

Jordan Bi- (Homomorphism, Left (Right) α -Centralizer) Mappings on a Prime Rings

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ABSTRACT

In this paper, we focus on the study of some classes of bi-derivation. In addition, there are new concepts that have been introduced, including functions bi- (homomorphism, α -centralizer) and Jordan bi- (homomorphism, α -centralizer) on the prime ring $R \times R$, particularly with reference to the elementary rings' permutation structure $L = R \times R$. The concepts of bi- (homomorphism, left (right) α -centralizer), Jordan bi- (homomorphism, left (right) α -centralizer), and triple bi- (homomorphism, left(right) α -centralizer) within the prime ring L are also introduced. Furthermore, we investigate an important condition that states if the ring is 2-torsion free ring, then any Jordan bi- (homomorphism, left α -centralizer) on L is necessarily a Jordan triple bi- (homomorphism, left α -centralizer).

MSC..

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1. Introduction

Throughout this paper, R is a ring and is called prime if $nRm = (0)$ implies $n = 0$ or $m = 0$, and R is called semi prime if $hRh = (0)$ implies $h = 0$, for all $h \in R$. An additive mapping $P : R \rightarrow R$ is called left (right) centralizer of R if $P(sb) = P(s)b$ is resp.) $P(sb) = sP(b)$.

Maksa G. [1] introduced the concept of a symmetric bi-derivation. It was shown in [2,3,4] that symmetric bi-derivation are related to general solution of some functional equations. Some results on symmetric bi-derivation in prime and semi prime rings can be found in [5, 6].

Also, Mahmood in [7], Mahmood and Hussein in [8] introduced a generalization of the traces of symmetric bi-derivation mappings, the effects must satisfy certain conditions at the ideal level of prime and semi-prime rings. A bi-additive mapping P is called symmetric, if $P(\omega, s) = P(s, \omega)$ where $\omega, s \in R$. In [9] a bi additive mapping P is called a symmetric left (right) bi-centralizer, if satisfy: $\Psi(gk, h) = \Psi(g, h)k$ and $\Psi(g, kh) = \Psi(g, k)h$ respectively $\Psi(gk, h) =$

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$g\Psi(k, h)$ and $\Psi(g, kh) = k\Psi(g, h)$ where $g, k, h \in R$. Yilmaz and Zekiye in [10], proved that every Jordan bi-derivation of prime ring R is an ordinary bi-derivation. Further in [10], a bi-additive mapping P is said to be a symmetric Jordan bi-derivation if satisfy: $\Psi(g^2, k) = \Psi(g, k)g + g\Psi(g, k)$ for all $g, k \in R$.

Then a bi-additive mapping $\mathcal{F}: R \times R \rightarrow R$ is named to be a bi-centralizer, if \mathcal{F} is a left and right bi-centralizer in both arguments. We follow in [11] a mapping $O: R \rightarrow R$ which is defined by $O(m) = P(m, m)$ is said to be trace of P for all $m \in R$, where $P: R \rightarrow R$ is bi-additive mapping, the relation between symmetric and trace is $O(u + b) = P(u + b, u + b) = O(u) + O(b) + P(u, b) + P(b, u)$ for all $u, b \in R$. Several studies have introduced new concepts about bi-effect linear functions. see [12,13]

In this paper we prove that every Jordan bi-(homomorphism, left α -centralizer) of 2-torsion free prime ring L is Jordan triple bi-(homomorphism, left α -centralizer) of L .

2. Bi- (Homomorphism, Left (Right) α -centralizer) on Prime Rings

Definition 2.1: Let Ψ be bi-additive mapping of the ring $L=R \times R$ then Ψ is named:

i) bi- (homomorphism, left(right) α -centralizer) of R if for each $v, k, e, g \in R$ then

$$\Psi(vk, eg) = (\Psi(v), \Psi(e))(\Psi(k), \alpha(g)).$$

$$(\text{resp. right } \Psi(vk, eg) = (\Psi(v), \alpha(e))(\Psi(k), \Psi(g)).$$

ii) Ψ is named Jordan bi- (homomorphism, left(right) α -centralizer) of R for each $v, e \in R$ then

$$\Psi(v^2, e^2) = (\Psi(v), \Psi(e))(\Psi(v), \alpha(e)).$$

iii) Ψ is named Jordan triple bi- (homomorphism, left(right) α -centralizer) of R if

for each $v, b, w, e \in R$.

