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Applications of Lucas-Balancing Polynomials to Estimate the Coefficients Bounds of analytic functions

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ABSTRACT

This work includes studying the existence of the first three coefficients of the Taylor coefficients series $|b_2|$ and $|b_3|$ of functions biunivalent related with polynomials are called Lucas Balancing. The existence of these bounds was proven and later used to find bounds on the Fecket-Szegö inequality. We obtained new and previous results by substituting appropriate values for the parameter values. The research topic is one of the new topics in dealing with the mentioned polynomials when associated with bi-univalent functions included in our work classes, namely $\mathcal{NH}_\Sigma(\alpha,\beta,\omega,\mathcal{R}(r,t))$, $\mathcal{AYA}_\Sigma(\vartheta,\gamma,\mathcal{R}(x,t))$ and $\mathcal{TA}_\Sigma(\mu,\mathcal{R}(x,t))$. which are new and defined by the concept of subordination.

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1. Introduction

Assume that the symbol $\mathcal{A}$ denote the set of all analytic functions h in the open unit disk $\mathcal{U}$ and defined by the formula

$$h(t) = t + \sum_{n=2}^{\infty} b_n t^n \ , \tag{1.1}$$

and satisfy the normalization conditions $h(0) = 0 = h'(0) - 1$, where $\mathcal{U} = \{t: t \in \mathbb{C}: |t| < 1\}$. Let

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the set of all univalent functions $h \in \mathcal{A}$ within $\mathcal{U}$ denote by $\mathcal{S}$. According to the Koebe one-quarter theorem [6], such that there exists an inverse function h^{-1} for every function $h \in \mathcal{S}$, satisfy

$$h^{-1}(h(t)) = t \quad ,(t \in \mathcal{U})$$

and

$$h^{-1}(h(v)) = v\left(|v| < v_0(h), v_0(h) \geq \frac{1}{4}\right),$$

where

$$h^{-1}(v) = g(v) = v - b_2v^2 + (2b_2^2 - b_3)v^3 - (5b_2^3 - 5b_2b_3 + b_4)v^4 + \cdots. \qquad (1.2)$$

For A function $h \in \mathcal{A}$,in case of each one of the functions h and h^{-1} ($i.e\ h^{-1}$ is a function inverse of h) are analytic and injective functions in $\mathcal{U}$, h is said to be bi-univalent in $\mathcal{U}$. The set of all bi-univalent functions in $\mathcal{U}$ denoted by Σ and defined as the form in (3.1).

The subject of subordination[13] is a mathematical dominance relationship between two functions, such as h and f, we say that a function h subordinate to f in this case ,this relationship is written in the form $h \prec f$ .This is achieved if there is a function such as $\mathcal{Z}(t)$, that is analytic in the unit disk ,such that $h(t) = f(\mathcal{Z}(t))$ and this function has the following properties $\mathcal{Z}(0) = 0$ and $|\mathcal{Z}(t)| < 1$ for every t that lies in the unit disk $\mathcal{U}$. Special case that if the function h in $\mathcal{U}$ has univalent property, then

$$h \prec f \Leftrightarrow h(0) = 0,$$

and

$$h(\mathcal{U}) \subset f(\mathcal{U}).$$

In this scientific article, we used an important polynomials in mathematics in general and important in geometric function theory in particular , called Lucas-Balancing ,where (B_n) called its Balancing numbers for n greater and equal zero, the first to introduce the topic of polynomials mentioned above were the researchers by Behera and Panda [3]. The recursive formula $B_{n+1} = 6B_n - B_{n-1}$ for n greater and equal 1 ,refers to the balancing numbers, with the primitive value set at $B_0 = 0$. The following expression $C_n\sqrt{8B_n^2 + 1}$ is a sequence of the Lucas-Balancing numbers, has garnered significant attention. It has a recursive relationship that takes the following form $C_{n+1} = 6C_n - C_{n-1}$ for n greater and equal 1, and have a first term as $C_0 = 0$ and $C_1 = 3$. Formore details, we refer readers to [7–10] ,[2,4,5,14,15,16]

Definition (1.1): (Lucas-Balancing polynomials, [11]) A polynomial have the form $C_n(r)$ called a Lucas-Balancing polynomials ,where r any complex number and n greater and equal to 2 , these polynomials are known by the following relationship:

$$C_n(r) = 6rC_{n-1}(r) - C_{n-2}(r)\ , \qquad (1.3)$$

using recurrence relation given by (1.3) we easily obtain

$$C_0(r)=1, \quad C_1(r)=3r, \quad C_2(r)=18r^2-1, \quad C_3(r)=108r^3-9r\,. \tag{1.4}$$

Other polynomial families are similar to the Lucas-Balancing polynomials, in that they can be generated using qualitative generating functions. Next lemma is considered as a first example of a generating function:

Lemma (1.1): [11] Lucas-Balancing polynomials have the following generator function

$$\mathcal{R}(r,t)=\sum_{n=0}^{\infty} C_n(r)t^n=\frac{1-3rt}{1-6rt+t^2}\,. \tag{1.5}$$

2. Estimation bounds and Fekete-Szegö problem for functions of the class $\mathcal{NH}_\Sigma(\alpha,\beta,\omega,\mathcal{R}(r,t))$

Definition (2.1): A function $h(t)$ of the form(1.1) that belonging to Σ is said to be in the subclass $\mathcal{NH}_\Sigma(\alpha,\beta,\omega,\mathcal{R}(r,t))$ and β,α are greater and equal zero, $\omega\in\mathbb{C}\setminus\{0\}$ and $r\in\mathbb{C}\setminus\left\{0,\pm\frac{1}{3}\sqrt{\frac{2\left((\beta+1)(\alpha+1)\right)^2}{4\omega(\beta+1)\prod_{j=1}^{2}(\beta+j)(\alpha+1)^2-(\alpha+\beta+(3\alpha+1))}}\right\}$. If the following conditions hold true:

$$1+\frac{1}{\omega}\left[(1-\alpha)\frac{th'(t)}{h(t)}\left(\frac{h(t)}{t}\right)^{\beta}+\alpha\left(1+\frac{th''(t)}{h'(t)}\right)\left(h'(t)\right)^{\beta}-1\right]\prec\mathcal{R}(r,t)\,, \tag{2.1}$$

and

$$1+\frac{1}{\omega}\left[(1-\alpha)\frac{vg'(v)}{g(v)}\left(\frac{g(v)}{v}\right)^{\beta}+\alpha\left(1+\frac{vg''(v)}{g'(v)}\right)\left(g'(v)\right)^{\beta}-1\right]\prec\mathcal{R}(r,v), \tag{2.2}$$

where g is the inverse of h and it is of the form (1.2).

