_{\ell=1}^{\vartheta} d_{n,\ell}\, s^n\right).$$

Since $k_\ell(s) \in \mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$. For each ($\ell = 1,2,\dots,\vartheta$), so based on Theorem (2.1), we get

$$\sum_{n=p+1}^{\infty}\left[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\left(\frac{1}{\vartheta}\sum_{\ell=1}^{\vartheta} d_{n,\ell}\right)$$

$$=\frac{1}{\vartheta}\sum_{\ell=1}^{\vartheta}\left(\sum_{n=p+1}^{\infty}\left[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)d_{n,\ell}\right)$$

$$\le \frac{1}{\vartheta}\sum_{\ell=1}^{\vartheta}(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)$$

$$=(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1).$$

This ends the proof.

Definition (5.1): Suppose that $k(s)$ and $g(s)$ belong to the class $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$. Then, the Weighted Mean $E_t(s)$ of $k(s)$ and $g(s)$ is given by

$$E_t(s) = \frac{1}{2}[(1-t)k(s)+(1+t)g(s)], \qquad (0<t<1).$$

Theorem (5.1): Let $k(s)$ and $g(s)$ be in the class $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$. Then the Weighted Mean of $k(s)$ and $g(s)$ is also belong to the class $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$.

Proof: By definition (5.1), we have

$$E_t(s) = \frac{1}{2}[(1-t)k(s)+(1+t)g(s)] = \frac{1}{2}\left[(1-t)\left(s^p-\sum_{n=1+p}^{\infty} d_n s^n\right)+(1+t)\left(s^p-\sum_{n=1+p}^{\infty} b_n s^n\right)\right]$$

$$= s^p - \sum_{n=p+1}^{\infty}\frac{1}{2}[(1-t)d_n+(1+t)b_n]s^n.$$

Since $k(s)$ and $g(s)$ are in the class $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$, so by Theorem (2.1), we get

$$\sum_{n=p+1}^{\infty}\left[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\, d_n$$

$$\leq (1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)$$

and

$$\sum_{n=p+1}^{\infty}\left[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\, b_n$$

$$\leq (1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1).$$

Hence,

$$\sum_{n=p+1}^{\infty}\left[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)$$

$$\times\left[\frac{1}{2}(1-t)d_n+\frac{1}{2}(1+t)b_n\right]$$

$$=\frac{1}{2}(1-t)\sum_{n=p+1}^{\infty}\left[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)d_n$$

$$+\frac{1}{2}(1+t)\sum_{n=p+1}^{\infty}\left[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)b_n$$

$$\leq\frac{1}{2}(1-t)[(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)]$$

$$+\frac{1}{2}(1+t)[(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)]$$

$$=(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1).$$

This shows that $E_t(s)\in\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$.

6. Integral operator

Theorem (6.1): Let $k(s)$ defined by (1.2) be in the class $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$, and let c be a real number such that $c>-p$. Then the function F(s) defined by

$$F(s)=\frac{c+p}{s^c}\int_0^s t^{c-1}k(t)dt \qquad (c>-p), \tag{6.1}$$

also belongs to the class $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$.

Proof: Form the representation of (6.1) of F(s), it follows from that

$$F(s)=\frac{c+p}{s^c}\int_0^s t^{c-1}\left(t^p-\sum_{n=1+p}^{\infty}d_n\, t^n\right)dt=\frac{c+p}{s^c}\int_0^s\left(t^{p+c-1}-\sum_{n=1+p}^{\infty}d_n\, t^{n+c-1}\right)dt$$

$$= s^p - \sum_{n=1+p}^{\infty} \left(\frac{c+p}{c+n}\right) d_n s^n = s^p - \sum_{n=1+p}^{\infty} g_n s^n \,,$$

where $g_n = \left(\frac{c+p}{c+n}\right) d_n$.