$$\Psi(vkv, ege) = (\Psi(v), \Psi(e))(\Psi(k), \alpha(g))(\Psi(v), \alpha(e))$$

(resp. right $\Psi(vkv, ege) = (\Psi(v), \alpha(e))(\Psi(k), \alpha(g))(\Psi(v), \Psi(e)).$

3. Main Results

Lemma 2.2: Let Ψ be a Jordan bi- (homomorphism, left (right) α -centralizer) of the ring $R \times R$, for every $v, k, e, g \in R$, and α -automorphism of R , then:

$$\Psi(vk + kv, eg + ge) = (\Psi(v), \Psi(e))(\Psi(k), \alpha(g)) + (\Psi(k), \Psi(g))(\Psi(v), \alpha(e))$$

Proof:

$$\Psi((v + k)(v + k), (e + g)(e + g))$$

$$= \Psi((v + k)^2, (e + g)^2)$$

$$= (\Psi(v + k), \Psi(e + g))(\Psi(v + k), \alpha(e + g))$$

Since Ψ be a Jordan bi- (homomorphism, left (right) α -centralizer)

$$= (\Psi(v) + \Psi(k), \Psi(e) + \Psi(g))(\Psi(v) + \Psi(k), \alpha(e + g))$$

$$= (\Psi(v), \Psi(e))(\Psi(v), \alpha(e)) + (\Psi(k), \Psi(g))(\Psi(v), \alpha(e))$$

$$+ (\Psi(v), \Psi(e))(\Psi(k), \alpha(g)) + (\Psi(k), \Psi(g))(\Psi(k), \alpha(g)) \tag{1}$$

On the other hand

$$\Psi((v + k)(v + k), (e + g)(e + g))$$

$$= \Psi(v^2 + vk + kv + k^2, e^2 + eg + ge + g^2)$$

$$= \Psi(v^2, e^2) + \Psi(k^2, g^2) + \Psi(vk + kv, eg + ge)$$

$$= (\Psi(v), \Psi(e))(\Psi(v), \alpha(e)) + (\Psi(k), \Psi(g))(\Psi(k), \alpha(g))$$

$$+ \Psi(vk + kv, eg + ge), \tag{2}$$

Comparing (1) and (2) we get:

$$\Psi(vk + kv, eg + ge) = (\Psi(v), \Psi(e))(\Psi(k), \alpha(g)) + (\Psi(k), \Psi(g))(\Psi(v), \alpha(e)).$$

Remark 2.3: Let Ψ be Jordan bi- (homomorphism, Left(right) α -Centralizer) of the ring R and

α -automorphism of R then we define φ on $R \times R$ by:

$$\varphi(v, k, e, g) = \Psi(vk, eg) - (\Psi(v), \Psi(e))(\Psi(k), \alpha(g)).$$

Lemma 2.4: Let Ψ be Jordan bi- (homomorphism, left(right) α -centralizer) of the ring R and φ automorphism of R then:

$$(1) - \varphi(v + x, k, e, g) = \varphi(v, k, e, g) + \varphi(x, k, e, g)$$

$$(2) - \varphi(v, k + x, e, g) = \varphi(v, k, e, g) + \varphi(v, x, e, g)$$

$$(3) - \varphi(v, k, e + x, g) = \varphi(v, k, e, g) + \varphi(v, k, x, g)$$

$$(4) - \varphi(v, k, e, g + x) = \varphi(v, k, e, g) + \varphi(v, k, e, x)$$

(5) $-\varphi(vk, eg) = -\varphi(kv, ge)$ where $v, k, e, g \in R$.