we have the following definition If we chose $\omega=1$ and $\alpha=0$ in Definition (2.1):

Definition (2.2): A function $h(t)\in\Sigma$ of the form(1.1) is in the subclass $\mathcal{AH}_\Sigma(\beta,\mathcal{R}(r,t))$, β greater and equal zero and $r\in\mathbb{C}\setminus\left\{0,\pm\frac{1}{3}\sqrt{\frac{2(\beta+1)}{4(\beta+1)-(\beta+2)}}\right\}$, if the following conditions hold true:

$$\frac{t^{1-\beta}h'(t)}{h(t)^{1-\beta}}\prec\mathcal{R}(r,t)\,, \tag{2.3}$$

and

$$\frac{v^{1-\beta}g'(v)}{g(v)^{1-\beta}}\prec\mathcal{R}(r,v)\,, \tag{2.4}$$

where g is the inverse of h and it is of the form (1.2).

We have the following definition If we chose $\alpha=1$ in Definition (2.1):

Definition (2.3): A function $h(t) \in \Sigma$ of the form(1.1) is in the subclass $\mathcal{ZH}_{\Sigma}(\beta, \omega, \mathcal{R}(r,t))$ for β greater and equal zero, $\omega \in \mathbb{C} \setminus \{0\}$ and $r \in \mathbb{C} \setminus \left\{0, \pm\frac{1}{3}\sqrt{\frac{4(\beta+1)^2}{8(\beta+1)^2-\omega(\beta+2)(2\beta+1)}}\right\}$, if the following conditions hold true:

$$1 + \left[\frac{1}{\omega}\left[\left(1 + \frac{th''(t)}{h'(t)}\right)\left(h'(t)\right)^{\beta} - 1\right]\right] \prec \mathcal{R}(r,t), \qquad (2.5)$$

and

$$1 + \left[\frac{1}{\omega}\left[\left(1 + \frac{vg''(v)}{g'(v)}\right)\left(g'(v)\right)^{\beta} - 1\right]\right] \prec \mathcal{R}(r,v)\,, \qquad (2.6)$$

where g is the inverse of h and it is of the form (1.2).

Remark (2.1):

1) If we take $\alpha = \omega = 1$ and $\beta = 0$ the class $\mathcal{NH}_{\Sigma}\left(\alpha, \beta, \omega, \mathcal{R}(r,t)\right)$ reduce to the class ${}_{\mathcal{LB}}\mathcal{C}_{\Sigma}\left(\mathcal{R}(x,z)\right)$ which was introduced by öztÜrk and Aktas [12].
2) If we take $\alpha = \beta = 0$ and $\omega = 1$, the class $\mathcal{NH}_{\Sigma}\left(\alpha, \beta, \omega, \mathcal{R}(x,t)\right)$ reduce to the class ${}_{\mathcal{LB}}\mathcal{S}^{\star}_{\Sigma}\left(\mathcal{R}(x,z)\right)$ which was introduced by öztÜrk and Aktas [12].
3) If we take $\alpha = 0$, and $\omega = \beta = 1$ the class $\mathcal{NH}_{\Sigma}\left(\alpha, \beta, \omega, \mathcal{R}(x,t)\right)$ reduce to the class $\mathfrak{C}^{\mathcal{LB}}_{\Sigma}(\mathcal{R}(x,z))$ which was introduced by Arzu and Akgül [1].

Theorem(2.1): If the function $h(t) \in \mathcal{NH}_{\Sigma}\left(\alpha, \beta, \omega, \mathcal{R}(r,t)\right)$, α, β are greater and equal zero, $\omega \in \mathbb{C} \setminus \{0\}$ and $r \in \mathbb{C} \setminus \left\{0, \pm\frac{1}{3}\sqrt{\frac{2(\beta+1)^2(\alpha+1)^2}{4\omega(\beta+1)(\alpha+1)^2\ \prod_{j=1}^{2}(\beta+2)-\left((3\alpha+1)+\alpha+\beta\right)}}\right\}$. Then

$$|b_2| \leq \frac{3|\omega||r|\sqrt{2|3r|}}{\sqrt{\left|\left(\sqrt{2}(\alpha+1)(\beta+1)\right)^2 + 9\left[\omega(\beta+2)(1+\alpha+(3\alpha+1)\beta) - \left(\sqrt{4}(\alpha+1)(\beta+1)\right)^2\right]r^2\right|}}\,, \qquad (2.7)$$

and

$$|b_3| \leq \frac{3|\omega||r|}{(\beta+2)(2\alpha+1)} + \frac{|\omega|^2 9|r|^2}{\left((\beta+1)(\alpha+1)\right)^2}, \qquad (2.8)$$

and for some $\eta \in \mathbb{R}$,

$|b_3 - \eta b_2^2|$

$$\leq \begin{cases} \frac{3|\omega||r|}{(\beta+2)(2\alpha+1)} \; if \\ |1-\eta| \leq \frac{\left|\left(\sqrt{2}(\beta+1)(\alpha+1)\right)^2+9\left[\omega(\beta+2)(1+\alpha+(3\alpha\beta+\beta))-\left(\sqrt{4}(\beta+1)(\alpha+1)\right)^2\right]r^2\right|}{18|\omega|(\beta+2)(2\alpha+1)|r^2|} \\ \frac{54|\omega|^2|r|^3|1-\eta|}{\left|9\left[\omega(\beta+2)(1+\alpha+(3\alpha\beta+\beta))-\left(\sqrt{4}(\beta+1)(\alpha+1)\right)^2\right]r^2+\left(\sqrt{2}(\beta+1)(\alpha+1)\right)^2\right|} \; if \\ |1-\eta| \geq \frac{\left|\left(\sqrt{2}(\beta+1)(\alpha+1)\right)^2+9\left[\omega(\beta+2)(1+\alpha+(3\alpha+1)\beta)-\left(\sqrt{4}(\beta+1)(\alpha+1)\right)^2\right]r^2\right|}{18|\omega|(\beta+2)(2\alpha+1)|r^2|} \end{cases}, \tag{2.9}$$

