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Some results about Subordination theory for Multivalent functions that defined by differential operator of Special Case of Jin-Owa integral operator

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ABSTRACT

For multivalent function h lie in $\mathcal{A}(\mathrm{n})$ defined by using differential operator for a special case of Jin and Owa in the unit disk, a new generalization of linear derivate operator $Z_{\mathrm{n}}^{\mu,\alpha}h(z)$ introduced in this current scientific discussion ,we have achieved some results about differential subordination and superordination. Through investigating the appropriate class of admissible functions we obtained these results and some important results about the sandwich theorem .

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1. Introduction

Let $\mathcal{A}(\mathrm{n})$ denote the class of all analytic functions of the form:

$$h(z) = z^{\mathrm{n}} + \sum_{l=1}^{\infty} s_{l+\mathrm{n}} z^{l+\mathrm{n}} \quad (z \in \mathfrak{U}), \tag{1.1}$$

where $\mathfrak{U} = \{z \in \mathbb{C}: |z| < 1\}$ is the unit disk and $\mathcal{A}(1) = \mathcal{A}$. Let $h(z)$ and $H(z)$ are two analytic functions in $\mathcal{H}(\mathfrak{U})$. We say that the function $h(z)$ is subordinate [14] to $H(z)$ or $H(z)$ is a superordinate to $h(z)$ in such a case we write $h(z) \prec H(z)$ if there exist analytic function $\mathcal{M}(z)$ in $\mathfrak{U}$ with the properties $\mathcal{M}(0) = 0$ and $|\mathcal{M}(z)| < 1 (z \in \mathfrak{U})$, such that $h(z) = H(\mathcal{M}(z))$. If $H(z)$ is univalent, then $h(z) \prec H(z)$ if and only if $h(0) = H(0)$ and $h(\mathfrak{U}) \subset H(\mathfrak{U})$. The Hadamard product of the functions $h(z)$ given be (1.1) and $\mathcal{T}(z) = z^{\mathrm{n}} + \sum_{l=1}^{\infty} \eta_{l+\mathrm{n}} z^{l+n}$ is defined by

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$$(h*\mathcal{T})(z)=z^{\mathfrak{n}}+\sum_{l=1}^{\infty}s_{l+\mathfrak{n}}\eta_{l+\mathfrak{n}}z^{l+\mathfrak{n}}=(\mathcal{T}*h)(z). \quad (1.2)$$

For a function $h(z)$ in the class $\mathcal{A}(n)$ given be (1.1), Liu- Owa [9] introduce the linear operator $Q^{\alpha}_{\gamma,\mathfrak{n}}:\mathcal{A}(\mathfrak{n})\to\mathcal{A}(\mathfrak{n})$ as the formula

$$Q^{\alpha}_{\gamma,\mathfrak{n}}h(z)=\binom{\mathfrak{n}+\alpha+\gamma-1}{\mathfrak{n}+\gamma-1}\frac{\alpha}{z^{\gamma}}\int_0^z\left(1-\frac{t}{z}\right)^{\alpha-1}t^{\gamma-1}h(t)dt\,,$$

$(\alpha>0\,;\gamma>-1\,;\mathfrak{n}\in\mathbb{N})$, such that

$$Q^{\alpha}_{\gamma,\mathfrak{n}}h(z)=z^{\mathfrak{n}}+\sum_{l=1}^{\infty}\frac{\Gamma(\mathfrak{n}+\alpha+\gamma)\Gamma(\alpha+\gamma+l)}{\Gamma(\mathfrak{n}+\gamma)\Gamma(\mathfrak{n}+\alpha+\gamma+l)}s_{l+\mathfrak{n}}z^{l+\mathfrak{n}},$$

$(\alpha\geq 0\,;\gamma>-1\,;\mathfrak{n}\in\mathbb{N})$,

so as a special case when $\gamma=1$ this operator will have the following formula

$$Q^{\alpha}_{1,\mathfrak{n}}h(z)=Q^{\alpha}_{\mathfrak{n}}h(z)=z^{\mathfrak{n}}+\sum_{l=1}^{\infty}\frac{\Gamma(\mathfrak{n}+\alpha+1)\Gamma(\alpha+l+1)}{\Gamma(\mathfrak{n}+1)\Gamma(\mathfrak{n}+\alpha+l+1)}s_{l+\mathfrak{n}}z^{l+\mathfrak{n}}\;.$$

So the differential operator $\mathcal{Z}^{\mu,\alpha}_{n,\gamma}h(z)$ for $Q^{\alpha}_{1,\mathfrak{n}}h(z)$ will define as

$$\mathcal{Z}^{0,\alpha}_{n}h(z)=d^{0}_{n}\left(Q^{\alpha}_{1,\mathfrak{n}}h(z)\right)=Q^{\alpha}_{1,\mathfrak{n}}h(z)=z^{\mathfrak{n}}+\sum_{l=1}^{\infty}\frac{\Gamma(\mathfrak{n}+\alpha+1)\Gamma(\alpha+l+1)}{\Gamma(\mathfrak{n}+1)\Gamma(\mathfrak{n}+\alpha+l+1)}s_{l+n}z^{l+\mathfrak{n}}$$

$$\mathcal{Z}^{1,\alpha}_{n}h(z)=z^{\mathfrak{n}}+\sum_{l=1}^{\infty}\frac{(l+n)\Gamma(\mathfrak{n}+\alpha+1)\Gamma(\alpha+l+1)}{\mathfrak{n}\Gamma(\mathfrak{n}+1)\Gamma(\mathfrak{n}+\alpha+l+1)}s_{l+n}z^{l+\mathfrak{n}}$$

$$\vdots$$

$$\mathcal{Z}^{\mu,\alpha}_{\mathfrak{n}}h(z)=z^{\mathfrak{n}}+\sum_{l=1}^{\infty}\frac{(l+n)^{\mu}\Gamma(l+\mathfrak{n}+1)\Gamma(\mathfrak{n}+\alpha+1)}{n^{\mu}\Gamma(l+\mathfrak{n}+\alpha+1)\Gamma(\mathfrak{n}+1)}s_{l+\mathfrak{n}}z^{l+\mathfrak{n}}. \quad (1.3)$$

Also, it is easily verified from (1.3) that

$$z\left(\mathcal{Z}^{\mu,\alpha}_{n}h(z)\right)'=\mathfrak{n}\alpha\mathcal{Z}^{\mu-1,\alpha}_{\mathfrak{n}}h(z)-\mathfrak{n}(\alpha-1)\mathcal{Z}^{\mu,\alpha}_{\mathfrak{n}}h(z)\;, \quad (1.4)$$

where $\mu\in\mathbb{N}_o=\mathbb{N}\cup\{0\},\alpha\geq 0,\mathfrak{n}\in\mathbb{N}$.

Denote by $\mathcal{D}$ the set of all functions $\mathfrak{B}(z)$ that are univalent on $\partial\mathfrak{U}\setminus\mathcal{E}(\mathfrak{B})$ where $\mathcal{E}(\mathfrak{B})=\{\delta\in\partial\mathfrak{U}:\lim_{z\to\delta}\mathfrak{B}(z)=\infty\}$ and $\mathfrak{B}'(\delta)\neq 0$ for $\delta\in\partial\mathfrak{U}\setminus\mathcal{E}(\mathfrak{B})$.

Definition 1.1[14]. Let $\aleph\subset\mathbb{C}$ and a function $\mathfrak{B}$ belonging to $\in\mathcal{D}$, $\mathfrak{n}\in\mathbb{N}$. The functions $\psi:\mathbb{C}^3\times\mathfrak{U}\to\mathbb{C}$ that belonging to the class $\Psi_{\mathfrak{n}}[\aleph,\mathfrak{B}]$ of admissible functions that satisfy the admissibility provision

$\psi(a,b,c;z)\notin\aleph$,

whenever $a=\mathfrak{B}(\delta),b=e\delta\mathfrak{B}'(\delta),\ \mathcal{R}\left\{\frac{c}{b}+1\right\}\geq e\mathcal{R}\left\{1+\frac{\delta\mathfrak{B}''(\delta)}{\mathfrak{B}'(\delta)}\right\},$

where $z\in\mathfrak{U},\delta\in\partial\mathfrak{U}\setminus\mathcal{E}(\mathfrak{B})$ and $e\geq\mathfrak{n}$. $\Psi[\aleph,\mathfrak{B}]$ mean the class $\Psi_1[\aleph,\mathfrak{B}]$.

when $\mathfrak{B}(z)=\mathscr{p}\frac{\mathscr{p}z+r}{\mathscr{p}+\bar{r}z}$, with $\mathscr{p}>0$ and $|r|<\mathscr{p}$ as a particular case, then $\mathfrak{B}(u)=\mathfrak{U}_{\mathscr{p}}=\{\wp:|\wp|<\mathscr{p}\}$, $\mathfrak{B}(0)=a$, $\mathcal{E}(\mathfrak{B})=\phi$ and $\mathfrak{B}\in\mathcal{D}$. So we set $\Psi_{\mathfrak{n}}[\aleph,\mathscr{p},a]=\Psi_{\mathfrak{n}}[\aleph,\mathfrak{B}]$ in this case, and in extra special case the class is written as $\Psi_{\mathfrak{n}}[\mathscr{p},a]$ when $\aleph=\mathfrak{U}_{\mathscr{p}}$.

Definition 1.2[15] Let $\aleph \subset \mathbb{C}$ and $\mathfrak{B}(z)$ lie in the class $\mathcal{H}[a,\mathrm{n}]$ with $\mathfrak{B}'(z) \neq 0$.The functions $\psi: \mathbb{C}^3 \times \overline{\mathfrak{U}} \to \mathbb{C}$ that belonging to the class $\Psi_{\mathrm{n}}{}'[\aleph,\mathfrak{B}]$ of admissible functions that satisfy the admissibility condition

$$\psi(a,b,c;\delta) \in \aleph,$$

whenever $a = \mathfrak{B}(z)$, $b = \frac{z\mathfrak{B}'(z)}{t}$, $\mathcal{R}\left\{\frac{c}{b}+1\right\} \leq \frac{1}{t}\mathcal{R}\left\{1+\frac{z\mathfrak{B}''(z)}{\mathfrak{B}'(z)}\right\}$,

where $z \in \mathfrak{U}, \delta \in \partial\mathfrak{U}$, and $t \geq \mathrm{n} \geq 1$. In particular, we will denote the class $\Psi_1{}'[\aleph,\mathfrak{B}]$ by $\Psi'[\aleph,\mathfrak{B}]$.