Proof (1)

$$\begin{aligned} \varphi(v, k + x, e, g) &= \Psi(v(k + x), eg) - (\Psi(v), \Psi(e))(\Psi(k + x), \alpha(g)). \text{ {by remark 2.3}} \\ &= \Psi(vk + vx, eg) - (\Psi(v) + \Psi(e), \Psi(e))(\Psi(k + x), \alpha(g)) \\ &= \Psi(vk, eg) - (\Psi(v), \Psi(e))(\Psi(k), \alpha(g)) + \Psi(vx, eg) - \\ &\quad (\Psi(v), \Psi(e))(\Psi(x), \alpha(g)) \\ &= \varphi(v, k, e, g) + \varphi(v, x, e, g). \end{aligned}$$

Proof (2)

$$\begin{aligned} \varphi(v, k + x, e, g) &= \Psi(v(k + x), eg) - (\Psi(v), \Psi(e))(\Psi(k + x), \alpha(g)) \text{ {by remark 2.3}} \\ &= \Psi(vk + vx, eg) - (\Psi(v) + \Psi(e), \Psi(e))(\Psi(k + x), \alpha(g)) \\ &= \Psi(vk, eg) - (\Psi(v), \Psi(e))(\Psi(k), \alpha(g)) + \Psi(vx, eg) - (\Psi(v), \Psi(e))(\Psi(x), \alpha(g)) \\ &= \varphi(v, k, e, g) + \varphi(v, x, e, g). \end{aligned}$$

Proof (3)

$$\begin{aligned} \varphi(v, k, e + x, g) &= \Psi(vk, (e + x)g) - (\Psi(v), \Psi(e + x))(\Psi(k), \alpha(g)) \text{ {by remark 2.3}} \\ &= \Psi(vk, eg + xg) - (\Psi(v), \Psi(e) + (\Psi(x)))(\Psi(k), \alpha(g)) \\ &= \Psi(vk, eg) - (\Psi(v), \Psi(e))(\Psi(k), \alpha(g)) + \Psi(vk, xg) \\ &\quad - (\Psi(v), \Psi(x))(\Psi(k), \alpha(g)) \\ &= \varphi(v, k, e, g) + \varphi(v, k, x, g). \end{aligned}$$

Proof (4)

$$\begin{aligned} \varphi(v, k, e, g + x) &= \Psi(vk +, e(g + x)) - (\Psi(v), \Psi(e))(\Psi(k), g + x) \text{ {by remark 2.3}} \\ &= \Psi(vk, eg + ex) - (\Psi(v) + \Psi(e), \Psi(e))(\Psi(k), g + x) \\ &= \Psi(vk, eg) - (\Psi(v), \Psi(e))(\Psi(k), g) + \Psi(vk, eg) - (\Psi(v), \Psi(e))(\Psi(g), x) \\ &= \varphi(v, k, e, g) + \varphi(v, k, e, x). \end{aligned}$$

Proof (5)

by using lemma 2.2 we have :

$$\begin{aligned} \Psi(vk + kv, eg + ge) &= (\Psi(v), \Psi(e))(\Psi(k), \alpha(g)) + (\Psi(k), \Psi(g))(\Psi(v), \alpha(e)) \\ \Psi(vk, eg) + \Psi(kv, ge) &= (\Psi(v), \Psi(e))(\Psi(k), \alpha(g)) + (\Psi(k), \Psi(g))(\Psi(v), \alpha(e)) \end{aligned}$$

Consequently,

$$\Psi(vk, eg) - (\Psi(v), \Psi(e))(\Psi(k), \alpha(g)) = -\Psi(kv, ge) + (\Psi(k), \Psi(g))(\Psi(v), \alpha(e))$$

That implies that

$$\varphi(vk, eg) = -\varphi(kv, ge).$$

Remark 2.5: Let R be a ring then Ψ is bi-(homomorphism, α -centralizer) on L iff $\varphi(vk, eg) = 0$, for each $v, k, e, g \in R$. where α -automorphism of R .