Proof: Assume that $h \in \mathcal{NH}_\Sigma(\alpha, \beta, \omega, \mathcal{R}(r,t))$. By virtue of Definition (2.1), form the relation (2.1) and (2.2) we can write that

$$1+\frac{1}{\omega}\left[(1-\alpha)\frac{th'(t)}{h(t)}\left(\frac{h(t)}{t}\right)^\beta+\alpha\left(1+\frac{th''(t)}{h'(t)}\right)\left(h'(t)\right)^\beta-1\right]=\mathcal{R}(r,m(t)), \tag{2.10}$$

and

$$1+\frac{1}{\omega}\left[(1-\alpha)\frac{vg'(v)}{g(v)}\left(\frac{g(v)}{v}\right)^\beta+\alpha\left(1+\frac{vg''(v)}{g'(v)}\right)\left(g'(v)\right)^\beta-1\right]=\mathcal{R}(r,n(v)), \tag{2.11}$$

where $m, n \in \mathcal{U}$ are given to be of the form

$$m(t) = m_1 t + m_2 t^2 + m_3 t^3 + \cdots, \tag{2.12}$$

$$n(v) = n_1 v + n_2 v^2 + n_3 v^3 + \cdots, \tag{2.13}$$

and

$m(0) = n(0) = 0$, $|m(t)| < 1$, $|n(v)| < 1$. It is known that if for $t, v \in \mathcal{U}$,

$$|m(t)| = \left|\sum_{i=1}^{n} m_i t^i\right| < 1,$$

and

$$|n(v)| = \left|\sum_{i=1}^{n} \psi_i v^i\right| < 1.$$

Thus

$$|m_i| < 1, \tag{2.14}$$

and

$$|n_i| < 1. \tag{2.15}$$

For all $i \in \mathbb{N}$. Now, some basic calculations yield that

$$\frac{1}{\omega}\left[(1-\alpha)\frac{th'(t)}{h(t)}\left(\frac{h(t)}{t}\right)^\beta+\alpha\left(h'(t)\right)^\beta\left(1+\frac{th''(t)}{h'(t)}\right)-1\right]+1$$

$$= C_0(r) + [C_1(r)m_1]t + [C_1(r)m_2 + C_2(r)m_1^2]t^2 + \cdots, \tag{2.16}$$

and

$$\frac{1}{\omega}\left[(1-\alpha)\frac{vg'(v)}{g(v)}\left(\frac{g(v)}{v}\right)^{\beta} + \alpha\big(g'(v)\big)^{\beta}\left(1+\frac{vg''(v)}{g'(v)}\right) - 1\right] + 1$$

$$= C_0(r) + [C_1(r)n_1]v + [C_1(r)n_2 + C_2(r)n_1^2]v^2 + \cdots. \tag{2.17}$$

Equating coefficients in (2.16) and (2.17), yields

$$\frac{1}{\omega}[(\beta+1)(\alpha+1)]b_2 = C_1(r)m_1 , \tag{2.18}$$

$$\frac{1}{\omega}\Big[\big((\beta+2)(2\alpha+1)\big)b_3 + \frac{1}{2}\big((\beta-1)(\beta+2)(3\alpha+1)\big)b_2^2\Big] = C_1(r)m_2 + C_2(r)m_1^2 , \tag{2.19}$$

$$-\frac{1}{\omega}\big[\big((\beta+1)(\alpha+1)\big)\big]b_2 = C_1(r)n_1 , \tag{2.20}$$

and

$$\frac{1}{\omega}\Big[\big((\beta+2)(2\alpha+1)(2b_2^2 - b_3)\big) + \frac{1}{2}\big((\beta-1)(\beta+2)(3\alpha+1)\big)b_2^2\Big] = C_1(r)n_2 + C_2(r)n_1^2 , \tag{2.21}$$

from (2.18) and (2.20), we have

$$m_1 = -n_1, \tag{2.22}$$

and

$$\frac{2\big((\beta+1)(\alpha+1)\big)^2}{\omega^2} b_2^2 = [C_1(r)]^2(m_1^2 + n_1^2). \tag{2.23}$$

By summing of the equations (2.19) to (2.21), we obtain

$$\frac{(\beta+2)(\alpha+(3\alpha+1)\beta+1)}{\omega} b_2^2 = C_1(r)(m_2+n_2) + C_2(r)(m_1^2+n_1^2). \tag{2.24}$$

Substituting form (2.23) that value $(m_1^2 + n_1^2)$ in the equation (2.24), we deduce that

$$\left[\frac{(\beta+2)(\alpha+(3\alpha+1)\beta+1)}{\omega} - \frac{2\big((\beta+1)(\alpha+1)\big)^2 C_2(r)}{\omega^2[C_1(r)]^2}\right] b_2^2 = C_1(r)(m_2+n_2). \tag{2.25}$$

Which yields

$$|b_2| \le \frac{3|\omega||r|\sqrt{2|3r|}}{\sqrt{\left|\left(\sqrt{2}(\beta+1)(\alpha+1)\right)^2 + 9\left[(\alpha+(3\alpha\beta+\beta)+1)\omega(\beta+2) - \left(\sqrt{4}(\beta+1)(\alpha+1)\right)^2\right]r^2\right|}},$$

subtract the equation in (2.21) from the equation in (2.19), we have

$$\frac{2(\beta+2)(2\alpha+1)(b_3-b_2^2)}{\omega} = C_1(r)(m_2-n_2) + C_2(r)(m_1^2-n_1^2). \tag{2.26}$$