Theorem 1.1[14] . Let $\Psi_{\mathrm{n}}[\aleph,\mathfrak{B}]$ be a class containing the functions ψ ,with $\mathfrak{B}(0) = a$. If the analytic function

$$\mathcal{T}(z) = a + a_{\mathrm{n}}z^{\mathrm{n}} + a_{\mathrm{n}+1}z^{\mathrm{n}+1} + \cdots \text{ , satisfies}$$

$$\psi(\mathcal{T}(z), z\mathcal{T}'(z), z^2\mathcal{T}''(z); z) \in \aleph,$$

then $\mathcal{T}(z) \prec \mathfrak{B}(z)$.

Theorem 1.2[15] . Let $\Psi'[\aleph,\mathfrak{B}]$ be a class containing the functions ψ with $\mathfrak{B}(0) = a$. If $\mathcal{T}(z) \in \mathcal{D}(a)$ and $\psi(\mathcal{T}(z), z\mathcal{T}'(z), z^2\mathcal{T}''(z); z)$ is analytic and injective in $\mathfrak{U}$, then

$$\aleph \subset \{\psi(\mathcal{T}(z), z\mathcal{T}'(z), z^2\mathcal{T}''(z); z) : z \in \mathfrak{U}\},$$

implies $\mathfrak{B}(z) \prec \mathcal{T}(z)$.

The present submition ,issues in the differential subordination and superordination [14] and [15], we discussed the implications of multivalent functions associated with the linear operator $\mathcal{Z}_n^{\mu,\alpha}$ hold that defined by an differential operator for a special case of Jin-Owa integral operator . A similar works to this work that include the multivalent analytic functions defined by some operators, we mention , generalization integral operator that studied by Atshan et al. [16], the Carlson-shaffer linear operator [4], the Ruschewcyh derivation operator [13],(see also [6],[7],[3],[1],[2],[7],[9],[5],[12],[10],[11], and [17]). Additionally, the corresponding differential superordination new results are investigated, and several sandwich-type results are obtained. I would like to point out that every analytic and injective function means a univalent function in this research.

2.Subordination results involving the linear operator $\mathcal{Z}_{\mathrm{n}}^{\mu,\alpha}$.

Definition 2.1. Suppose that $\aleph \subset \mathbb{C}$ and $\mathfrak{B}(z) \in \mathcal{D}_0 \cap \mathcal{H}[0,\mathrm{n}]$. The functions $\mathcal{X}: \mathbb{C}^3 \times \mathfrak{U} \to \mathbb{C}$ that belonging to the class $\Phi_{\mathcal{Z}_{\mathrm{n}}}[\aleph,\mathfrak{B}]$ of admissible functions that satisfy the admissibility provision

$$\mathcal{X}(u, v, w; z) \notin \aleph,$$

whenever $u = \mathfrak{B}(\delta)$, $v = \frac{e\delta\mathfrak{B}'(\delta) + \mathrm{n}(\alpha+1)\mathfrak{B}(\delta)}{\mathrm{n}\alpha}$,

$$\mathcal{R}\left\{\frac{(\mathrm{n}\alpha)^2 w - \mathrm{n}^2(\alpha+1)^2 u}{\mathrm{n}\alpha v - \mathrm{n}(\alpha+1)u} - 2\mathrm{n}(\alpha+1)\right\} \geq e\mathcal{R}\left\{\frac{\delta\mathfrak{B}''(\delta)}{\mathfrak{B}'(\delta)}+1\right\},$$

where $z \in \mathfrak{U}, \delta \in \partial\mathfrak{U} \setminus \mathcal{E}(\mathfrak{B})$, and $e \geq \mathrm{n}$.

Theorem 2.1. Consider $\mathcal{X} \in \Phi_{\mathcal{Z}_n}[\aleph,\mathfrak{B}]$. If $h(z) \in \mathcal{A}(\mathrm{n})$ satisfies

$$\left\{\mathcal{X}\left(\mathcal{Z}_n^{\mu,\alpha}h(z), \mathcal{Z}_n^{\mu-1,\alpha}h(z), \mathcal{Z}_n^{\mu-2,\alpha}h(z); z\right) : z \in \mathfrak{U}\right\} \subset \aleph, (\mu > 2, \alpha \geq 0, \mathrm{n} \in \mathbb{N}) \qquad (2.1)$$

then $\mathcal{Z}_{\mathrm{n}}^{\mu,\alpha}h(z) \prec \mathfrak{B}(z)$.

Proof. Suppose that $\mathcal{T}(z)$ be analytic function in $\mathfrak{U}$ have the form

$$\mathcal{T}(z) = \mathcal{Z}_{\mathrm{n}}^{\mu,\alpha} h(z), \tag{2.2}$$

so with respect to z if we differentiating (2.2) and using the relation in (1.4), we get

$$\frac{z\mathcal{T}'(z)+\mathrm{n}(\alpha-1)\mathcal{T}(z)}{\mathrm{n}\alpha} = \mathcal{Z}_{\mathrm{n}}^{\mu-1,\alpha} h(z). \tag{2.3}$$

Further computations show that

$$\frac{z^2\mathcal{T}''(z)+\left(1+2\mathrm{n}(\alpha-1)\right)z\mathcal{T}'(z)+\mathrm{n}^2(\alpha-1)^2\mathcal{T}(z)}{(\mathrm{n}\alpha)^2} = \mathcal{Z}_{\mathrm{n}}^{\mu-2,\alpha} h(z). \tag{2.4}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u(a,b,c) = a, v(a,b,c) = \frac{b+\mathrm{n}(\alpha-1)a}{\mathrm{n}\alpha}, w(a,b,c) = \frac{c+\left(1+2\mathrm{n}(\alpha-1)\right)b+\mathrm{n}^2(\alpha-1)^2 a}{(\mathrm{n}\alpha)^2}. \tag{2.5}$$

Let

$$\psi(a,b,c;z) = \mathcal{X}(u,v,w;z) = \mathcal{X}\left(a, \frac{b+\mathrm{n}(\alpha-1)a}{\mathrm{n}\alpha}, \frac{c+\left(1+2\mathrm{n}(\alpha-1)\right)b+\mathrm{n}^2(\alpha-1)^2 a}{(\mathrm{n}\alpha)^2}; z\right). \tag{2.6}$$

By theorem 1.1 and Equations (2.2)- (2.4), we obtain

$$\psi(\mathcal{T}(z), z\mathcal{T}'(z), z^2\mathcal{T}''(z); z) = \mathcal{X}\left(\mathcal{Z}_{\mathrm{n}}^{\mu,\alpha} h(z), \mathcal{Z}_{\mathrm{n}}^{\mu-1,\alpha} h(z), \mathcal{Z}_{\mathrm{n}}^{\mu-2,\alpha} h(z); z\right). \tag{2.7}$$

Hence (2.1), becomes

$$\psi(\mathcal{T}(z), z\mathcal{T}'(z), z^2\mathcal{T}''(z); z) \in \aleph.$$

Note that

$$\frac{c}{b} + 1 = \frac{(\mathrm{n}\alpha)^2 w - \mathrm{n}^2(\alpha-1)^2 u}{\mathrm{n}\alpha v - \mathrm{n}(\alpha-1)u} - 2\mathrm{n}(\alpha-1).$$

Hence the admissibility conditions for $\mathcal{X} \in \Phi_{\mathcal{Z}_{\mathrm{n}}}[\aleph, \mathfrak{B}]$ and for ψ are amounting to the admissibility condition as given in definition 1.1. Hence $\psi \in \Psi_{\mathcal{Z}_{\mathrm{n}}}[\aleph, \mathfrak{B}]$. According to Theorem 1.1,

$$\mathcal{T}(z) \prec \mathfrak{B}(z) \text{ or } \mathcal{Z}_{\mathrm{n}}^{\mu,\alpha} h(z) \prec \mathfrak{B}(z).$$

Then $\aleph = \mathcal{F}(\mathfrak{U})$ for some conformal mapping $\mathcal{F}(z)$ of $\mathfrak{U}$ onto $\aleph$ this case when $\aleph \neq \mathbb{C}$ is a simply connected domain, and we write the class $\Phi_{\mathcal{Z}_{\mathrm{n}}}[\mathcal{F}(\mathfrak{U}), \mathfrak{B}]$ as $\Phi_{\mathcal{Z}_{\mathrm{n}}}[\mathcal{F}, \mathfrak{B}]$.

As a result of the above theorem, we get the following result :

Theorem 2.2. Assume that $\mathcal{X} \in \Phi_{\mathcal{Z}_{\mathrm{n}}}[\mathcal{F}, \mathfrak{B}]$. If $h(z) \in \mathcal{A}(\mathrm{n})$ satisfies

$$\mathcal{X}\left(\mathcal{Z}_{\mathrm{n}}^{\mu,\alpha} h(z), \mathcal{Z}_{\mathrm{n}}^{\mu-1,\alpha} h(z), \mathcal{Z}_{\mathrm{n}}^{\mu-2,\alpha} h(z); z\right) \prec \mathcal{F}(z), \quad (\mu > 2, \alpha \geq 0, \mathrm{n} \in \mathbb{N}), \tag{2.8}$$

then $\mathcal{Z}_{\mathrm{n}}^{\mu,\alpha} h(z) \prec \mathfrak{B}(z)$.

To discuss the unknown behavior of $\mathfrak{B}(z)$ on $\partial\mathfrak{U}$, the following corollary will illustrates this :

Corollary 2.1. Let $\mathcal{X} \in \Phi_{\mathcal{Z}_{\mathrm{n}}}[\aleph, \mathfrak{B}_{k}]$ for some $k \in (0,1)$, $\mathfrak{B}_{k}(z) = \mathfrak{B}(kz)$, and $\mathfrak{B}(z)$ be an analytic and injective function in $\mathfrak{U}$ such that where $\mathfrak{B}(0) = 0$ and $\aleph \subset \mathbb{C}$. If $h(z) \in \mathcal{A}(\mathrm{n})$ and

$$\mathcal{X}\left(\mathcal{Z}_{\mathrm{n}}^{\mu,\alpha} h(z), \mathcal{Z}_{\mathrm{n}}^{\mu-1,\alpha} h(z), \mathcal{Z}_{\mathrm{n}}^{\mu-2,\alpha} h(z); z\right) \in \aleph, (\mu > 2, \alpha \geq o, \mathrm{n} \in \mathbb{N}),$$

then $\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha} h(z) \prec \mathfrak{B}(z)$.