Therefore, we have

$$\sum_{n=1+p}^{\infty} \left[(\delta n - \delta + 1)\big(n(1+\alpha)\big) - (\alpha p + \gamma)(\delta n^2 - n^2\tau - \delta n + 2n\tau - \tau + 1)\right] \Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v) g_n$$

$$= \sum_{n=1+p}^{\infty} \left[(\delta n - \delta + 1)\big(n(\alpha+1)\big) - (\alpha p + \gamma)(\delta n^2 - n^2\tau - \delta n + 2n\tau - \tau + 1)\right] \Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v) \left(\frac{c+p}{c+n}\right) d_n$$

$$\leq \sum_{n=1+p}^{\infty} \left[(\delta n - \delta + 1)\big(n(1+\alpha)\big) - (\alpha p + \gamma)(\delta n^2 - n^2\tau - \delta n + 2n\tau - \tau + 1)\right] \Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v) d_n$$

$$\leq (1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1),$$

since $k(s) \in \mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$.

Hence by (2.1), $F(s) \in \mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$.

7. Integral Means Inequalities

Theorem (7.1): Let $\omega > 0$. If $k(s) \in \mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ and suppose that $k_c(s)$ is defined by

$$k_c(s) = s^p - \frac{(1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1)}{\left[(\delta c - \delta + 1)\big(c(1+\alpha)\big) - (\alpha p + \gamma)(\delta c^2 - c^2\tau - \delta c + 2c\tau - \tau + 1)\right] \Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)} s^c.$$

$$(c \geq p+1; p \in \mathbb{N}),$$

If a holomorphic function $w(s)$ is created and defined based on

$$\big(w(s)\big)^{c-p} = \frac{\left[(\delta c - \delta + 1)\big(c(1+\alpha)\big) - (\alpha p + \gamma)(\delta c^2 - c^2\tau - \delta c + 2c\tau - \tau + 1)\right] \Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{(1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1)} \sum_{n=p+1}^{\infty} d_n s^{n-p}.$$

Then, for $s = re^{i\theta}$ and $(0 < r < 1)$,

$$\int_0^{2\pi} |k(s)|^{\omega} \, d\theta \leq \int_0^{2\pi} |k_c(s)|^{\omega} \, d\theta \,, \qquad (\omega > 0). \tag{7.1}$$

Proof: We show that

$$\int_0^{2\pi} \left| 1 - \sum_{n=p+1}^{\infty} d_n s^{n-p} \right|^{\omega} d\theta$$

$$\leq \int_0^{2\pi} \left| 1 - \frac{(1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1)}{[(\delta c - \delta + 1)(c(1+\alpha)) - (\alpha p + \gamma)(\delta c^2 - c^2\tau - \delta c + 2c\tau - \tau + 1)]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)} s^{c-p} \right|^{\omega} d\theta.$$

By applying Lemma (1.3), it suffices to show that

$$1 - \sum_{n=p+1}^{\infty} d_n s^{n-p} \prec 1 - \frac{(1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1)}{[(\delta c - \delta + 1)(c(1+\alpha)) - (\alpha p + \gamma)(\delta c^2 - c^2\tau - \delta c + 2c\tau - \tau + 1)]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)} s^{c-p}.$$

Set

$$1 - \sum_{n=p+1}^{\infty} d_n s^{n-p}$$

$$= 1 - \frac{(1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1)}{[(\delta c - \delta + 1)(c(1+\alpha)) - (\alpha p + \gamma)(\delta c^2 - c^2\tau - \delta c + 2c\tau - \tau + 1)]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)} (w(s))^{c-p}.$$

We find that

$$(w(s))^{c-p} = \frac{[(\delta c - \delta + 1)(c(1+\alpha)) - (\alpha p + \gamma)(\delta c^2 - c^2\tau - \delta c + 2c\tau - \tau + 1)]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{(1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1)} \sum_{n=p+1}^{\infty} d_n s^{n-p},$$

which readily yield $w(0) = 0$.

Furthermore using (2.1), we obtain

$$|w(s)|^{c-p} = \left| \frac{[(\delta c - \delta + 1)(c(1+\alpha)) - (\alpha p + \gamma)(\delta c^2 - c^2\tau - \delta c + 2c\tau - \tau + 1)]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{(1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1)} \sum_{n=p+1}^{\infty} d_n s^{n-p} \right|$$

$$\leq |s| \left| \sum_{n=p+1}^{\infty} \frac{[(\delta c - \delta + 1)(c(1+\alpha)) - (\alpha p + \gamma)(\delta c^2 - c^2\tau - \delta c + 2c\tau - \tau + 1)]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{(1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1)} d_n \right| \leq |s| < 1.$$

Next is the proof for the first derivative.