Then in view of (2.22) and (2.23), equation (2.26) becomes

$$b_3 = \frac{(m_2-n_2)C_1(r)\omega}{(2\beta+4)(2\alpha+1)} + \frac{(m_1^2+n_1^2)[C_1(r)]^2\omega^2}{\left(\sqrt{2}(\beta+1)(\alpha+1)\right)^2}.$$

Thus applying (1.4), we get

$$|b_3| \le 3\frac{|\omega|}{(\beta+2)(2\alpha+1)}|r| + 9\frac{|\omega|^2}{\left((\beta+1)(\alpha+1)\right)^2}|r|^2 .$$

It follows form (2.25) and (2.26), we get

$$b_3-\eta b_2^2 = \frac{\omega(m_2-n_2)C_1(r)}{2(\beta+2)(2\alpha+1)} + \frac{(1-\eta)\omega^2(m_2+n_2)[C_1(r)]^3}{\omega\left(\alpha+(\beta+1)(3\alpha+1)\right)[C_1(r)]^2(\beta+2)-2\left((\beta+1)(\alpha+1)\right)^2C_2(r)}$$

$$= \omega C_1(r)\left[\left(\psi(\eta,r)+\frac{1}{2(\beta+2)(2\alpha+1)}\right)m_2 + \left(\psi(\eta,r)-\frac{1}{2(\beta+2)(2\alpha+1)}\right)n_2\right],$$

where

$$\psi(\eta,x) = \frac{\omega(1-\eta)[C_1(r)]^2}{\omega(\beta+2)[C_1(r)]^2\left(1+\alpha+\beta(3\alpha+1)\right)-2\left((\alpha+1)(\beta+1)\right)^2C_2(r)}.$$

In view of (1.4). Conclude that

$$|b_3-\eta b_2^2| \le \begin{cases} \frac{|\omega||C_1(r)|}{(\beta+2)(2\alpha+1)} & if\ \ 0 \le |\psi(\eta,r)| \le \frac{1}{2(\beta+2)(2\alpha+1)} \\ 2|\omega||C_1(r)||\psi(\eta,r)| & if\ \ |\psi(\eta,r)| \ge \frac{1}{2(\beta+2)(2\alpha+1)} \end{cases} .$$

This completes the proof of Theorem (2.1).

Therefore the following fact is a consequence of the above theorem by setting $\omega = 1$ and $\alpha = 0$:

Corollary (2.1): If $h(t)$ given by (1.1) is in the class $\mathcal{AH}_\Sigma(\beta,\mathcal{R}(r,t))$, $\beta \ge 0$ and $r \in \mathbb{C}\backslash\left\{0, \pm\frac{1}{3}\sqrt{\frac{2(\beta+1)}{4(\beta+1)-(\beta+2)}}\right\}$. Then

$$|b_2| \le \frac{3|r|\sqrt{2|3r|}}{\sqrt{|(-3\beta^2-5\beta-2)9r^2+2\beta^2+4\beta+2|}},$$

and

$$|b_3| \le \frac{3|r|}{(\beta+2)} + \frac{9|r|^2}{\beta^2+2\beta+1},$$

and for some $\eta \in \mathbb{R}$,

$$|b_3-\eta b_2^2| \leq \begin{cases} \dfrac{3|r|}{(\beta+2)} \ if \\ |1-\eta| \leq \dfrac{|(-3\beta^2-5\beta-2)9r^2+2\beta^2+4\beta+2|}{(\beta+2)18|r^2|} \\ \dfrac{54|1-\eta||r|^3}{|(-3\beta^2-5\beta-2)9r^2+2\beta^2+4\beta+2|} \ if \\ |1-\eta| \geq \dfrac{|(-3\beta^2-5\beta-2)9r^2+2\beta^2+4\beta+2|}{(\beta+2)18|r^2|} \end{cases}.$$

The following corollary consequence by setting $\alpha = 1$ in the above theorem :

Corollary (2.2): If $h(t)$ is in the class $\mathcal{ZH}_\Sigma(\beta,\omega,\mathcal{R}(r,t))$ given by (1.1) , $\beta \geq 0$, $\omega \in \mathbb{C} \setminus \{0\}$ and $r \in \mathbb{C} \setminus \left\{0, \pm\frac{1}{3}\sqrt{\frac{4(\beta+1)^2}{8(\beta+1)^2-\omega(\beta+2)(2\beta+1)}}\right\}$. Then

$$|b_2| \leq \frac{3|\omega||r|\sqrt{2|3r|}}{\sqrt{|[2\omega(2\beta^2+5\beta+2)-16(\beta^2+2\beta+1)]9r^2+8(\beta^2+2\beta+1)|}},$$

and

$$|b_3| \leq \frac{|\omega||r|}{(\beta+2)} + \frac{|\omega|^2\, 9|r|^2}{4(\beta^2+2\beta+1)} \quad ,$$

and for some $\eta \in \mathbb{R}$,

$$|b_3-\eta b_2^2| \leq \begin{cases} \dfrac{|\omega||r|}{(\beta+2)} \ if \\ |1-\eta| \leq \dfrac{|[2\omega(2\beta^2+5\beta+2)-16(\beta^2+2\beta+1)]9r^2+8(\beta^2+2\beta+1)|}{|\omega|(\beta+2)54|r^2|} \\ \dfrac{54|\omega|^2|1-\eta||r|^3}{|[2\omega(2\beta^2+5\beta+2)-16(\beta^2+2\beta+1)]9r^2+8(\beta^2+2\beta+1)|} \ if \\ |1-\eta| \geq \dfrac{|[2\omega(2\beta^2+5\beta+2)-16(\beta^2+2\beta+1)]9r^2+8(\beta^2+2\beta+1)|}{|\omega|(\beta+2)54|r^2|} \end{cases}.$$

Remark (2.2): If we take the following options in Theorem (2.1).