Lemma 2.6

Let Ψ be a Jordan bi- (homomorphism, left(right) α - centralizer)

on L and let $\alpha: R \rightarrow R$ be an automorphism of R then:

$$\varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(v), \Psi(e))(\Psi(k), \alpha(g))] = 0, \text{ for every } v, k, e, g \in R.$$

Proof

Take $a = (vkjkv + kvjvk, eguge + geueg)$

$$\begin{aligned} \Psi(a) &= \Psi(v(kjk)v + k(vjv)k, e(gug)e + g(eue)g) \\ &= \Psi(v(kjk)v, e(gug)e) + \Psi(k(vjv)k, \alpha(g(eue)g)) \\ &= (\Psi(v), \Psi(e))(\Psi(kjk), \alpha(gug))(\Psi(v), \alpha(e)) \\ &\quad + (\Psi(k), \Psi(g))(\Psi(vjv), \alpha(eue))(\Psi(k), \alpha(g)) \\ &= (\Psi(v), \Psi(e))(\Psi(k)\Psi(j)\Psi(k), \alpha(gug))(\Psi(v), \alpha(e)) \\ &\quad + (\Psi(k), \Psi(g))(\Psi(v)\Psi(j)\Psi(v), \alpha(eue))(\Psi(k), \alpha(g)) \\ &= (\Psi(v), \Psi(e))(\Psi(k), \alpha(g))(\Psi(j), \alpha(u))(\Psi(k), \alpha(g))(\Psi(v), \alpha(e)) \\ &\quad + (\Psi(k), \Psi(g))((\Psi(v), \alpha(e))(\Psi(j), \alpha(u))(\Psi(v), \alpha(e))(\Psi(k), \alpha(g))), \end{aligned}$$

$$\text{for each } v, k, e, g, j, u \in R \quad (1)$$

On the other hand

$$\begin{aligned} a &= ((vk)j(kv), ((eg)u(ge)) + ((kv)j(vk)), ((ge)u(eg)) \\ \Psi(a) &= \Psi(((vk)j(kv), ((eg)u(ge)) + ((kv)j(vk)), ((ge)u(eg))) \\ &= \Psi(((vk)j(kv), ((eg)u(ge)))) + \Psi(((kv)j(vk)), ((ge)u(eg))) \\ &= (\Psi(vk), \Psi(eg))(\Psi(j), \alpha(u))(\Psi(kv), \alpha(ge)) \\ &\quad + (\Psi(kv), \Psi(ge))(\Psi(j), \alpha(u))(\Psi(vk), \alpha(eg)) \\ &= (\Psi(vk), \Psi(eg))(\Psi(j), \alpha(u))(\Psi(k)\Psi(v), \alpha(ge)) \\ &\quad + (\Psi(kv), \Psi(ge))(\Psi(j), \alpha(u))(\Psi(vk), \alpha(eg)) \\ &= (\Psi(vk), \Psi(eg))(\Psi(j), \alpha(u))(\Psi(kv), \alpha(ge)) \\ &\quad + (\Psi(kv), \Psi(ge))(\Psi(j), \alpha(u))(\Psi(vk), \alpha(eg)) \end{aligned}$$

$$\begin{aligned}
 &= (\Psi(v), \Psi(e))(\Psi(v), \Psi(g))(\Psi(k), \Psi(e))(\Psi(k), \Psi(g)) \\
 &= (\Psi(vk), \Psi(eg))(\Psi(j), \alpha(u)) (\Psi(k), \alpha(g))(\Psi(v), \alpha(e)) \\
 &\quad + (\Psi(kv), \Psi(ge))(\Psi(j), \alpha(u)) (\Psi(v), \alpha(e))(\Psi(k), \alpha(g)), \\
 &\qquad\qquad\qquad \text{for every } v, k, e, g, j, u \in R \qquad (2)
 \end{aligned}$$

Compare (1) and (2) we get:

$$\begin{aligned}
 0 &= (\Psi(vk), \Psi(eg)) - (\Psi(v), \Psi(e))(\Psi(k), \alpha(g))(\Psi(j), \alpha(u))(\Psi(k), \alpha(g))(\Psi(v), \alpha(e)) \\
 &\quad + (\Psi(kv), \Psi(ge)) - (\Psi(k), \Psi(g)) (\Psi(v), \alpha(e))(\Psi(j), \alpha(u))(\Psi(v), \alpha(e)) (\Psi(k), \alpha(g)) \\
 0 &= \varphi(v, k, e, g)(\Psi(j), \alpha(u))(\Psi(k), g)(\Psi(v), \alpha(e)) + \varphi(k, v, g, e)(\Psi(j), u)(\Psi(v), e)\Psi(k), \alpha(g)) \\
 0 &= \varphi(vk, eg)(\Psi(j), \alpha(u))[(\Psi(v), e)(\Psi(k), \alpha(g))].
 \end{aligned}$$