Proof. Theorem 1 imply that $\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha} h(z) \prec \mathfrak{B}_{\hbar}(z)$. From the fact that

$$\mathfrak{B}_{\hbar}(z) \prec \mathfrak{B}(z).$$

Hence $\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha} h(z) \prec \mathfrak{B}(z)$.

Theorem 2.3. Suppose that $\mathcal{F}(z)$ and $\mathfrak{B}(z)$ be two univalent functions lie in $\mathfrak{U}$, with $\mathfrak{B}(0) = 0$ and $\mathfrak{B}_{\hbar}(z) = \mathfrak{B}(\hbar z)$, $\mathcal{F}_{\hbar}(z) = \mathcal{F}(\hbar z)$. If the function $\mathcal{X}: \mathbb{C}^3 \times \mathfrak{U} \to \mathbb{C}$ fulfil one of the requirement:

(1) $\mathcal{X} \in \Phi_{\mathcal{Z}_n}[\mathcal{F}, \mathfrak{B}_{\hbar}]$, for some $\hbar \in (0,1)$, or

(2) If $h(z) \in \mathcal{A}(\mathfrak{n})$ satisfy (2.8) and there exists $\hbar_0 \in (0,1)$ such that

(3) $\mathcal{X} \in \Phi_{\mathcal{Z}_{\mathfrak{n}}}[\mathcal{F}_{\hbar}, \mathfrak{B}_{\hbar}]$, for all $\hbar \in (\hbar_0, 1)$, then

$$\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha} h(z) \prec \mathfrak{B}(z).$$

Proof.

Case (1). From Theorem 1.1,we get $\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha} h(z) \prec \mathfrak{B}_{\hbar}(z)$, since

$$\mathfrak{B}_{\hbar}(z) \prec \mathfrak{B}(z),$$

then $\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha} h(z) \prec \mathfrak{B}(z)$.
Case (2). If $\mathcal{T}(z) = \mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha} h(z)$ and $\mathcal{T}_{\hbar}(z) = \mathcal{T}(\hbar z)$, so

$$\mathcal{X}(\mathcal{T}_{\hbar}(z), z\mathcal{T}_{\hbar}'(z), z^2\mathcal{T}_{\hbar}''(z); \hbar z) = \mathcal{X}(\mathcal{T}(\hbar z), (\hbar z)\mathcal{T}'(\hbar z), (\hbar z)^2\mathcal{T}''(\hbar z), \hbar z) \in \mathcal{F}_{\hbar}(\mathfrak{U}),$$

by Theorem 1.1 with

$\mathcal{X}\left(\mathcal{T}(z), z\mathcal{T}'(z), z^2\mathcal{T}''(z); \wp(z)\right) \in \aleph$, where $\wp(\mathfrak{U}) = \mathfrak{U}$ is any mapping with $\wp(z) = \hbar z$, therefore

$$\mathcal{T}_{\hbar}(z) \prec \mathfrak{B}_{\hbar}(z).$$

For $\hbar \in (\hbar_o, 1)$, by letting $\hbar \to 1$, we get

$$\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha} h(z) \prec \mathfrak{B}(z).$$

The best dominant for (2.8) is obtained according to the content in the following theorem:

Theorem 2.4. Suppose that $\mathcal{X}: \mathbb{C}^3 \times \mathfrak{U} \to \mathbb{C}$ be a function and $\mathcal{F}(z)$ is a univalent in $\mathfrak{U}$ and $\mathfrak{B}(z)$ is a solution of the differential equation

$$\mathcal{X}\left(\mathfrak{B}(z), \frac{z\mathfrak{B}'(z)+\mathfrak{n}(\alpha-1)\mathfrak{B}(z)}{\mathfrak{n}\alpha}, \frac{z^2\mathfrak{B}''(z)+(1+2\mathfrak{n}(\alpha-1))z\mathfrak{B}'(z)+\mathfrak{n}^2(\alpha-1)^2\mathfrak{B}(z)}{(\mathfrak{n}\alpha)^2}; z\right) = \mathcal{F}(z). \tag{2.9}$$

Such that $\mathfrak{B}(0) = 0$ and satisfy one of the following requirements :

(1) $\mathfrak{B}(z) \in \mathcal{D}_o$ and $\mathcal{X} \in \Phi_{\mathcal{Z}_{\mathfrak{n}}}[\mathcal{F}, \mathfrak{B}]$.

(2) $\mathfrak{B}(z)$ is univalent in $\mathfrak{U}$ and $\mathcal{X} \in \Phi_{\mathcal{Z}_{\mathfrak{n}}}[\mathcal{F}, \mathfrak{B}_{\hbar}]$, for some $0 < \hbar < 1$,

or

(3) $\mathfrak{B}(z)$ is analytic and injective in $\mathfrak{U}$ and $\exists \hbar_o \in (0,1)$,where

$\mathcal{X} \in \Phi_{\mathcal{Z}_n}[\mathcal{F}_\hbar, \mathfrak{B}_\hbar]$ $\forall \hbar \in (\hbar_o, 1)$. If $h(z) \in \mathcal{A}(n)$ fulfil the relation in (2.8), then

$$\mathcal{Z}_n^{\mu,\alpha} h(z) \prec \mathfrak{B}(z),$$

thus the best dominant is $\mathfrak{B}(z)$.

Proof. According to the facts in Theorems 2.2 and 2.3,we conclude that $\mathfrak{B}(z)$ is a dominant of (2.8), Also the function $\mathfrak{B}(z)$ satisfies (2.9) it is also a solution of (2.8) , therefore $\mathfrak{B}(z)$ will be dominated by all dominants. Consequently $\mathfrak{B}(z)$ is the best dominant.

when $\mathfrak{B}(z) = \wp z, \wp > 0$, in the special case and through definition 2.1, the class of admissible functions $\Phi_{\mathcal{Z}_n}[\aleph, \mathfrak{B}]$, denoted by $\Phi_{\mathcal{Z}_n}[\aleph, \wp]$, is explain below.

Definition 2.2. Let $\wp > 0$ and $\aleph \subset \mathbb{C}$ be a set. The functions $\mathcal{X}: \mathbb{C}^3 \times \mathfrak{U} \to \mathbb{C}$ belonging to the class $\Phi_{\mathcal{Z}_n}[\aleph, \wp]$ of admissible functions such that

$$\mathcal{X}\left(\wp e^{i\phi}, \frac{(e+n(\alpha-1))\wp e^{i\phi}}{n\alpha}, l + \left(e + n(\alpha-1)\left(1 + 2(e + n(\alpha-1))\right)\right)\wp e^{i\phi}; z \notin \aleph\right), \quad (2.10)$$

whenever $z \in \mathfrak{U}$, $\phi \in \mathbb{R}$, $\mathcal{R}(le^{-i\phi}) \geq (e-1)e\wp$ ($\forall \phi \in \mathbb{R}$ and $e \geq n$).

Corollary 2.2. Let $\mathcal{X} \in \Phi_{\mathcal{Z}_n}[\aleph, \wp]$. If $h(z) \in \mathcal{A}(n)$ satisfies

$$\mathcal{X}\left(\mathcal{Z}_n^{\mu,\alpha} h(z), \mathcal{Z}_n^{\mu-1,\alpha} h(z), \mathcal{Z}_n^{\mu-2,\alpha} h(z); z\right) \in \aleph, \ (\mu > 2, \alpha \geq 0, n \in \mathbb{N})$$

then $\left|\mathcal{Z}_n^{\mu,\alpha} h(z)\right| < \wp$.

When $\aleph = \mathfrak{B}(\mathfrak{U}) = \{\wp : |\wp| < \wp\}$ as a special case, the class $\Phi_{\mathcal{Z}_n}[\aleph, \wp]$ is usually denoted by $\Phi_{\mathcal{Z}_n}[\wp]$.

Corollary 2.3. Let $\mathcal{X} \in \Phi_{\mathcal{Z}_n}[\wp]$. If $h(z) \in \mathcal{A}(n)$ satisfy the inequality

$$\left|\mathcal{X}\left(\mathcal{Z}_n^{\mu,\alpha} h(z), \mathcal{Z}_n^{\mu-1,\alpha} h(z), \mathcal{Z}_n^{\mu-2,\alpha} h(z); z\right)\right| < \wp, \quad (\mu > 2, \alpha \geq 0, n \in \mathbb{N})$$

then $\left|\mathcal{Z}_n^{\mu,\alpha} h(z)\right| < \wp$.

Corollary 2.4. Let $\wp > 0$ and $h(z) \in \mathcal{A}(n)$ satisfies

$$\left|(n\alpha)^2 \mathcal{Z}_n^{\mu-2,\alpha} h(z) - (n\alpha)\mathcal{Z}_n^{\mu-1,\alpha} h(z) - n^2(\alpha-1)^2 \mathcal{Z}_n^{\mu-2,\alpha} h(z)\right| < (2n(\alpha-1)+1)n + n(n-1)\wp,$$

then

$$\left|\mathcal{Z}_n^{\mu,\alpha} h(z)\right| < \wp. \quad (2.11)$$

Proof. By taking $\mathcal{X}(u, v, w; z) = (n\alpha)^2 w - n\alpha v - n^2(\alpha+1)^2 u$,

and $\aleph = \mathcal{F}(\mathfrak{U})$,where

$$\mathcal{F}(z) = \left[n\left((2n(\alpha-1)+1) + (n-1)\right)\right]\wp z, \ \wp > 0$$

since

$$\left|l + (2n(\alpha-1)+1)e\wp e^{i\phi}\right| \geq \left[n\left((2n(\alpha-1)+1) + (n-1)\right)\right]\wp.$$

Thus $\mathcal{X} \in \Phi_{Z_n}[\aleph, \wp]$.

Hence admissible condition (2.10) is satisfied,

By using corollary 2.2, we get the required result.