1) $\alpha = \delta = 1$ and $\omega = 1$, then we get the results established by öztÜrk and Aktas [12].
2) $\alpha = \delta = 0$ and $\omega = 1$, then we get the results established by öztÜrk and Aktas [12].
3) $\alpha = 0$ and $\lambda = \omega = 1$, and then we get the results established by Arzu and Akgül [1].

3. Boundary inequalities for coefficients of functions in the class $\mathcal{AYA}_\Sigma(\vartheta,\gamma,\mathcal{R}(x,t))$

Definition (3.1): A function $h(t)$ of the form(1.1) that belonging to Σ is said to be in the subclass $\mathcal{AYA}_\Sigma(\beta,\vartheta,\mathcal{R}(r,t))$ and $\beta,\vartheta \geq 0$, $r \in \mathbb{C} \setminus \left\{0, \pm\frac{1}{3}\sqrt{\frac{((\beta+1)(\vartheta+1))^2}{2((\beta+1)(\vartheta+1))^2-\frac{1}{2}(2+\vartheta(\vartheta+3)+(\vartheta+1)2\beta)}}\right\}$,if the following conditions hold true:

$$\left(\frac{h(t)}{t}\right)^{\vartheta}\frac{th'(t)}{h(t)}+\beta\left[1+\vartheta\left(\frac{th'(t)}{h(t)}-1\right)+\frac{th''(t)}{h'(t)}-\frac{th'(t)}{h(t)}\right]\prec\mathcal{R}(r,t)\,, \qquad (3.1)$$

and

$$\left(\frac{g(v)}{v}\right)^{\vartheta}\frac{vg'(v)}{g(v)}+\beta\left[1+\vartheta\left(\frac{vg'(v)}{g(v)}-1\right)+\frac{vg''(v)}{g'(v)}-\frac{vg'(v)}{g(v)}\right]\prec\mathcal{R}(r,v), \qquad (3.2)$$

where g is the inverse of h and it is of the form (1.2) .

Remark(3.1):

1) If we take $\beta = 1$ and $\vartheta = 0$ the class $\mathcal{AYA}_{\Sigma}(\beta,\vartheta,\mathcal{R}(r,t))$ reduce to the class ${}_{\mathcal{LB}}\mathcal{C}_{\Sigma}\left(\mathcal{R}(x,z)\right)$ which was introduced by öztÜrk and Aktas [12].
2) If we take $\beta = 0$ and $\vartheta = 0$, the class $\mathcal{AYA}_{\Sigma}(\beta,\vartheta,\mathcal{R}(r,t))$ reduce to the class ${}_{\mathcal{LB}}\mathcal{S}^{\star}_{\Sigma}\left(\mathcal{R}(x,z)\right)$ which was introduced by öztÜrk and Aktas [12].
3) If we take $\beta = 0$ and $\vartheta = 1$ the class $\mathcal{AYA}_{\Sigma}(\beta,\vartheta,\mathcal{R}(r,t))$ reduce to the class $\mathfrak{C}^{\mathcal{LB}}_{\Sigma}\left(\mathcal{R}(x,z)\right)$ which was introduced by Arzu and Akgül [1].

Theorem(3.1): If the function $h\in\mathcal{AYA}_{\Sigma}(\beta,\vartheta,\mathcal{R}(r,t))$, $\beta,\vartheta\geq 0$, and $r\in\mathbb{C}\backslash\left\{0,\pm\frac{1}{3}\sqrt{\frac{((\vartheta+1)(\beta+1))^2}{2((\beta+1)(\vartheta+1))^2-\left(1+\frac{\vartheta}{2}(\vartheta+3)+\beta(\vartheta+1)\right)}}\right\}$. Then

$$|b_2|\leq\frac{3\sqrt{|3r|}\,|r|}{\sqrt{\left|9\left[\frac{1}{2}(2+\vartheta(\vartheta+3)+2(\vartheta+1)\beta)-2((\vartheta+1)(\beta+1))^2\right]r^2+((\vartheta+1)(\beta+1))^2\right|}}\,, \qquad (3.3)$$

and

$$|b_3|\ \leq\frac{3|r|}{(\vartheta+2)(2\beta+1)}+\frac{9|r|^2}{((\vartheta+1)(\beta+1))^2}, \qquad (3.4)$$

and for some $\eta\in\mathbb{R}$,

$|b_3-\eta b_2^2|$

$$\leq\begin{cases}\dfrac{3|r|}{(\vartheta+2)(2\beta+1)} \quad if \\ |1-\eta|\leq\dfrac{\left|9\left[\left(1+\frac{1}{2}(\vartheta+3)\vartheta+(\vartheta+1)\beta\right)-\left(\sqrt{2}(\vartheta+1)(\beta+1)\right)^2\right]r^2+((\beta+1)(\vartheta+1))^2\right|}{9(\vartheta+2)(2\beta+1)|r^2|} \\ \dfrac{27|r|^3|1-\eta|}{\left|9\left[1+\frac{1}{2}\vartheta(\vartheta+3)+(\vartheta+1)\beta-\left(\sqrt{2}(\vartheta+1)(\beta+1)\right)^2\right]r^2+((\beta+1)(\vartheta+1))^2\right|} \quad if \\ |1-\eta|\geq\dfrac{\left|9\left[1+\frac{1}{2}\vartheta(\vartheta+3)+\beta(\vartheta+1)-2((\vartheta+1)(\beta+1))^2\right]r^2+((\beta+1)(\vartheta+1))^2\right|}{9(2\beta+1)(\vartheta+2)|r^2|}\end{cases}. \qquad (3.5)$$