Lemma 2.7: Let Ψ be a Jordan bi left- (homomorphism, α -centralizer) on the ring R and let $\alpha: R \rightarrow R$ be an automorphism of R then:

$$\varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(n), \Psi(y))(\Psi(i), \alpha(t))] = 0. \text{ for all } v, k, e, g, n, y, i, t \in R.$$

Proof:

Replace v by $v + n$ in Lemma 2.6 we get:

$$\begin{aligned}
 &\varphi((v + n), k, e, g)(\Psi(j), \alpha(u))(\Psi(v + n), \Psi(e)), (\Psi(k), \alpha(g)), (\Psi(k), \Psi(g)) \\
 &\quad (\Psi(v + n), \alpha(e))) = 0
 \end{aligned}$$

That is

$$\begin{aligned}
 &\varphi((v + n), k, e, g)(\Psi(j), \alpha(u))[(\Psi(v + n), \alpha(e)), (\Psi(k), \alpha(g))] \\
 &= \varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(e)), (\Psi(k), \alpha(g))] + \\
 &\varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(n), \alpha(e)), (\Psi(k), \alpha(g))] + \\
 &\varphi(n, k, e, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(e)), (\Psi(k), \alpha(g))] + \\
 &\varphi(n, k, e, g)(\Psi(j), \alpha(u))[(\Psi(n), \alpha(e)), (\Psi(k), \alpha(g))]
 \end{aligned}$$

By lemma 2.6 we get

$$\begin{aligned}
 &\varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(n), \Psi(e)), (\Psi(k), \alpha(g))] + \\
 &\varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(n), \Psi(e)), (\Psi(k), \alpha(g))] = 0.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 &\varphi((v, k, e, g)(\Psi(j), \alpha(u))((\Psi(v), \Psi(e))((\Psi(k), \alpha(g)), (\Psi(k), \Psi(g)) \\
 &\quad + \varphi((v, k, e, g)(\Psi(j), \alpha(u))((\Psi(n), \Psi(e))(\Psi(k), \alpha(g)), (\Psi(k), \Psi(g))(\Psi(n), \alpha(e)))) \\
 &\varphi(n, k, e, g)(\Psi(j), \alpha(u))((\Psi(v), \Psi(e))(\Psi(k), g), (\Psi(k), \Psi(g))(\Psi(v), \alpha(e))) + \\
 &\varphi(n, k, e, g)(\Psi(j), \alpha(u)) \left((\Psi(n), \Psi(e))(\Psi(k), \alpha(g)), (\Psi(k), \alpha(g))(\Psi(n), \alpha(e)) \right) = 0
 \end{aligned}$$

by using Lemma 2.6 implies that:

$$\begin{aligned} & \varphi(v, k, e, g)(\Psi(j), \alpha(u))((\Psi(n), \Psi(e))(\Psi(k), \alpha(g)), (\Psi(k), \Psi(g))(\Psi(n), \alpha(e))) + \\ & \varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(v+n), \alpha(e)), (\Psi(k), \alpha(g))] = 0 \end{aligned} \tag{1}$$

Therefore,

$$\begin{aligned} & \varphi(v, k, e, g)(\Psi(j), \alpha(u))((\Psi(n), \Psi(e))(\Psi(k), \alpha(g)), (\Psi(k), \Psi(g))(\Psi(n), \alpha(e))) \\ & (\Psi(j), \alpha(u))\varphi(n, k, e, g)(\Psi(j), \alpha(u))((\Psi(n), \Psi(e))(\Psi(k), \alpha(g)), (\Psi(k), \Psi(g))(\Psi(n), \alpha(e))) = 0 \end{aligned}$$