Definition 2.3. Suppose that $\mathfrak{B}(z) \in \mathcal{D}_o \cap \mathcal{H}_o$ and a set $\aleph \subset \mathbb{C}$. The functions $\mathcal{X}: \mathbb{C}^3 \times \mathfrak{U} \to \mathbb{C}$ belonging in the class $\Phi_{Z_n,1}[\aleph, \mathfrak{B}]$ of admissible functions that content the admissibility essential provision

$$\mathcal{X}(u, v, w; z) \notin \aleph,$$

whenever $u = \mathfrak{B}(\delta), v = \frac{k\delta\mathfrak{B}'(\delta)+\mathfrak{n}(\alpha-1)\mathfrak{B}(\delta)}{\mathfrak{n}\alpha}$,

$$\mathcal{R}\left\{\frac{(\alpha\mathfrak{n})^2 w-(\mathfrak{n}\alpha-1)^2 u}{\mathfrak{n}\alpha v-(\mathfrak{n}\alpha)u} - 2(\mathfrak{n}\alpha - 1)\right\} \geq \wp\mathcal{R}\left\{1 + \frac{\delta\mathfrak{B}''(\delta)}{\mathfrak{B}'(\delta)}\right\},$$

$$(z \in \mathfrak{U}, \delta \in \partial\mathfrak{U} \setminus \mathcal{E}(\mathfrak{B}), \wp \geq 1).$$

Theorem 2.5. Assume that $h(z) \in \mathcal{A}(\mathfrak{n})$ and $\mathcal{X} \in \Phi_{Z_n,1}[\aleph, \mathfrak{B}]$.If $h(z)$ satisfies

$$\left\{\mathcal{X}\left(\frac{Z_{\mathfrak{n}}^{\mu,\alpha}h(z)}{z^{\mathfrak{n}-1}}, \frac{Z_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{z^{\mathfrak{n}-1}}, \frac{Z_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}{z^{\mathfrak{n}-1}}; z\right): z \in \mathfrak{U}\right\} \subset \aleph \text{ },(\mu > 2, \alpha \geq 0, \mathfrak{n} \in \mathbb{N}) \tag{2.12}$$

then $\frac{Z_{\mathfrak{n}}^{\mu,\alpha}h(z)}{z^{\mathfrak{n}-1}} \prec \mathfrak{B}(z)$.

Proof. Define the function

$$\mathcal{T}(z) = \frac{Z_{\mathfrak{n}}^{\mu,\alpha}h(z)}{z^{\mathfrak{n}-1}}. \tag{2.13}$$

Which is analytic in $\mathfrak{U}$. By differentiating (2.13) and using (1.4),

$$\frac{Z_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{z^{\mathfrak{n}-1}} = \frac{z\mathcal{T}'(z)+(\mathfrak{n}(\alpha-1)+\mathfrak{n}-1)\mathcal{T}(z)}{\mathfrak{n}\alpha}. \tag{2.14}$$

Further computations show that

$$\frac{Z_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}{z^{\mathfrak{n}-1}} = \frac{z^2\mathcal{T}''(z)+(2\mathfrak{n}\alpha-1)z\mathcal{T}'(z)+(\mathfrak{n}\alpha-1)^2\mathcal{T}(z)}{(\mathfrak{n}\alpha)^2}. \tag{2.15}$$

Suppose that the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ is defined as

$$u(a, b, c) = a, \qquad v(a, b, c) = \frac{b+\mathfrak{n}\alpha a}{\mathfrak{n}\alpha}, \quad w(a, b, c) = \frac{c+(2\mathfrak{n}\alpha-1)b+(\mathfrak{n}\alpha-1)^2 a}{(\mathfrak{n}\alpha)^2}. \tag{2.16}$$

Let

$$\psi(a, b, c; z) = \mathcal{X}(u, v, w; z) = \mathcal{X}\left(a, \frac{b + \mathfrak{n}\alpha a}{\mathfrak{n}\alpha}, \frac{c + (2\mathfrak{n}\alpha - 1)b + (\mathfrak{n}\alpha - 1)^2 a}{(\mathfrak{n}\alpha)^2}; z\right). \tag{2.17}$$

By using Theorem 1.1 and equations (2.13), (2.14), (2.15), and from (2.17), we obtain

$$\psi(\mathcal{T}(z), z\mathcal{T}'(z), z^2\mathcal{T}''(z); z) = \mathcal{X}\left(\frac{Z_{\mathfrak{n}}^{\mu,\alpha}h(z)}{z^{\mathfrak{n}-1}}, \frac{Z_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{z^{\mathfrak{n}-1}}, \frac{Z_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}{z^{\mathfrak{n}-1}}; z\right). \tag{2.18}$$

Hence (2.12), becomes

$$\psi(\mathcal{T}(z), z\mathcal{T}'(z), z^2\mathcal{T}''(z); z) \in \aleph.$$

Note that

$$\frac{c}{b}+1=\frac{(\mathfrak{n}\alpha)^2 w-(\mathfrak{n}\alpha-1)^2 u}{\mathfrak{n}\alpha v-(\mathfrak{n}\alpha)u}-2(\mathfrak{n}\alpha-1).$$

Hence the admissibility conditions for $\mathcal{X} \in \Phi_{Z_n,1}[\aleph, \mathfrak{B}]$ and for ψ are amounting to the admissibility condition as given in a definition 1.1.

Thus $\psi \in \Psi[\aleph, \mathfrak{B}]$. In view of justification of Theorem 1.1 that imply

$$\mathcal{T}(z) \prec \mathfrak{B}(z) \qquad \text{or} \qquad \frac{Z_{\mathfrak{n}}^{\mu,\alpha}h(z)}{z^{\mathfrak{n}-1}} \prec \mathfrak{B}(z).$$

We deduce that for some conformal mapping $\mathcal{F}(z)$ of $\mathfrak{U}$ onto $\aleph$, $\aleph = \mathcal{F}(\mathfrak{U})$ If a domain $\aleph$ is a simply connected not equal $\mathbb{C}$. In this case the class $\Phi_{Z_n,1}[\mathcal{F}(\mathfrak{U}), \mathfrak{B}]$ is written as $\Phi_{Z_n,1}[\mathcal{F}, \mathfrak{B}]$.

A class of admissible function $\Phi_{Z_{\mathfrak{n}},1}[\aleph, \mathfrak{B}]$ will denoted by $\Phi_{Z_n,1}[\aleph, \wp]$ as a particular case when $\mathfrak{B}(z) = \wp z, \wp > 0$. The following Theorem is an immediate consequence of Theorem 2.5.

Theorem 2.6. Assume that $\mathcal{X} \in \Phi_{Z_n,1}[\mathcal{F}, \mathfrak{B}]$. If $h(z) \in \mathcal{A}(\mathfrak{n})$ satisfies

$$\left\{\mathcal{X}\left(\frac{Z_{\mathfrak{n}}^{\mu,\alpha}h(z)}{z^{\mathfrak{n}-1}}, \frac{Z_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{z^{\mathfrak{n}-1}}, \frac{Z_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}{z^{\mathfrak{n}-1}}; z\right): z \in \mathfrak{U}\right\} \prec \mathcal{F}(z), \tag{2.19}$$

then $\frac{Z_{\mathfrak{n}}^{\mu,\alpha}h(z)}{z^{\mathfrak{n}-1}} \prec \mathfrak{B}(z)$.

Definition 2.4. If $\wp > 0$ and $\aleph \subset \mathbb{C}$ be a set. The functions $\mathcal{X}: \mathbb{C}^3 \times \mathfrak{U} \to \mathbb{C}$ belonging to the class $\Phi_{Z_n,1}[\aleph, \wp]$ of admissible functions, where

$$\mathcal{X}\left(\wp e^{i\phi}, \frac{[e+(\mathfrak{n}\alpha-1)]\wp e^{i\phi}}{\mathfrak{n}\alpha}, \frac{l+[2(\mathfrak{n}\alpha-1)e+(\mathfrak{n}\alpha-1)^2]\wp e^{i\phi}}{(\mathfrak{n}\alpha)^2}; z\right) \notin \aleph, \tag{2.20}$$

whenever $z \in \mathfrak{U}$, $\phi \in \mathbb{R}$, $\mathcal{R}(le^{-i\phi}) \geq (e-1)e\wp$ for all real ϕ and $e \geq 1$.

Corollary 2.5. Let $\mathcal{X} \in \Phi_{Z_n,1}[\aleph, \wp]$. If $h(z) \in \mathcal{A}(\mathfrak{n})$ satisfies

$$\mathcal{X}\left(\frac{Z_{\mathfrak{n}}^{\mu,\alpha}h(z)}{z^{\mathfrak{n}-1}}, \frac{Z_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{z^{\mathfrak{n}-1}}, \frac{Z_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}{z^{\mathfrak{n}-1}}; z\right) \in \aleph, (\mu > 2, \alpha \geq 0, \mathfrak{n} \in \mathbb{N})$$

then $\left|\frac{Z_{n,\gamma}^{\mu,\alpha}h(z)}{z^{\mathfrak{n}-1}}\right| < \wp$.

The class $\Phi_{Z_n,1}[\aleph, \wp]$ is simply denoted by $\Phi_{Z_n,1}[\wp]$ as a special case if $\aleph = \Upsilon(\mathfrak{U}) = \{\wp: |\wp| < \wp\}$.

Corollary 2.6. Let $\mathcal{X} \in \Phi_{Z_{\mathfrak{n}},1}[\wp]$. If $h(z) \in \mathcal{A}(\mathfrak{n})$ satisfies

$$\left|\mathcal{X}\left(\frac{Z_{\mathfrak{n}}^{\mu,\alpha}h(z)}{z^{\mathfrak{n}-1}}, \frac{Z_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{z^{\mathfrak{n}-1}}, \frac{Z_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}{z^{\mathfrak{n}-1}}; z\right)\right| < \wp, (\mu > 2, \alpha \geq 0, \mathfrak{n} \in \mathbb{N}),$$

then $\left|\frac{Z_{\mathfrak{n}}^{\mu,\alpha}h(z)}{z^{\mathfrak{n}-1}}\right| < \wp$.