Theorem (7.2): Suppose that $\omega > 0$. If $k(s) \in \mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ and

$$k_c(s) = s^p - \frac{(1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1)}{[(\delta c - \delta + 1)(c(1+\alpha)) - (\alpha p + \gamma)(\delta c^2 - c^2\tau - \delta c + 2c\tau - \tau + 1)]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)} s^c,$$

$$(c \geq p+1; p \in \mathbb{N}).$$

Then, for $s = re^{i\theta}$ and $(0 < r < 1)$,

$$\int_0^{2\pi} |k'(s)|^{\omega} d\theta \leq \int_0^{2\pi} |k_c'(s)|^{\omega} d\theta, \qquad (\omega > 0). \tag{7.2}$$

Proof: It's sufficient to show that

$$1 - \sum_{n=p+1}^{\infty} \frac{n}{p} d_n s^{n-p}$$

$$\prec 1-\frac{c\big((1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)\big)}{p\big[(\delta c-\delta+1)\big(c(1+\alpha)\big)-(\alpha p+\gamma)(\delta c^2-c^2\tau-\delta c+2c\tau-\tau+1)\big]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}s^{c-p}.$$

This follows because

$$|w(s)|^{c-p}=\left|\frac{p\big[(\delta c-\delta+1)\big(c(1+\alpha)\big)-(\alpha p+\gamma)(\delta c^2-c^2\tau-\delta c+2c\tau-\tau+1)\big]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{c\big((1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)\big)}\sum_{n=p+1}^{\infty}\frac{n}{p}d_n s^{n-p}\right|$$

$$\le |s|\left|\sum_{n=p+1}^{\infty}\frac{\big[(\delta c-\delta+1)\big(c(1+\alpha)\big)-(\alpha p+\gamma)(\delta c^2-c^2\tau-\delta c+2c\tau-\tau+1)\big]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)}d_n\right|\le|s|<1.$$

Theorem (7.3): Let $g(s)=s^p-\sum_{n=p+1}^{\infty}b_n s^n$, $(s\in\mathbb{U}; b_n\ge 0; n\ge p+1; p\in\mathbb{N})$

and $k(s)\in\ \mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ for $c\in\mathbb{N}$,

$$\frac{Q_c}{b_c}=\min_{n=p+1}\frac{Q_n}{b_n},$$

where

$$Q_n=\frac{\big[(\delta n-\delta+1)\big(n(1+\alpha)\big)-(\alpha p+\gamma)(\delta n^2-n^2\tau-\delta n+2n\tau-\tau+1)\big]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)}.$$

Also, for $c\in\mathbb{N}$, the functions k_c and g_c be defined by

$$k_c(s)=s^p-\frac{(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)}{\big[(\delta c-\delta+1)\big(c(1+\alpha)\big)-(\alpha p+\gamma)(\delta c^2-c^2\tau-\delta c+2c\tau-\tau+1)\big]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}s^c,$$

and

$$g_c(s)=s^p-b_c s^c. \tag{7.3}$$

If there exists a holomorphic function

$$\big(w(s)\big)^{c-p}=\frac{\big[(\delta c-\delta+1)\big(c(1+\alpha)\big)-(\alpha p+\gamma)(\delta c^2-c^2\tau-\delta c+2c\tau-\tau+1)\big]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)\ b_c}\sum_{n=p+1}^{\infty}d_n b_n s^{n-p}.$$

Then, for $\omega>0, s=re^{i\theta}$ and $(0<r<1)$,

$$\int_0^{2\pi}|(k*g)(s)|^{\omega}\,d\theta\le\int_0^{2\pi}|(k_c*g_c)(s)|^{\omega}\,d\theta,\qquad(\omega>0).$$

Proof: Since

$$(k*g)(s)=s^p-\sum_{n=p+1}^{\infty}d_n b_n s^n.$$

from (7.3), we have

$$(k_c * g_c)(s) = s^p - \frac{(1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1)\, b_c}{\left[(\delta c - \delta + 1)\big(c(1+\alpha)\big) - (\alpha p + \gamma)(\delta c^2 - c^2\tau - \delta c + 2c\tau - \tau + 1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)} s^c.$$