By using (1), we get:

$$\begin{aligned} & \varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(k), \Psi(g))(\Psi(v), \alpha(e))] = 0, \\ & -\varphi(v, k, e, g)(\Psi(j), \alpha(u))((\Psi(n), \Psi(e))(\Psi(k), \alpha(g)), (\Psi(k), \Psi(g))(\Psi(n), \alpha(e))) \\ & (\Psi(j), \alpha(u))\varphi(n, k, e, g)(\Psi(j), \alpha(u))((\Psi(n), \Psi(e))(\Psi(k), \alpha(g)), (\Psi(k), \Psi(g))(\Psi(n), \alpha(e))) = 0 \end{aligned}$$

For prime ness of R, implies that:

$$\varphi(k, e, g)(\Psi(j), \alpha(u))(\Psi(n), \Psi(e))(\Psi(k), \alpha(g)), (\Psi(k), \Psi(g))(\Psi(n), \alpha(e)) = 0. \tag{2}$$

Replace e by $e + i$ in lemma 2.6

$$\begin{aligned} & \varphi(v, k, (e + i), g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(e + i)), (\Psi(k), \alpha(g))] = 0 \\ & \varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(e)), \Psi(k), \alpha(g)] + \\ & \varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(i)), \Psi(k), \alpha(g)] + \\ & \varphi(v, k, i, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(e)), \Psi(k), \alpha(g)] + \\ & \varphi(v, k, i, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(i)), \Psi(k), \alpha(g)] = 0 \end{aligned}$$

By lemma 2.6 we have

$$\begin{aligned} & \varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(i)), \Psi(k), \alpha(g)] + \\ & \varphi(v, k, i, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(e)), \Psi(k), \alpha(g)] = 0 \\ & \varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(i)), \Psi(h), \alpha(c)] \\ & \varphi(v, k, i, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(e)), \Psi(k), \alpha(g)] = 0 \\ & \varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(i)), \Psi(h), \alpha(c)] \\ & \varphi(v, k, i, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(e)), (\Psi(k), \alpha(g))] = 0 \end{aligned}$$

By prime ring we get

$$\varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(i)), (\Psi(k), \alpha(g))] = 0 \tag{3}$$

Once more , we putting k by $k + y$ in Lemma 2.6 we get :

$$\varphi(v, k + y, e, g)(\Psi(j), \alpha(u))[(\Psi(n), \alpha(e)), (\Psi(k + y), \alpha(g))] = 0$$

$$\begin{aligned} & \varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(n), \alpha(e)), \Psi(k), \alpha(g)] + \\ & \varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(n), \alpha(e)), \Psi(y), \alpha(g)] + \\ & \varphi(v, y, e, g)(\Psi(j), \alpha(u))[(\Psi(n), \alpha(e)), \Psi(k), \alpha(g)] + \\ & \varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(n), \alpha(e)), \Psi(y), \alpha(g)] = \mathbf{0} \end{aligned}$$

By lemma 2. 6

$$\begin{aligned} & \varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(n), \alpha(e)), \Psi(y), \alpha(g)] + \\ & \varphi(v, y, e, g)(\Psi(j), \alpha(u))[(\Psi(n), \alpha(e)), \Psi(k), \alpha(g)] = \mathbf{0} \\ & \varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(n), \alpha(e)), \Psi(y), \alpha(g)] \\ & (\Psi(h), \alpha(t))\varphi(v, y, e, g)(\Psi(j), \alpha(u))[(\Psi(n), \alpha(e)), \Psi(k), \alpha(g)] = \mathbf{0} \end{aligned}$$

By prime ring

$$\varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(n), \alpha(e)), (\Psi(y), \alpha(g))] = \mathbf{0} \quad (4)$$