Corollary 2.7. If $\wp > 0$ and $h(z) \in \mathcal{A}(\mathfrak{n})$ satisfies

$$\left|\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{z^{\mathfrak{n}-1}}\right| < 1 \quad (\mu > 1, \alpha \geq 0, \mathfrak{n} \in \mathbb{N}),$$

then $\left|\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)}{z^{\mathfrak{n}-1}}\right| < \wp.$

Corollary 2.8. Suppose that $\wp > 0$ and $\mathrm{h(z)} \in \mathcal{A}(\mathfrak{n})$ satisfies

$$\left|\mathcal{X}\left(\wp e^{i\phi}, \frac{[e+(\mathfrak{n}\alpha-1)]\wp e^{i\phi}}{\alpha\gamma}, \frac{l+[2(\mathfrak{n}\alpha-1)e+(\mathfrak{n}\alpha-1)^2]\wp e^{i\phi}}{(\mathfrak{n}\alpha)^2}; z\right)\right| < (3\mathfrak{n}(\mathfrak{n}\alpha-1)-\mathfrak{n}\alpha)\wp,$$

then $\left|\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)}{z^{\mathfrak{n}-1}}\right| < 1.$ (2.21)

Proof. Let $\mathcal{X}(u, v, w; z) = (\mathfrak{n}\alpha)^2 w + \mathfrak{n}\alpha v - (\mathfrak{n}\alpha-1)^2 u$

and $\aleph = \mathcal{F}(\mathfrak{U})$, where $\mathcal{F}(z) = (3\mathfrak{n}(\mathfrak{n}\alpha-1)-\mathfrak{n}\alpha)\wp z,\ \wp > 0$. since

$$\left|\mathcal{X}\left(\wp e^{i\phi}, \frac{[e+(\mathfrak{n}\alpha-1)]\wp e^{i\phi}}{\mathfrak{n}\alpha}, \frac{l+[(2(\mathfrak{n}\alpha-1)e+(\mathfrak{n}\alpha-1)^2]\wp e^{i\phi}}{(\mathfrak{n}\alpha)^2}; z\right)\right|$$

$$\left|l + \left(2\mathfrak{n}\alpha-1)e + (\mathfrak{n}\alpha-1)\right)\wp e^{i\phi}\right| \geq \left|le^{-i\phi} + (2\mathfrak{n}\alpha-1)e + (\mathfrak{n}\alpha-1)\wp\right|$$

$$\geq \mathcal{R}\left(le^{-i\phi}\right) + [(e-1)(\mathfrak{n}\alpha-1) - \mathfrak{n}\alpha]\wp$$

$$\geq e(e-1)\wp + [(2\mathfrak{n}\alpha-1)e + (\mathfrak{n}\alpha-1)]\wp$$

$$\geq (3\mathfrak{n}\alpha-2)\wp\ .$$

$z \in \mathfrak{U}, \phi \in \mathbb{R}, \mathcal{R}\left(le^{-i\phi}\right) \geq (e-1)e\wp$ for all real ϕ and $e \geq 1$.

Hence $\mathcal{X} \in \Phi_{\mathcal{Z}_{\mathfrak{n}},1}[\aleph, \wp]$, that is the admissible condition (2.20) is satisfies.

We deduce the required result by corollary 2.5.

Definition 2.5. Assume that $\aleph$ be a set in $\mathbb{C}$ and $\mathfrak{B}(z) \in \mathcal{D}_1 \cap \mathcal{H}$. The functions $\mathcal{X}: \mathbb{C}^3 \times \mathfrak{U} \to \mathbb{C}$ belonging to the class $\Phi_{\mathcal{Z}_{\mathfrak{n}},2}[\aleph, \mathfrak{B}]$ of admissible functions that satisfy the admissibility condition

$\mathcal{X}(u, v, w; z) \notin \aleph$,

whenever $u = \mathfrak{B}(\delta), v = \mathfrak{B}(\delta) + \frac{e\delta\mathfrak{B}'(\delta)}{\mathfrak{n}\alpha\,\mathfrak{B}(\delta)}$

$$(\mathfrak{B}(\delta) \neq 0)\mathcal{R}\left\{\frac{\mathfrak{n}\alpha\left(wv - 3vu + 2u^2\right)}{v-u}\right\} \geq e\mathcal{R}\left\{1 + \frac{\delta\mathfrak{B}''(\delta)}{\mathfrak{B}'(\delta)}\right\},$$

$(z \in \mathfrak{U}, \delta \in \partial\mathfrak{U} \setminus \mathcal{E}(\mathfrak{B})$, and $e \geq 1)$.

Theorem 2.7. Suppose that $\mathcal{X} \in \Phi_{\mathcal{Z}_{\mathfrak{n}},2}[\aleph, \mathfrak{B}]$ and $\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)$ not equal zero . If $h(z) \in \mathcal{A}(\mathfrak{n})$ satisfies

$$\left\{\mathcal{X}\left(\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)}, \frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}, \frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-3,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}; z\right) : z \in \mathfrak{U}\right\} \subset \aleph\,, (\,\mu > 3, \alpha \geq 0, \mathfrak{n} \in \mathbb{N}) \quad (2.22)$$

then $\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)} \prec \mathfrak{B}(z).$

Proof. Define the function

$$\mathcal{T}(z) = \frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha} h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha} h(z)}. \tag{2.23}$$

Which is analytic in $\mathfrak{U}$. By differentiating (2.23) logarithmically with respect to z and using (2.2), we have

$$\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-2,\alpha} h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha} h(z)} = \frac{z\mathcal{T}'(z)}{\mathfrak{n}\alpha\, \mathcal{T}(z)} + \mathcal{T}(z). \tag{2.24}$$

Differentiating (2.24) logarithmically with respect to z and using (2.2), we conclude that

$$\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-3,\alpha} h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu-2,\alpha} h(z)} = \mathcal{T}(z) + \frac{z\mathcal{T}'(z)}{\mathfrak{n}\alpha\, \mathcal{T}(z)} + \frac{z\mathcal{T}'(z) + \frac{z\mathcal{T}'(z)}{\mathfrak{n}\alpha\, \mathcal{T}(z)} - \frac{1}{\mathfrak{n}\alpha}\left(\frac{z\mathcal{T}'(z)}{\mathcal{T}(z)}\right)^2 + \frac{1}{\mathfrak{n}\alpha}\frac{z^2\mathcal{T}''(z)}{\mathcal{T}(z)}}{\mathfrak{n}\alpha\, \mathcal{T}(z) + \frac{z\mathcal{T}'(z)}{\mathfrak{n}\alpha\, \mathcal{T}(z)}}. \tag{2.25}$$

Now ,the following conversions from $\mathbb{C}^3$ to $\mathbb{C}$ are known as

$u(a,b,c) = a,\ v(a,b,c) = a + \frac{b}{\mathfrak{n}\alpha a}$,

$$w(a,b,c) = a + \frac{b}{\mathfrak{n}\alpha a} + \left(\frac{b + \frac{b}{\mathfrak{n}\alpha a} - \frac{1}{\mathfrak{n}\alpha}\left(\frac{b}{a}\right)^2 + \frac{c}{\mathfrak{n}\alpha a}}{\mathfrak{n}\alpha a + \frac{b}{a}}\right). \tag{2.26}$$

Let $\psi(a,b,c;z) = \mathcal{X}(u,v,w;z) =$

$$\mathcal{X}\left(a,\ \left(a + \frac{b}{\mathfrak{n}\alpha a}\right),\ \left(a + \frac{b}{\mathfrak{n}\alpha a} + \frac{b + \frac{b}{\mathfrak{n}\alpha a} - \frac{1}{\mathfrak{n}\alpha}\left(\frac{b}{a}\right)^2 + \frac{c}{\mathfrak{n}\alpha a}}{\mathfrak{n}\alpha a + \frac{b}{a}}\right)\ ; z\right). \tag{2.27}$$

By using Theorem 1.1 and equations (2.23)—(2.25), also from (2.27), we have

$$\psi(\mathcal{T}(z), z\mathcal{T}'(z), z^2\mathcal{T}''(z); z) = \mathcal{X}\left(\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha} h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha} h(z)}, \frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-2,\alpha} h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha} h(z)}, \frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-3,\alpha} h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu-2,\alpha} h(z)}; z\right). \tag{2.28}$$

Therefore (2.22), becomes

$$\psi(\mathcal{T}(z), z\mathcal{T}'(z), z^2\mathcal{T}''(z); z) \in \aleph.$$

Note that

$$\frac{c}{b} + 1 = \frac{\mathfrak{n}\alpha\left(wv - 3vu + 2u^2\right)}{v - u}.$$

The admissible conditions for $\mathcal{X} \in \Phi_{\mathcal{Z}_{\mathfrak{n}},2}[\aleph, \mathfrak{B}]$ and for ψ are amounting to the admissibility condition as given in definition 1.1.

Therefore $\psi \in \Psi[\aleph, \mathfrak{B}]$. So by Theorem 1.1,

$$\mathcal{T}(z) \prec \mathfrak{B}(z) \text{ or } \frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha} h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha} h(z)} \prec \mathfrak{B}(z)\,.$$

Then for some conformal mapping $\mathcal{F}(z)$ of $\mathfrak{U}$ onto σ ,we get $\aleph = \mathcal{F}(\mathfrak{U})$. In this case the domain $\aleph$ not equal $\mathbb{C}$,and $\aleph$ is a simply connected , the class $\Phi_{\mathcal{Z}_{\mathfrak{n}},2}[\mathcal{F}, \mathfrak{B}]$ meaning $\Phi_{\mathcal{Z}_{\mathfrak{n}},2}[\mathcal{F}(\mathfrak{U}), \mathfrak{B}]$ in simple form.

In the special case $\mathfrak{B}(z) = 1 + \wp z, \wp > 0$. The class $\Phi_{\mathcal{Z}_{\mathfrak{n}},2}[\aleph, \mathfrak{B}]$ of admissible functions become $\Phi_{\mathcal{Z}_{\mathfrak{n}},2}[\aleph, \wp]$. The next theorem is an direct result of Theorem 2.7.

Theorem 2.8. Assume that $\mathcal{X} \in \Phi_{\mathcal{Z}_{\mathfrak{n}},2}[\mathcal{F}, \mathfrak{B}]$. If $h(z) \in \mathcal{A}(\mathfrak{n})$ fulfill the following

$$\mathcal{X}\left(\frac{Z_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{Z_{\mathfrak{n}}^{\mu,\alpha}h(z)},\frac{Z_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}{Z_{\mathfrak{n}}^{\mu-1,\alpha}h(z)},\frac{Z_{\mathfrak{n}}^{\mu-3,\alpha}h(z)}{Z_{\mathfrak{n}}^{\mu-2,\alpha}h(z)};z\right)\prec\mathcal{F}(z),(\mu>3,\alpha\geq 0,\mathfrak{n}\in\mathbb{N}) \tag{2.29}$$

then $\frac{Z_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{Z_{\mathfrak{n}}^{\mu,\alpha}h(z)}\prec\mathfrak{B}(z)$.