We need to prove the Theorem by taking $\omega > 0, s = re^{i\theta}$ and $(0 < r < 1)$ such that

$$\int_0^{2\pi} \left|1 - \sum_{n=p+1}^{\infty} d_n b_n s^{n-p}\right|^{\omega} d\theta \leq$$

$$\int_0^{2\pi} \left|1 - \frac{\big((1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1)\big)\, b_c}{\left[(\delta c - \delta + 1)\big(c(1+\alpha)\big) - (\alpha p + \gamma)(\delta c^2 - c^2\tau - \delta c + 2c\tau - \tau + 1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)} s^{c-p}\right|^{\omega} d\theta.$$

Lemma (1.3) may be used to demonstrate that

$$1 - \sum_{n=p+1}^{\infty} d_n b_n s^{n-p}$$

$$\prec 1 - \frac{\big((1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1)\big)\, b_c}{\left[(\delta c - \delta + 1)\big(c(1+\alpha)\big) - (\alpha p + \gamma)(\delta c^2 - c^2\tau - \delta c + 2c\tau - \tau + 1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)} s^{c-p}. \tag{7.4}$$

If (7.4), holds, then there exists a holomorphic function w(s)

$$1 - \sum_{n=p+1}^{\infty} d_n b_n s^{n-p}$$

$$= 1 - \frac{\big((1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1)\big)\, b_c}{\left[(\delta c - \delta + 1)\big(c(1+\alpha)\big) - (\alpha p + \gamma)(\delta c^2 - c^2\tau - \delta c + 2c\tau - \tau + 1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)} \big(w(s)\big)^{c-p}.$$

We have

$$\big(w(s)\big)^{c-p} = \frac{\left[(\delta c - \delta + 1)\big(c(1+\alpha)\big) - (\alpha p + \gamma)(\delta c^2 - c^2\tau - \delta c + 2c\tau - \tau + 1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{\big((1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1)\big)\, b_c} \sum_{n=p+1}^{\infty} d_n b_n s^{n-p},$$

Then $w(0) = 0$.

From (2.1) we have

$$|w(s)|^{c-p} = \left|\frac{\left[(\delta c - \delta + 1)\big(c(1+\alpha)\big) - (\alpha p + \gamma)(\delta c^2 - c^2\tau - \delta c + 2c\tau - \tau + 1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{\big((1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1)\big)\, b_c} \sum_{n=p+1}^{\infty} d_n b_n s^{n-p}\right|$$

$$\leq |s| \left|\frac{\left[(\delta c - \delta + 1)\big(c(1+\alpha)\big) - (\alpha p + \gamma)(\delta c^2 - c^2\tau - \delta c + 2c\tau - \tau + 1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}{\big((1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1)\big)\, b_c} \sum_{n=p+1}^{\infty} d_n b_n\right| \leq |s| < 1.$$

8. Neighborhood Property

Now we define the $(n-\varepsilon)$ −neighborhoods for the function $k(s) \in \mathcal{M}_p$ by

$$N_{n,\varepsilon}(k)=\left\{g\in\mathcal{M}_p: g(s)=s^p-\sum_{n=p+1}^{\infty}b_ns^n \ \ and \ \sum_{n=p+1}^{\infty}n|d_n-b_n|\le\varepsilon\ ,0\le\varepsilon<1\right\}. \quad (8.1)$$

For identity function $e(s)=s^p, \quad (p\in\mathbb{N})$

$$N_{n,\varepsilon}(e)=\left\{g\in\mathcal{M}_p: g(s)=s^p-\sum_{n=p+1}^{\infty}b_ns^n \ \ and \ \sum_{n=p+1}^{\infty}n|b_n|\le\varepsilon\ ,0\le\varepsilon<1\right\}. \quad (8.2)$$

The concept of neighborhoods was first introduced by Goodman [5] and the generalized by Ruscheweyh [10].