By the same we can prove that:

$$\begin{aligned} & \varphi(v, k, e, g)(\Psi(j), \alpha(u))((\Psi(n), \Psi(e)) \\ & \quad (\Psi(y), \alpha(g)), (\Psi(y), \Psi(g))(\Psi(n), \alpha(e))) = \mathbf{0}. \quad (5) \end{aligned}$$

$$\begin{aligned} & \varphi(v, k, e, g)(\Psi(j), \alpha(u))((\Psi(v), \Psi(y)) \\ & \quad (\Psi(e), \alpha(t)), (\Psi(e), \Psi(t))(\Psi(v), \alpha(y))) = \mathbf{0}. \quad (6) \end{aligned}$$

$$\begin{aligned} & \varphi(v, k, e, g)(\Psi(j), \alpha(u))((\Psi(n), \Psi(y)) \\ & \quad (\Psi(i), \alpha(g)), (\Psi(i), \Psi(g))(\Psi(n), \alpha(y))) = \mathbf{0}. \quad (7) \\ & \varphi(v, k, e, g)(\Psi(j), \alpha(u))((\Psi(n), \Psi(e)) \end{aligned}$$

$$(\Psi(i), \alpha(t)), (\Psi(i), \Psi(t))(\Psi(n), \alpha(e))) = \mathbf{0}. \quad (8)$$

$$\begin{aligned} & \varphi(v, k, e, g)(\Psi(j), \alpha(u))((\Psi(v), \Psi(e)) \\ & \quad (\Psi(i), \alpha(t)), (\Psi(i), \Psi(t))(\Psi(v), \alpha(e))) = \mathbf{0}. \quad (9) \end{aligned}$$

$$\begin{aligned} & \varphi(v, k, e, g)(\Psi(j), \alpha(u))((\Psi(v), \Psi(y)) \\ & \quad (\Psi(i), \alpha(t)), (\Psi(i), \Psi(t))(\Psi(v), \alpha(y))) = \mathbf{0}. \quad (10) \end{aligned}$$

Replace $g + t$ by g in lemma 2.6

$$\begin{aligned} & \varphi(v, k, e, g + t)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(e)), (\Psi(k), \alpha(g + t))] = \mathbf{0} \\ & \varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(e)), \Psi(k), \alpha(g)] + \\ & \varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(i)), \Psi(k), \alpha(t)] + \\ & \varphi(v, k, e, t)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(e)), \Psi(k), \alpha(g)] + \\ & \varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(e)), \Psi(k), \alpha(g)] = \mathbf{0} \end{aligned}$$

By lemma 2. 6 we have

$$\begin{aligned} &\varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(e)), \Psi(k), \alpha(t)] + \\ &\varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(e)), \Psi(k), \alpha(g)] = 0 \\ &\varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(e)), \Psi(k), \alpha(t)] \\ &(\Psi(j), \alpha(u))\varphi(v, k, e, t)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(e)), (\Psi(k), \alpha(g))] = 0 \\ &-\varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(e)), \Psi(k), \alpha(g)](\Psi(j), \alpha(u)) \\ &\varphi(v, k, e, t)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(e)), (\Psi(k), \alpha(g))] = 0 \end{aligned}$$

By prime ring

$$\begin{aligned} &\varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(v), \alpha(e)), (\Psi(k), \alpha(g))] = 0 \\ &\varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(v + n), \alpha(e + i)), (\Psi(k + y), \alpha(g + t))] = 0 \\ &-\varphi(v, k, e, g)(\Psi(j), \alpha(u))((\Psi(n), \Psi(y))(\Psi(i), \alpha(t)), (\Psi(i), \Psi(t))(\Psi(n), \alpha(y))) \\ &(\Psi(j), \alpha(u)) \varphi(n, k, e, g)(\Psi(j), \alpha(u)) \\ &((\Psi(v), \Psi(y))(\Psi(i), \alpha(t)), (\Psi(i), \Psi(t))(\Psi(v), \alpha(y))) = 0. \end{aligned}$$

$$\varphi(v, k, e, g)(\Psi(j), \alpha(u))((\Psi(n), \alpha(i))(\Psi(y), \alpha(t))) = 0.$$

Since R is prime ring then

$$\varphi(v, k, e, g)(\Psi(j), \alpha(u))[(\Psi(n), \Psi(y))(\Psi(i), \alpha(t))] = 0.$$

Theorem 2.8: Every Jordan Bi-(homomorphism, α -centralizer) of 2-torsion free prime ring R is bi-(homomorphism, α -Centralizer) of R and let $\alpha: R \rightarrow R$ automorphism of R.