Definition 2.6. Suppose that $\aleph\subset\mathbb{C}$ be a set. The functions $\mathcal{X}:\mathbb{C}^3\times\mathfrak{U}\to\mathbb{C}$ belonging to the class $\Phi_{Z_{\mathfrak{n}},2}[\aleph,\wp]$ of admissible functions ,such that

$$\mathcal{X}\left(1+\wp e^{i\phi},1+\left(\frac{e+\mathfrak{n}\alpha(1+\wp e^{i\phi})}{\mathfrak{n}\alpha(1+\wp e^{i\phi})}\right)\wp e^{i\phi},\right.$$
$$\left.1+\left(\frac{e+\mathfrak{n}\alpha(1+\wp e^{i\phi})}{\mathfrak{n}\alpha(1+\wp e^{i\phi})}\right)\wp e^{i\phi}+\frac{(e^{-i\phi}+\wp)[le^{-i\phi}+e\wp(\mathfrak{n}\alpha(1+\wp e^{i\phi})-1)]-e^2\wp^2}{(e^{-i\phi}+\wp)[\mathfrak{n}\alpha e\wp+(\mathfrak{n}\alpha)^2\wp(2+\wp e^{i\phi})+(\mathfrak{n}\alpha)^2e^{-i\phi}]};z\notin\aleph\right), \tag{2.30}$$

whenever $z\in\mathfrak{U}$, $\phi\in\mathbb{R}$, $\mathcal{R}(le^{-i\phi})\geq(e-1)e\wp$ for all real ϕ and $e\geq 1$.

Corollary 2.9. Let $\mathcal{X}\in\Phi_{Z_{\mathfrak{n}},2}[\aleph,\wp]$. If $h(z)\in\mathcal{A}(\mathfrak{n})$ satisfies

$$\mathcal{X}\left(\frac{Z_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{Z_{\mathfrak{n}}^{\mu,\alpha}h(z)},\frac{Z_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}{Z_{\mathfrak{n}}^{\mu-1,\alpha}h(z)},\frac{Z_{\mathfrak{n}}^{\mu-3,\alpha}h(z)}{Z_{\mathfrak{n}}^{\mu-2,\alpha}h(z)};z\right)\in\aleph,(\mu>3,\alpha\geq 0,\mathfrak{n}\in\mathbb{N})$$

then $\frac{Z_{n,\gamma}^{\mu-1,\alpha}h(z)}{Z_{n,\gamma}^{\mu,\alpha}h(z)}<1+\wp e^{i\phi}$.

The case if $\aleph=\mathfrak{B}(\mathfrak{U})=\{\wp:|\wp-1|<\wp\}$, The class $\Phi_{Z_{\mathfrak{n}},2}[\aleph,\wp]$ is denoted by $\Phi_{Z_{\mathfrak{n}},2}[\wp]$.

Corollary 2.10. Let $\mathcal{X}\in\Phi_{Z_{\mathfrak{n}},2}[\wp]$. If $h(z)\in\mathcal{A}(n)$ satisfies

$$\left|\mathcal{X}\left(\frac{Z_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{Z_{\mathfrak{n}}^{\mu,\alpha}h(z)},\frac{Z_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}{Z_{\mathfrak{n}}^{\mu-1,\alpha}h(z)},\frac{Z_{\mathfrak{n}}^{\mu-3,\alpha}h(z)}{Z_{\mathfrak{n}}^{\mu-2,\alpha}h(z)};z\right)-1\right|<\wp,(\mu>3,\alpha\geq 0,\mathfrak{n}\in\mathbb{N})$$

then $\left|\frac{Z_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{Z_{\mathfrak{n}}^{\mu,\alpha}h(z)}-1\right|<\wp$.

Corollary 2.11. Suppose that $\wp>0$ and h(z) $\in\mathcal{A}(n)$ satisfies

$$\left|\frac{Z_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}{Z_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}-\frac{Z_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{Z_{\mathfrak{n}}^{\mu,\alpha}h(z)}\right|<\frac{\wp}{\mathfrak{n}\alpha(\wp+1)},\quad(\mu>2,\alpha\geq 0,\mathfrak{n}\in\mathbb{N})$$

then $\left|\frac{Z_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{Z_{\mathfrak{n}}^{\mu,\alpha}h(z)}-1\right|<\wp$.

Proof. Suppose that $\mathcal{X}(u,v,w;z)=v-u$ and $\aleph=\mathcal{F}(u)$

where $\mathcal{F}(z)=\frac{\wp}{\mathfrak{n}\alpha(\wp+1)}z$, $\wp>0$. Since by corollary 2.9,

$$\frac{e\wp}{\mathfrak{n}\alpha(k+1)}>\frac{\wp}{\mathfrak{n}\alpha(k+1)}.$$

3. Superordination of Linear Operator $Z_{\mathfrak{n}}^{\mu,\alpha}$ Transformation

In this section we discuse the results about problems of the differential superordination of the linear transformation for analytic function.

Definition 3.1. Assume that $\mathfrak{B}(z) \in \mathcal{H}[0,\mathfrak{n}]$ and $\aleph \subset \mathbb{C}$ be a set, with $z\mathfrak{B}'(z) \neq 0$. The functions $\mathcal{X}: \mathbb{C}^3 \times \overline{\mathfrak{U}} \to \mathbb{C}$ that belonging to the class $\Phi'_{\mathcal{Z}_\mathfrak{n}}[\aleph, \mathfrak{B}]$ of admissible functions that fulfil the following admissibility condition

$$\mathcal{X}(u, v, w; \delta) \in \aleph,$$

whenever $u = \mathfrak{B}(z), v = \frac{\mathfrak{n}\alpha\mathfrak{B}(z) + \frac{z\mathfrak{B}'(z)}{y}}{\mathfrak{n}\alpha y}$,

$$\mathcal{R}\left\{\frac{\mathfrak{n}\alpha(w - 2v + u)}{v - u}\right\} \geq \frac{1}{y}\mathcal{R}\left\{\frac{z\mathfrak{B}''(z)}{\mathfrak{B}'(z)} + 1\right\},$$

where $z \in \mathfrak{U}, \delta \in \partial\mathfrak{U}$, and $y \geq \mathfrak{n}$.

Theorem 3.1. Let $\mathcal{X} \in \Phi'_{\mathcal{Z}_\mathfrak{n}}[\aleph, \mathfrak{B}]$. If $h(z) \in \mathcal{A}(\mathfrak{n})$, $\mathcal{Z}_\mathfrak{n}^{\mu,\alpha}h(z) \in \mathcal{D}_0$ and

$$\mathcal{X}\left(\mathcal{Z}_\mathfrak{n}^{\mu,\alpha}h(z), \mathcal{Z}_\mathfrak{n}^{\mu-1,\alpha}h(z), \mathcal{Z}_\mathfrak{n}^{\mu-2,\alpha}h(z); z\right), \quad (\mu > 2, \alpha \geq 0, \mathfrak{n} \in \mathbb{N})$$

is univalent in $\mathfrak{U}$, then

$$\aleph \subset \left\{\mathcal{X}\left(\mathcal{Z}_\mathfrak{n}^{\mu,\alpha}h(z), \mathcal{Z}_\mathfrak{n}^{\mu-1,\alpha}h(z), \mathcal{Z}_\mathfrak{n}^{\mu-2,\alpha}h(z); z\right): z \in \mathfrak{U}\right\}, \tag{3.1}$$

implies

$$\mathfrak{B}(z) \prec \mathcal{Z}_\mathfrak{n}^{\mu,\alpha}h(z).$$

Proof. From the relation in (2.7) and (3.1), we obtain

$$\aleph \subset \{\psi(\mathcal{T}(z), z\mathcal{T}'(z), z^2\mathcal{T}''(z); z): z \in \mathfrak{U}\}.$$

Through the definition of a transformation in (2.5), we see that the admissibility condition for functions $\mathcal{X} \in \Phi'_{\mathcal{Z}_\mathfrak{n}}[\aleph, \mathfrak{B}]$ and for ψ are amounting to the admissibility condition as given in definition 1.2. there for $\psi \in \Psi_\mathfrak{n}[\aleph, \mathfrak{B}]$, since by Theorem 1.2,we get

$$\mathfrak{B}(z) \prec \mathcal{T}(z) \quad \text{or} \quad \mathfrak{B}(z) \prec \mathcal{Z}_\mathfrak{n}^{\mu,\alpha}h(z).$$

Then $\aleph = \mathcal{F}(\mathfrak{U})$ for function $\mathcal{F}(z)$, where $\mathcal{F}(z)$ conformal mapping of $\mathfrak{U}$ onto $\aleph$ when a domain $\aleph$ is a simply connected not equal $\mathbb{C}$, so the class $\Phi'_{\mathcal{Z}_\mathfrak{n}}[\mathcal{F}(\mathfrak{U})\mathfrak{B}]$ will be symbolized by $\Phi'_{\mathcal{Z}_\mathfrak{n}}[\mathcal{F}, \mathfrak{B}]$. As a consequence of Theorem 3.1, we have the following result:

Theorem 3.2. Assume that $\mathfrak{B}(z) \in \mathcal{H}[0,\mathcal{T}]$ and the analytic function $\mathcal{F}(z)$ on $\mathcal{Z}_\mathfrak{n}^{\mu,\alpha}$, $\mathcal{X} \in \Phi'_{\mathcal{Z}_\mathfrak{n}}[\mathcal{F}, \mathfrak{B}]$. If $h(z) \in \mathcal{A}(\mathfrak{n})$, $\mathcal{Z}_\mathfrak{n}^{\mu,\alpha}h(z) \in \mathcal{D}_0$ and

$$\mathcal{X}\left(\mathcal{Z}_\mathfrak{n}^{\mu,\alpha}h(z), \mathcal{Z}_\mathfrak{n}^{\mu-1,\alpha}h(z), \mathcal{Z}_\mathfrak{n}^{\mu-2,\alpha}h(z); z\right) \ (\mu > 2, \alpha \geq 0, \mathfrak{n} \in \mathbb{N}),$$

is univalent in $\mathfrak{U}$, then

$$\mathcal{F}(z) \prec \mathcal{X}\left(\mathcal{Z}_\mathfrak{n}^{\mu,\alpha}h(z), \mathcal{Z}_\mathfrak{n}^{\mu-1,\alpha}h(z), \mathcal{Z}_\mathfrak{n}^{\mu-2,\alpha}h(z); z\right), \tag{3.2}$$

implies

$$\mathfrak{B}(z) \prec \mathcal{Z}_\mathfrak{n}^{\mu,\alpha}h(z).$$

We can acquire the subordination of differential superordination of the form (3.1) or (3.2) according to only Theorem 3.1 and Theorem 3.2. For certain $\mathcal{X}$, the next fact shows the presence of the ideal subordinant of (3.2).