Definition (8.1): A function $k(s)\in\mathcal{M}_p$ is said to be in the class $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$, if there exist a function $g(s)\in\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ such that

$$\left|\frac{k(s)}{g(s)}-1\right|<p-\sigma \qquad (s\in\mathbb{U},0\le\sigma<1).$$

Theorem (8.1): If $g(s)\in\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ and

$$\sigma=p-\frac{\varepsilon\Big([(1+\delta p)(p+1)(1+\alpha)-(\alpha p+\gamma)(\delta p^2-\tau(p+1)^2+\delta p+2(p+1)\tau-\tau+1)]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\Big)}{(p+1)\left[\begin{array}{c}[(1+\delta p)(p+1)(1+\alpha)-(\alpha p+\gamma)(\delta p^2-\tau(p+1)^2+\delta p+2(p+1)\tau-\tau+1)]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\\-(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)\end{array}\right]}.$$

Then $N_{n,\varepsilon}(g)\subset\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$.

Proof: Let $k(s)\in N_{n,\varepsilon}(g)$. Then we have from (8.1) that

$$\sum_{n=p+1}^{\infty}n|d_n-b_n|\le\varepsilon,$$

this indicates the following coefficient inequality with ease.

$$\sum_{n=p+1}^{\infty}|d_n-b_n|\le\frac{\varepsilon}{p+1}.$$

Next, since $g(s)\in\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$, we have from Theorem (2.1)

$$\sum_{n=p+1}^{\infty}b_n\le\frac{(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)}{[(1+\delta p)(p+1)(1+\alpha)-(\alpha p+\gamma)(\delta p^2-\tau(p+1)^2+\delta p+2(p+1)\tau-\tau+1)]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)}.$$

so that

$$\left|\frac{k(s)}{g(s)}-1\right|\le\frac{\sum_{n=1+p}^{\infty}|d_n-b_n|}{1-\sum_{n=1+p}^{\infty}b_n}$$

$$\le\frac{\varepsilon\Big([(\delta p+1)(1+p)(1+\alpha)-(\alpha p+\gamma)(\delta p^2-\tau(p+1)^2+\delta p+2(p+1)\tau-\tau+1)]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\Big)}{(p+1)\left[\begin{array}{c}[(1+\delta p)(p+1)(1+\alpha)-(\alpha p+\gamma)(\delta p^2-\tau(p+1)^2+\delta p+2(p+1)\tau-\tau+1)]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\\-(1+\alpha)(\delta p^2-\delta p+p)-(\alpha p+\gamma)(\delta p^2-\tau p^2-\delta p+2\tau p-\tau+1)\end{array}\right]}$$

$=p-\sigma.$

Then by Definition (8.1), $k(s) \in \mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ for each σ given by (8.3).

9. Convex Set

Theorem (9.1): The class $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$ is convex set.

proof: Suppose that the functions (k and g) be in the class $\mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma)$. Then for every $0 \le \Gamma \le 1$ we must show that

$$(1-\Gamma)k(s) + \Gamma g(s) \in \mathcal{RK}_{\lambda,\mu}^{m,p}(\alpha,\delta,\tau,\gamma). \tag{9.1}$$

We have

$$(1-\Gamma)k(s) + \Gamma g(s) = s^p - \sum_{n=p+1}^{\infty} [(1-\Gamma)d_n + \Gamma b_n]s^n.$$

So by Theorem (2.1), we get

$$\sum_{n=p+1}^{\infty} \left[(\delta n - \delta + 1)\big(n(1+\alpha)\big) - (\alpha p + \gamma)(\delta n^2 - n^2\tau - \delta n + 2n\tau - \tau + 1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,[(1-\Gamma)d_n + \Gamma b_n]$$

$$(1-\Gamma)\sum_{n=p+1}^{\infty} \left[(\delta n - \delta + 1)\big(n(1+\alpha)\big) - (\alpha p + \gamma)(\delta n^2 - n^2\tau - \delta n + 2n\tau - \tau + 1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)d_n$$

$$+\Gamma\sum_{n=p+1}^{\infty} \left[(\delta n - \delta + 1)\big(n(1+\alpha)\big) - (\alpha p + \gamma)(\delta n^2 - n^2\tau - \delta n + 2n\tau - \tau + 1)\right]\Omega_{\lambda,\mu}^{m,p}(\xi,\beta,\eta,v)\,b_n$$

$$\le (1-\Gamma)[(1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1)]$$

$$+\Gamma[(1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1)]$$

$$= (1+\alpha)(\delta p^2 - \delta p + p) - (\alpha p + \gamma)(\delta p^2 - \tau p^2 - \delta p + 2\tau p - \tau + 1).$$

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