Proof

Because Ψ is Jordan bi-(homomorphism, left (right) α -centralizer) of L and L is prime ring then by lemma 2.7 we have:

either $\varphi(v, k, e, g) = 0$ for every $v, k, e, g \in R$

or $[(\Psi(n), \Psi(y))(\Psi(i), \alpha(t))] = 0$, for every $n, y, i, t \in R$.

If $[(\Psi(n), \Psi(y))(\Psi(i), \alpha(t))] \neq 0, n, i, t, y \in R$.

We obtain $\varphi(v, k, e, g) = 0$ for every $v, k, e, g \in R$

by Remark 2.5

we have Ψ is bi- (homomorphism, left(right) α -centralizer) of L.

On the other hand

If $[(\Psi(n), \Psi(y))(\Psi(i), \alpha(t))] = 0$ for every $n, y, i, t \in R$ then L is commutative.

Now by using Lemma 2.2 we get:

$$\Psi(vk + kv, eg + ge) = (\Psi(v), \Psi(e))(\Psi(k), \alpha(g)) + (\Psi(k), \Psi(g))(\Psi(v), \alpha(e)).$$

So,

$$2\Psi(vk, eg) = 2(\Psi(v), \Psi(e))(\Psi(k), \alpha(g)),$$

Since L is 2-torsion free we have,

$$\Psi(v, k, e, g) = (\Psi(v), \Psi(e))(\Psi(k), \alpha(g))$$

Therefore Ψ is bi- (homomorphism, centralizer) of L.

Proposition 2.9: Each Jordan bi- (homomorphism, α -centralizer) of prime ring L is Jordan triple bi- (homomorphism, α -centralizer) of L and let $\alpha: R \rightarrow R$ automorphism of R.

Proof

Let Ψ be Jordan left bi- (homomorphism, α -centralizer), then for every $v, k, e, g \in R$, we get:

By replacing k by $vk + kv$ and g with $eg + ge$ in definition 2.1.i, we obtain:

$$\begin{aligned} \Psi(v(vk + kv), e(eg + ge)) &= (\Psi(v), \Psi(e))(\Psi(vk + kv), \alpha(eg + ge)) \\ &= (\Psi(v), \Psi(e))(\Psi(vk) + \Psi(kv), \alpha(eg + ge)) \\ &= (\Psi(v), \Psi(e))(\Psi(v)\Psi(k) + \Psi(k)\Psi(v), \alpha(eg + ge)) \\ &= (\Psi(v), \Psi(e))(\Psi(v), \alpha(e))(\Psi(k), \alpha(g)) \\ &\quad + (\Psi(v), \Psi(e))(\Psi(k), \alpha(g))(\Psi(v), \alpha(e)), \quad (1) \end{aligned}$$

On the other hand

$$\begin{aligned} \Psi(v(vk + kv), e(eg + ge)) &= \Psi(vvk + vkv, eeg + ege) \\ &= \Psi(vvk, eeg) + \Psi(vkv, ege) \\ &= (\Psi(v), \Psi(e))(\Psi(v), \alpha(e))(\Psi(k), \alpha(g)) + \Psi(vkv, ege), \quad (2) \end{aligned}$$

Comparing (1) and (2) we have,

$$\Psi(vkv, ege) = (\Psi(v), \Psi(e))(\Psi(k), \alpha(g))(\Psi(v), \alpha(e)).$$

Thus, Ψ is Jordan triple Bi- (homomorphism, left(right) α -Centralizer) on L.

3. Conclusions

In this paper we have proven that every Jordan bi- (homomorphism, left(resp. right) α -centralizer) of 2- torsion free prime ring L is bi-(homomorphism, left(resp. right) α -centralizer) of L and prove every Jordan bi- (homomorphism, left(resp. right) α -centralizer) of prime ring L is Jordan triple bi- (homomorphism, left(resp. right) α -centralizer) of L. Some results have been demonstrated along the lines of the well-known which have been demonstrated in the case of bi-centralizer mappings on a prime rings by several authors.

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