Theorem 3.3. Suppose that $\mathcal{X}: \mathbb{C}^3 \times \overline{\mathfrak{U}} \to \mathbb{C}$ and $\mathcal{F}(z)$ be analytic in $\mathfrak{U}$. Suppose that the differential equation

$$\mathcal{X}\left(\mathfrak{B}(z), \frac{z\mathfrak{B}'(z)+n\alpha\mathfrak{B}(z)}{n\alpha}, \frac{z^2\mathfrak{B}''(z)+(1+2n\alpha)z\mathfrak{B}'(z)+(n\alpha)^2\mathfrak{B}(z)}{(n\alpha)^2}; z\right) = \mathcal{F}(z), \tag{3.3}$$

has a solution $\mathfrak{B}(z) \in \mathcal{D}_0$. If $\mathcal{X} \in \Phi'_{\mathcal{Z}_n}[\mathcal{F}, \mathfrak{B}]$, $h(z) \in \mathcal{A}(n)$, $\mathcal{Z}_n^{\mu,\alpha} h(z) \in \mathcal{D}_0$ and

$$\mathcal{X}\left(\mathcal{Z}_n^{\mu,\alpha} h(z), \mathcal{Z}_n^{\mu-1,\alpha} h(z), \mathcal{Z}_n^{\mu-2,\alpha} h(z); z\right)$$

$$(\mu > 2, \alpha \geq 0, n \in \mathbb{N}),$$

is univalent in $\mathfrak{U}$, then

$$\mathcal{F}(z) \prec \mathcal{X}\left(\mathcal{Z}_n^{\mu,\alpha} h(z), \mathcal{Z}_n^{\mu-1,\alpha} h(z), \mathcal{Z}_n^{\mu-2,\alpha} h(z); z\right),$$

implies

$$\mathfrak{B}(z) \prec \mathcal{Z}_n^{\mu,\alpha} h(z).$$

Thus the best subordination is $\mathfrak{B}(z)$.

Proof. The proof is comparable to the proof of theorem 2.4.

The Next result is a consequence of amalgamate Theorems 2.2 and 3.2.

Corollary 3.1. For a univalent function $\mathcal{F}_2(z)$ in $\mathfrak{U}$, and two analytic functions $\mathcal{F}_1(z)$ and $\mathfrak{B}_1(z)$ in $\mathfrak{U}$, such that $\mathfrak{B}_2(z) \in \mathcal{D}_o$ with $\mathfrak{B}_1(0) = \mathfrak{B}_2(0) = 0$ and $\mathcal{X} \in \Phi_{\mathcal{Z}_{n,\gamma}}[\mathcal{F}_2, \mathfrak{B}_2] \cap \Phi'_{\mathcal{Z}_{n,\gamma}}[\mathcal{F}_1, \mathfrak{B}_1]$. If $h(z) \in \mathcal{A}(n)$, $\mathcal{Z}_{n,\gamma}^{\mu,\alpha} h(z) \in \mathcal{H}[0,n] \cap \mathcal{D}_0$ and

$$\mathcal{X}\left(\mathcal{Z}_n^{\mu,\alpha} h(z), \mathcal{Z}_n^{\mu-1,\alpha} h(z), \mathcal{Z}_n^{\mu-2,\alpha} h(z); z\right) (\mu > 2, \alpha \geq 0, n \in \mathbb{N}),$$

is univalent in $\mathfrak{U}$, then

$$\mathcal{F}_1(z) \prec \mathcal{X}\left(\mathcal{Z}_n^{\mu,\alpha} h(z), \mathcal{Z}_n^{\mu-1,\alpha} h(z), \mathcal{Z}_n^{\mu-2,\alpha} h(z); z\right) \prec \mathcal{F}_2(z),$$

implies

$$\mathfrak{B}_1(z) \prec \mathcal{Z}_n^{\mu,\alpha} h(z) \prec \mathfrak{B}_2(z).$$

Definition 3.2. Let $\mathfrak{B}(z) \in \mathcal{H}_o$ and $\aleph$ be a set in $\mathbb{C}$,with $z\mathfrak{B}'(z) \neq 0$. The class of admissible functions $\Phi'_{\mathcal{Z}_n,1}[\aleph, \mathfrak{B}]$ consists of those functions $\mathcal{X}: \mathbb{C}^3 \times \overline{\mathfrak{U}} \to \mathbb{C}$ that satisfy the admissibility provision

$\mathcal{X}(u, v, w; \delta) \in \aleph$,

where $u = \mathfrak{B}(z)$, $v = \frac{z\mathfrak{B}'(z)+y(n\alpha-1)\mathfrak{B}(z)}{n\alpha y}$,

$\mathcal{R}\left\{\frac{(n\alpha)^2 w-(n\alpha-1)^2 u}{n\alpha v-(n(\alpha+2)-1)u} - 2(n\alpha-1)\right\} \leq \frac{1}{y}\mathcal{R}\left\{1 + \frac{z\mathfrak{B}''(z)}{\mathfrak{B}'(z)}\right\}$,

$(z \in \mathfrak{U}, \delta \in \partial\mathfrak{U}$, and $y \geq 1)$.

Now we will clarify the dual result of theorem 2.5 for differential superordination in the next result:

Theorem 3.4. Suppose that $\mathcal{X} \in \Phi'_{\mathcal{Z}_n,1}[\aleph, \mathfrak{B}]$. If $h(z)$ imply in $\mathcal{A}(n)$, $\frac{\mathcal{Z}_n^{\mu,\alpha} h(z)}{z^{n-1}} \in \mathcal{D}_0$ and

$$\mathcal{X}\left(\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)}{z^{n-1}},\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{z^{n-1}},\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}{z^{n-1}};z\right),\quad (\mu>2,\alpha\geq 0,\mathfrak{n}\in\mathbb{N})$$

is univalent in $\mathfrak{U}$, then

$$\aleph\subset\left\{\mathcal{X}\left(\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)}{z^{n-1}},\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{z^{n-1}},\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}{z^{n-1}};z\right):z\in\mathfrak{U}\right\},\tag{3.4}$$

implies

$$\mathfrak{B}(z)\prec\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)}{z^{n-1}}.$$

Proof. From the results in (2.18) and (3.4), we get

$$\aleph\subset\{\psi(\mathcal{T}(z),z\mathcal{T}'(z),z^2\mathcal{T}''(z);z):z\in\mathfrak{U}\}.$$

From the function in the relation (2.16), we vision that the admissibility condition for $\mathcal{X}\in\Phi'_{\mathcal{Z}_{\mathfrak{n}},1}[\aleph,\mathfrak{B}]$ is equivalent to the admissibility condition for ψ as given in definition 1.2. Let σ be a set in $\mathbb{C}$ and $\mathfrak{B}(z)\in\mathcal{H}[a,\mathfrak{n}]$ with $\mathfrak{B}'(z)\neq 0$. The functions $\mathcal{X}:\mathbb{C}^3\times\overline{\mathfrak{U}}\to\mathbb{C}$ belonging to the class $\Psi'_{\mathfrak{n}}[\aleph,\mathfrak{B}]$ of admissible functions that satisfy the admissibility condition

$$\psi(a,b,c;\delta)\in\aleph,$$

whenever $a=\mathfrak{B}(z)$, $b=\frac{z\mathfrak{B}'(z)}{y}$, $\mathcal{R}\left\{1+\left(\frac{c}{b}\right)\right\}\leq\frac{1}{y}\mathcal{R}\left\{1+\frac{z\mathfrak{B}''(z)}{\mathfrak{B}'(z)}\right\}$,

where $z\in\mathfrak{U},\delta\in\partial\mathfrak{U}$, and $y\geq\mathfrak{n}\geq 1$. In exceptional , $\Psi'[\aleph,\mathfrak{B}]$will denoted by $\Psi'_1[\aleph,\mathfrak{B}]$ .Theorem 1.2 imply that $\psi\in\Psi'[\aleph,\mathfrak{B}]$, Let $\psi\in\Psi_{\mathfrak{n}}[\aleph,\mathfrak{B}]$ with $\mathfrak{B}(0)=0$. If $\mathcal{T}(z)\in\mathcal{D}(a)$ and $\psi(\mathcal{T}(z),z\mathcal{T}'(z),z^2\mathcal{T}''(z);z)$ is univalent in $\mathfrak{n}$, then

$$\aleph\subset\{\psi(\mathcal{T}(z),z\mathcal{T}'(z),z^2\mathcal{T}''(z);z):z\in\mathfrak{U}\},$$

implies

$$\mathfrak{B}(z)\prec\mathcal{T}(z)\text{ or }\mathfrak{B}(z)\prec\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)}{z^{n-1}}.$$

For some conformal mapping $\mathcal{F}(z)$ of $\mathfrak{U}$ onto $\aleph$,we conclude that $\aleph=\mathcal{F}(\mathfrak{U})$ and $\Phi'_{\mathcal{Z}_{\mathfrak{n}},1}[\mathcal{F}(\mathfrak{U}),\mathfrak{B}]$ will denoted by $\Phi'_{\mathcal{Z}_{\mathfrak{n}},1}[\mathcal{F},\mathfrak{B}]$, this because $\aleph$ is a domain which is simple connected not equal $\mathbb{C}$. The next result is a direct consequence of Theorem 3.4.