Proof: Assume that $h\in\mathcal{AYA}_{\Sigma}(\beta,\vartheta,\mathcal{R}(r,t))$. By virtue of Definition (3.1), form the relation (3.1) and (3.2) we can write that

$$\left(\frac{h(t)}{t}\right)^{\vartheta}\frac{th'(t)}{h(t)}+\beta\left[1+\frac{th''(t)}{h'(t)}+\vartheta\left(\frac{th'(t)}{h(t)}-1\right)-\frac{th'(t)}{h(t)}\right]=\mathcal{R}(r,m(t))\,, \tag{3.6}$$

and

$$\left(\frac{g(v)}{v}\right)^{\vartheta}\frac{vg'(v)}{g(v)}+\beta\left[1+\frac{vg''(v)}{g'(v)}+\vartheta\left(\frac{vg'(v)}{g(v)}-1\right)-\frac{vg'(v)}{g(v)}\right]=\mathcal{R}(r,n(v)), \tag{3.7}$$

where $m,n\in\mathcal{U}$ are given to be of the form

$$m(t)=m_1t+m_2t^2+m_3t^3+\cdots, \tag{3.8}$$

$$n(v)=n_1v+n_2v^2+n_3v^3+\cdots, \tag{3.9}$$

and $m(0)=n(0)=0$, $|m(t)|<1$, $|n(v)|<1$. It is known that if for $t,v\in\mathcal{U}$,

$$|m(t)|=\left|\sum_{i=1}^{n}m_it^i\right|<1\,,$$

and

$$|n(v)|=\left|\sum_{i=1}^{n}n_iv^i\right|<1\,.$$

Thus

$$|m_i|<1, \tag{3.10}$$

and

$$|n_i|<1\,. \tag{3.11}$$

For all $i\in\mathbb{N}$. Now, Some basic calculations yield that

$$\frac{th'(t)}{h(t)}\left(\frac{h(t)}{t}\right)^{\vartheta}+\beta\left[1+\frac{th''(t)}{h'(t)}-\frac{th'(t)}{h(t)}+\vartheta\left(\frac{th'(t)}{h(t)}-1\right)\right]$$

$$=C_0(r)+[C_1(r)m_1]t+\ [C_1(r)m_2+C_2(r)m_1^2]\,t^2+\cdots, \tag{3.12}$$

and

$$\frac{vg'(v)}{g(v)}\left(\frac{g(v)}{v}\right)^{\vartheta}+\beta\left[1+\frac{vg''(v)}{g'(v)}-\frac{vg'(v)}{g(v)}+\vartheta\left(\frac{vg'(v)}{g(v)}-1\right)\right]$$

$$=C_0(r)+[C_1(r)n_1]v+\ [C_1(r)n_2+C_2(r)n_1^2]\,v^2+\cdots. \tag{3.13}$$

Equating coefficients in (3.12) and (3.13), yields

$$(\vartheta+1)(\beta+1)b_2=C_1(r)m_1, \tag{3.14}$$

$$(\vartheta+2)(2\beta+1)b_3+\frac{[\vartheta^2+\vartheta(1-2\beta)-2(3\beta+1)]}{2}b_2^2=C_1(r)m_2+C_2(r)m_1^2, \qquad (3.15)$$

$$-(\vartheta+1)(\beta+1)b_2=C_1(r)n_1\,, \qquad (3.16)$$

and

$$-(\vartheta+2)(2\beta+1)b_3+\frac{[\vartheta^2+5(\vartheta+2\beta)+6(\vartheta\beta+1)]}{2}b_2^2=C_1(x)n_2+C_2(x)n_1^2\,. \qquad (3.17)$$

From (3.14) and (3.16), we have

$$m_1=-n_1, \qquad (3.18)$$

and

$$2\big((\vartheta+1)(\beta+1)\big)^2 b_2^2=[C_1(r)]^2(m_1^2+n_1^2)\,. \qquad (3.19)$$

By summing of the equations (3.15) to (3.17), we obtain

$$[\vartheta(\vartheta+3)+2\beta(\vartheta+1)+2]b_2^2=C_1(r)(m_2+n_2)+C_2(r)(m_1^2+n_1^2). \qquad (3.20)$$

Substituting form (3.19) that value $(m_1^2+n_1^2)$ in the equation (3.20), we deduce that

$$\left[2+(\vartheta^2+3\vartheta)+(2\vartheta+2)\beta-\frac{\big(\sqrt{2}(\vartheta+1)(\beta+1)\big)^2C_2(r)}{[C_1(r)]^2}\right]b_2^2=C_1(r)(m_2+n_2). \qquad (3.21)$$

Which yields

$$|b_2|\le\frac{3|r|\sqrt{|3r|}}{\sqrt{\left|9\left[\frac{1}{2}(2+\vartheta(\vartheta+3)+2(\vartheta+1)\beta)-2\big((\vartheta+1)(\beta+1)\big)^2\right]r^2+\big((\vartheta+1)(\beta+1)\big)^2\right|}}\;,$$

we subtract the equation (3.17) from (3.18), we have

$$2(\vartheta+2)(2\beta+1)(b_3-b_2^2)=C_1(r)(m_2-n_2)+C_2(r)(m_1^2-n_1^2). \qquad (3.22)$$

Then In view of (3.18) and (3.19), equation (3.22) becomes

$$b_3=\frac{(m_2-n_2)C_1(r)}{2(\vartheta+2)(2\beta+1)}+\frac{(m_1^2+n_1^2)[C_1(r)]^2}{2\big((\vartheta+1)(\beta+1)\big)^2}.$$

Thus applying (1.4), we get

$$|b_3|\le 3\frac{|r|}{(\vartheta+2)(2\beta+1)}+9\frac{|r|^2}{\big((\vartheta+1)(\beta+1)\big)^2}.$$

It follows form (3.21) and (3.22), we get

$$b_3-\eta b_2^2=\frac{C_1(r)(m_2-n_2)}{2(2\beta+1)(\vartheta+2)}+\frac{(1-\eta)[C_1(r)]^3(m_2+n_2)}{[C_1(r)]^2(2+(\vartheta^2+3\vartheta)+\beta(2\vartheta+2))-\left(\sqrt{2}(\beta+1)(\vartheta+1)\right)^2C_2(r)}$$

$$=C_1(r)\left[\left(\psi(\eta,r)+\frac{1}{2(\vartheta+2)(2\beta+1)}\right)m_2+\left(\psi(\eta,r)-\frac{1}{2(\vartheta+2)(2\beta+1)}\right)n_2\right],$$

where

$$\psi(\eta,r)=\frac{(1-\eta)[C_1(r)]^2}{[C_1(r)]^2(2+\vartheta\,(\vartheta+3)+2(\vartheta+1)\beta)-2\left((\vartheta+1)(\beta+1)\right)^2C_2(r)}\,.$$

In view of (1.4). Conclude that

$$|b_3-\eta b_2^2|\le\begin{cases}\dfrac{|C_1(r)|}{(\vartheta+2)(2\beta+1)} & if\ \ 0\le|\psi(\eta,r)|\le\dfrac{1}{2(\vartheta+2)(2\beta+1)}\\ 2|C_1(r)||\psi(\eta,r)| & if\ \ |\psi(\eta,r)|\ge\dfrac{1}{2(\vartheta+2)(2\beta+1)}\end{cases}.$$

Thus the proof is complete ■

Remark (3.2): Suppose that we take the following options in Theorem (3.1).

1) $\beta=1$ and $\vartheta=0$, then we get the results established by öztÜrk and Aktas [12].
2) $\beta=0$ and $\vartheta=0$, then we get the results established by öztÜrk and Aktas [12].
3) $\beta=0$ and $\vartheta=1$, then we get the results established by Arzu and Akgül [1].

4. Fekete-Szegö inequality and coefficient bounds for the class $\mathcal{TA}_\Sigma(\mu,\mathcal{R}(x,t))$

Definition(4.1): We say that a function $g(t)$ of the form(1.1) that belonging to Σ be in the subclass $\mathcal{TA}_\Sigma(\mu,\mathcal{R}(r,t))$ for $0\le\mu\le1$ and $r\in\mathbb{C}\backslash\left\{0,\pm\frac{1}{3}\sqrt{\frac{(1+3\mu-2\mu^2)}{(-12\mu^4+28\mu^3-19\mu^2+4\mu+1)}}\right\}$, if the following conditions hold true:

$$\frac{th'(t)+(2\mu^2-\mu)t^2h''(t)}{4(\mu-\mu^2)t+(2\mu^2-\mu)tg'(t)+(2\mu^2-3\mu+1)h(t)}\prec\mathcal{R}(r,t)\,, \tag{4.1}$$

and

$$\frac{vg'(v)+(2\mu^2-\mu)v^2g''(v)}{4(\mu-\mu^2)v+(2\alpha^2-\mu)vg'(v)+(2\mu^2-3\mu+1)g(v)}\prec\mathcal{R}(r,v), \tag{4.2}$$

where g is the inverse of h and it is of the form (1.2).

Remark (4.1): The class $\mathcal{TA}_\Sigma(\mu,\mathcal{R}(r,t))$ reduce to the class ${}_{\mathcal{LB}}\mathcal{S}^{\star}_{\Sigma}(\mathcal{R}(x,z))$ which was introduced by öztÜrk and Aktas [12] If we take $\mu=0$ in the previous theorem .

Theorem(4.1): If the function $h\in\mathcal{TA}_\Sigma(\mu,\mathcal{R}(x,t))$, $0\le\mu\le1$ and $r\in\mathbb{C}\backslash\left\{0,\pm\frac{1}{3}\sqrt{\frac{(1+3\mu-2\mu^2)}{(-12\mu^4+28\mu^3-19\mu^2+4\mu+1)}}\right\}$. Then

$$|b_2| \leq \frac{3|r|\sqrt{|3r|}}{\sqrt{|[12\mu^4-28\mu^3+19\mu^2-4\mu-1]9r^2+(1+3\mu-2\mu^2)|}}, \tag{4.3}$$

and

$$|b_3| \leq \frac{3|r|}{2(2\mu^2+1)} + \frac{9|r|^2}{(1+3\mu-2\mu^2)}, \tag{4.4}$$

and for some $\eta \in \mathbb{R}$,

$$|b_3 - \eta b_2^2| \leq \begin{cases} \frac{3|r|}{2(2\mu^2+1)} \; if \\ |1-\eta| \leq \frac{|[12\mu^4-28\mu^3+19\mu^2-4\mu-1]9r^2+(1+3\mu-2\mu^2)|}{(2\mu^2+1)18|r^2|} \\ \frac{27|r|^3|1-\eta|}{|[12\mu^4-28\mu^3+19\mu^2-4\mu-1]9r^2+(1+3\mu-2\mu^2)|} \; if \\ |1-\eta| \geq \frac{|[12\mu^4-28\mu^3+19\mu^2-4\mu-1]9r^2+(1+3\mu-2\mu^2)|}{(2\mu^2+1)18|r^2|} \end{cases}, \tag{4.5}$$

Proof: Assume that $h \in \mathcal{T}\mathcal{A}_\Sigma(\mu, \mathcal{R}(r,t))$. By virtue of Definition (3.1), form the relations (4.1) and (4.2) , we can be write that

$$\frac{th'(t)+(2\mu^2-\mu)t^2h''(t)}{4(\mu-\mu^2)t+(2\mu^2-\mu)tg'(t)+(2\mu^2-3\mu+1)h(t)} \prec \mathcal{R}(r, m(t)), \tag{4.6}$$

and

$$\frac{vg'(v)+(2\mu^2-\mu)v^2g''(v)}{4(\mu-\mu^2)v+(2\alpha^2-\mu)vg'(v)+(2\mu^2-3\mu+1)g(v)} \prec \mathcal{R}(r, n(v)), \tag{4.7}$$