Theorem 3.5. Suppose that the function $\mathfrak{B}(z)$ belonging to $\in\mathcal{H}_0$, $\mathcal{F}(z)$ is analytic on $\mathfrak{U}$ and $\mathcal{X}\in\Phi_{\mathcal{Z}_{\mathfrak{n}},1}[\mathcal{F},\mathfrak{B}]$. If $h(z)\in\mathcal{A}(\mathfrak{n})$, $\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)\in\mathcal{D}_0$ and

$$\mathcal{X}\left(\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)}{z^{n-1}},\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{z^{n-1}},\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}{z^{n-1}};z\right),\ (\mu>2,\alpha\geq 0,\mathfrak{n}\in\mathbb{N}),$$

is univalent in $\mathfrak{U}$, then

$$\mathcal{F}(z)\prec\mathcal{X}\left(\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)}{z^{n-1}},\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{z^{n-1}},\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}{z^{n-1}};z\right),\tag{3.5}$$

implies

$$\mathfrak{B}(z)\prec\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)}{z^{n-1}}.$$

When we composite theorems 2.6 and 3.5, we get the next important corollary:

Corollary 3.2. Suppose that $\mathcal{F}_1(z)$ and $\mathfrak{B}_1(z)$ two analytic function in $\mathfrak{U}$, $\mathcal{F}_2(z)$ be univalent function in $\mathfrak{U}$, $\mathfrak{B}_2(z) \in \mathcal{D}_0$ with $\mathfrak{B}_1(z) = \mathfrak{B}_2(z) = 0$ and $\mathcal{X} \in \Phi_{\mathcal{Z}_n,1}[\mathcal{F}_2, \mathfrak{B}_2] \cap \Phi'_{\mathcal{Z}_n,1}[\mathcal{F}_1, \mathfrak{B}_1]$. If $h(z) \in \mathcal{A}(n)$, $\frac{\mathcal{Z}_{n,\gamma}^{\mu,\alpha} h(z)}{z^{n-1}} \in \mathcal{H}_0 \cap \mathcal{D}_0$ and

$$\mathcal{X}\left(\frac{\mathcal{Z}_n^{\mu,\alpha} h(z)}{z^{n-1}}, \frac{\mathcal{Z}_n^{\mu-1,\alpha} h(z)}{z^{n-1}}, \frac{\mathcal{Z}_n^{\mu-2,\alpha} h(z)}{z^{n-1}}; z\right),\ (\mu > 2, \alpha \geq 0, n \in \mathbb{N}),$$

is univalent in $\mathfrak{U}$, then

$$\mathcal{F}_1(z) \prec \mathcal{X}\left(\frac{\mathcal{Z}_n^{\mu,\alpha} h(z)}{z^{n-1}}, \frac{\mathcal{Z}_n^{\mu-1,\alpha} h(z)}{z^{n-1}}, \frac{\mathcal{Z}_n^{\mu-2,\alpha} h(z)}{z^{n-1}}; z\right) \prec \mathcal{F}_2(z),$$

implies

$$\mathfrak{B}_1(z) \prec \frac{\mathcal{Z}_n^{\mu,\alpha} h(z)}{z^{n-1}} \prec \mathfrak{B}_2(z).$$

Now, for the differential superordination we will give the duplex result of Theorem 2.7.

Definition 3.3. Let $\mathfrak{B}(z) \neq 0$, $z\mathfrak{B}'(z) \neq 0$ and $\aleph$ be a set in $\mathbb{C}$, and $\mathfrak{B}(z) \in \mathcal{H}$.The class of admissible functions $\Phi'_{\mathcal{Z}_{n,\gamma},2}[\aleph, \mathfrak{B}]$ consists of those functions $\mathcal{X}: \mathbb{C}^3 \times \overline{\mathfrak{U}} \to \mathbb{C}$ that satisfy the admissibility condition

$$\mathcal{X}(u, v, w; \delta) \in \aleph,$$

whenever $u = \mathfrak{B}(z)$, $v = \frac{z\mathfrak{B}'(z)}{yn\alpha\mathfrak{B}(z)} + \mathfrak{B}(z)$,

$$\mathcal{R}\left\{\frac{n\alpha(wv - 3vu + 2u^2)}{v - u}\right\} \leq \frac{1}{y}\mathcal{R}\left\{\frac{z\mathfrak{B}''(z)}{\mathfrak{B}'(z)} + 1\right\},$$

where $z \in \mathfrak{U}, \delta \in \partial\mathfrak{U}$, and $y \geq 1$.

Theorem 3.6. Let $\mathcal{X} \in \Phi'_{\mathcal{Z}_n,2}[\aleph, \mathfrak{B}]$. If $h(z) \in \mathcal{A}(n)$, $\frac{\mathcal{Z}_n^{\mu-1,\alpha} h(z)}{\mathcal{Z}_n^{\mu,\alpha} h(z)} \in \mathcal{D}_1$ and

$$\mathcal{X}\left(\frac{\mathcal{Z}_n^{\mu-1,\alpha} h(z)}{\mathcal{Z}_n^{\mu,\alpha} h(z)}, \frac{\mathcal{Z}_n^{\mu-2,\alpha} h(z)}{\mathcal{Z}_n^{\mu-1,\alpha} h(z)}, \frac{\mathcal{Z}_n^{\mu-3,\alpha} h(z)}{\mathcal{Z}_n^{\mu-2,\alpha} h(z)}; z\right)\ (\mu > 3, \alpha \geq 0, n \in \mathbb{N}),$$

is univalent in $\mathfrak{U}$, then

$$\aleph \subset \left\{\mathcal{X}\left(\frac{\mathcal{Z}_n^{\mu-1,\alpha} h(z)}{\mathcal{Z}_n^{\mu,\alpha} h(z)}, \frac{\mathcal{Z}_n^{\mu-2,\alpha} h(z)}{\mathcal{Z}_n^{\mu-1,\alpha} h(z)}, \frac{\mathcal{Z}_n^{\mu-3,\alpha} h(z)}{\mathcal{Z}_n^{\mu-2,\alpha} h(z)}; z\right): z \in \mathfrak{U}\right\}, \tag{3.6}$$

implies

$$\mathfrak{B}(z) \prec \frac{\mathcal{Z}_n^{\mu-1,\alpha} h(z)}{\mathcal{Z}_n^{\mu,\alpha} h(z)}.$$

Proof. We conclude from (2.28) and (3.6) that

$$\aleph \subset \{\psi(\mathcal{T}(z), z\mathcal{T}'(z), z^2\mathcal{T}''(z); z): z \in \mathfrak{U}\}.$$

From (2.27), the admissibility conditions for $\mathcal{X} \in \Phi'_{\mathcal{Z}_n,2}[\aleph, \mathfrak{B}]$ and ψ are synonymous to the admissibility condition as given in definition 1.2. Hence $\psi \in \Psi'[\aleph, \mathfrak{B}]$, Thus by Theorem 1.2 ,we obtain

$$\mathfrak{B}(z) \prec \mathcal{T}(z) \text{ or } \mathfrak{B}(z) \prec \frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)}.$$

then for some conformal mapping $\mathcal{F}(z)$ of $\mathfrak{U}$ onto $\aleph$ we have $\aleph = \mathcal{F}(\mathfrak{U})$,when $\aleph$ is a domain which is simply connected not equal $\mathbb{C}$, we will denote the class $\Phi'_{\mathcal{Z}_{\mathfrak{n}},2}[\mathcal{F},\mathfrak{B}]$ by the symbol class $\Phi'_{\mathcal{Z}_{\mathfrak{n}},2}[\mathcal{F}(\mathfrak{U}),\mathfrak{B}]$. The following result is a direct consequence of the previous theorem.

Theorem 3.7. Suppose that $\mathcal{X}$ lie in the class $\Phi'_{\mathcal{Z}_{\mathfrak{n}},2}[\mathcal{F},\mathfrak{B}]$, $\mathcal{F}(\mathrm{z})$ is analytic function in $\mathfrak{U}$. If $h(z) \in \mathcal{A}(n)$, $\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)} \in \mathcal{D}_1$ and

$$\mathcal{X}\left(\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)},\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)},\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-3,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu-2,\alpha}h(z)};z\right), \quad (\mu > 3, \alpha \geq 0, \mathfrak{n} \in \mathbb{N})$$

is univalent in $\mathfrak{U}$, then

$$\mathcal{F}(\mathrm{z}) \prec \mathcal{X}\left(\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)},\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)},\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-3,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu-2,\alpha}h(z)};z\right),$$

implies

$$\mathfrak{B}(z) \prec \frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)}.$$

We conclude The following sandwich theorem, if we commingle Theorems 2.8 and 3.7.

Corollary 3.3. Let $\mathcal{F}_2(z)$ be univalent function in $\mathfrak{U}$, $\mathfrak{B}_2(z) \in \mathcal{D}_{\mathcal{Z}_{\mathfrak{n}}}$ with $\mathfrak{B}_1(0) = \mathfrak{B}_2(0) = 1$, $\mathcal{F}_1(\mathrm{z})$ and $\mathfrak{B}_1(z)$ be analytic function in $\mathfrak{U}$, and $\mathcal{X} \in \Phi_{\mathcal{Z}_{\mathfrak{n}},2}[\mathcal{F}_2,\mathfrak{B}_2] \cap \Phi'_{\mathcal{Z}_{\mathfrak{n}},2}[\mathcal{F}_1,\mathfrak{B}_1]$. If $h(z) \in \mathcal{A}(\mathfrak{n})$, $\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)} \in \mathcal{H} \cap \mathcal{D}_1$, $\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z) \neq 0$ and

$$\mathcal{X}\left(\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)},\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)},\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-3,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu-2,\alpha}h(z)};z\right), (\mu > 3, \alpha \geq 0, \mathfrak{n} \text{ is a positive integer })$$

is analytic and injective in $\mathfrak{U}$, therefore

$$\mathcal{F}_1(\mathrm{z}) \prec \mathcal{X}\left(\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)},\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-2,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)},\frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-3,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu-2,\alpha}h(z)};z\right) \prec \mathcal{F}_2(\mathrm{z}),$$

implies

$$\mathfrak{B}_1(z) \prec \frac{\mathcal{Z}_{\mathfrak{n}}^{\mu-1,\alpha}h(z)}{\mathcal{Z}_{\mathfrak{n}}^{\mu,\alpha}h(z)} \prec \mathfrak{B}_2(z).$$

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