where $m, n \in \mathcal{U}$ are given to be of the form

$$m(t) = m_1t + m_2t^2 + m_3t^3 + \cdots, \tag{4.8}$$

$$n(v) = n_1v + n_2v^2 + n_3v^3 + \cdots, \tag{4.9}$$

and $m(0) = n(0) = 0$, $|m(t)| < 1$, $|n(v)| < 1$. It is known that if for $t, v \in \mathcal{U}$,

$$|m(t)| = \left|\sum_{i=1}^{n} m_i t^i\right| < 1,$$

with

$$|n(v)| = \left|\sum_{i=1}^{n} n_i v^i\right| < 1.$$

Therefore

$$|m_i| < 1, \tag{4.10}$$

and

$$|n_i| < 1\,. \tag{4.11}$$

For all $i \in \mathbb{N}$. Now, Some basic calculations yield that

$$\frac{th'(t) + (2\mu^2 - \mu)t^2h''(t)}{4(\mu - \mu^2)t + (2\mu^2 - \mu)tg'(t) + (2\mu^2 - 3\mu + 1)h(t)}$$

$$= C_0(r) + [C_1(r)m_1]t + [C_1(r)m_2 + C_2(r)m_1^2]t^2 + \cdots, \tag{4.12}$$

and

$$\frac{vg'(v) + (2\mu^2 - \mu)v^2g''(v)}{4(\mu - \mu^2)v + (2\alpha^2 - \mu)vg'(v) + (2\mu^2 - 3\mu + 1)g(v)}$$

$$= C_0(r) + [C_1(r)n_1]t + [C_1(r)n_2 + C_2(r)n_1^2]t^2 + \cdots. \tag{4.13}$$

Equating coefficients in (3.12) and (3.13), yields

$$(1 + 3\mu - 2\mu^2)b_2 = C_1(r)m_1\ , \tag{4.14}$$

$$(12\mu^4 - 28\mu^3 + 11\mu^2 + 2\mu - 1)b_2^2 + (4\mu^2 + 2)b_3 = C_1(r)m_2 + C_2(r)m_1^2, \tag{4.15}$$

$$-(1 + 3\mu - 2\mu^2)b_2 = C_1(r)n_1, \tag{4.16}$$

and

$$(12\mu^4 - 28\mu^3 + 19\mu^2 + 2\mu + 3)b_2^2 - (4\mu^2 + 2)b_3 = C_1(r)n_2 + C_2(r)n_1^2, \tag{4.17}$$

from (4.14) and (4.16), we have

$$m_1 = -n_1, \tag{4.18}$$

and

$$2(1 + 3\mu - 2\mu^2)^2b_2^2 = [C_1(r)]^2(m_1^2 + n_1^2). \tag{4.19}$$

By summing of the equations (4.15) to (4.17), we obtain

$$(24\mu^4 - 56\mu^3 + 30\mu^2 + 4\mu + 2)b_2^2 = C_1(r)(m_2 + n_2) + C_2(r)(m_1^2 + n_1^2). \tag{4.20}$$

Substituting form (4.19) that value $(m_1^2 + n_1^2)$ in the equation (4.20), we deduce that

$$\left[(24\mu^4 - 56\mu^3 + 30\mu^2 + 4\mu + 2) - \frac{2(1+3\mu-2\mu^2)^2C_2(r)}{[C_1(r)]^2}\right]b_2^2 = C_1(r)(m_2 + n_2), \tag{4.21}$$

which yields

$$|b_2| \leq \frac{3|r|\sqrt{|3r|}}{\sqrt{|[12\mu^4 - 28\mu^3 + 19\mu^2 - 4\mu - 1]9r^2 + (1 + 3\mu - 2\mu^2)|}},$$

we subtract the equation (4.17) from (4.18), we have

$$4(2\mu^2 + 1)(b_3 - b_2^2) = C_1(r)(m_2 - n_2) + C_2(r)(m_1^2 - n_1^2). \tag{4.22}$$

Then In view of (4.18) and (4.19), equation (4.22) becomes

$$b_3 = \frac{C_1(r)(m_2 - n_2)}{4(2\mu^2 + 1)} + \frac{[C_1(r)]^2(m_1^2 + n_1^2)}{2(1 + 3\mu - 2\mu^2)^2}.$$

Thus applying (1.4), we get

$$|b_3| \leq \frac{3|r|}{2(2\mu^2 + 1)} + \frac{9|r|^2}{(1 + 3\mu - 2\mu^2)}.$$

It follows form (4.21) and (4.22), we get

$$b_3 - \eta b_2^2 = \frac{C_1(r)(m_2 - n_2)}{4(2\mu^2 + 1)} + \frac{[C_1(r)]^3(1 - \eta)(m_2 + n_2)}{(24\mu^4 - 56\mu^3 + 30\mu^2 + 4\mu + 2)[C_1(r)]^2 - 2(1 + 3\mu - 2\alpha^2)^2 C_2(r)}$$

$$= \frac{C_1(r)}{2}\left[\left(\psi(\eta, r) + \frac{1}{2(2\mu^2+1)}\right) m_2 + \left(\psi(\eta, r) - \frac{1}{2(2\mu^2+1)}\right) n_2\right],$$

where

$$\psi(\eta, r) = \frac{[C_1(r)]^2(1 - \eta)}{(12\mu^4 - 28\mu^3 + 15\mu^2 + 2\mu + 1)[C_1(r)]^2 - (1 + 3\mu - 2\mu^2)^2 C_2(r)}.$$

In view of (1.4). Conclude that

$$|b_3 - \eta b_2^2| \leq \begin{cases} \dfrac{|C_1(r)|}{2(2\mu^2 + 1)} & if \ \ 0 \leq |\psi(\eta, r)| \leq \dfrac{1}{2(2\mu^2 + 1)} \\ |C_1(r)||\psi(\eta, r)| & if \ \ |\psi(\eta, r)| \geq \dfrac{1}{2(2\mu^2 + 1)} \end{cases}.$$

Thus we have the required result .

Remark (4.2): We obtain the result that introduced by öztÜrk and Aktas in [12] when we take the option $\mu = 0$ in Theorem (4.1).

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