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The Properties (Raw) and (gRaw) as a Generalization of The Weyl's Theorem

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ABSTRACT

In this paper, we introduce a novel generalization of Weyl's theorem of bounded linear operators, which dented by ($\mathcal{R}aw$) and ($g\mathcal{R}aw$). Moreover, we demonstrate that this generalization encompasses and extends the properties (aw) and (gaw) by the researchers in [1]. Our approach utilizes key characteristics of the relatively regular alongside the properties of constrained operators to explore and verify the proposed generalization. Several illustrative examples are provided to support the theoretical results and to highlight the applicability of the presented theorems.

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1. Introduction

The space X represents an infinite dimensional Banach space. The following notations are used in this paper: $\mathcal{L}(X)$ is a space of all bounded linear operators on X, *if* $\Upsilon \in \mathcal{L}(X)$, then $\sigma(\Upsilon), \rho(\Upsilon), \mathrm{N}(\Upsilon), \mathcal{R}(\Upsilon), ker\ \Upsilon, \dot{\alpha}(\Upsilon)$ *and* $\dot{\beta}(\Upsilon)$ mean the spectrum, the resolvent, the null space , the range, the kernel and the nullity of Υ, respectively. The finite ascent and finite descent of operator $\Upsilon \in \mathcal{L}(X)$ denoted by $p(\Upsilon)$ and $q(\Upsilon)$ are defined $p \in Z^+, \bigcup_{n\in N} \ker \Upsilon^n = ker\ \Upsilon^{p}$ and smallest $q \in Z, \bigcap_{n\in N} \ker \Upsilon^n = \mathcal{R}(\Upsilon^{q})$, respectively [2]. For each operator $\Upsilon \in \mathcal{L}(X)$, the following statements are always hold: :

$p(\Upsilon) < \infty$ leads to $\dot{\alpha}(\Upsilon) \le \dot{\beta}(\Upsilon)$;

$q(\Upsilon) < \infty$ leads to $\dot{\beta}(\Upsilon) \le \dot{\alpha}(\Upsilon)$;

$p(\Upsilon) = q(\Upsilon) < \infty$ leads to $\dot{\alpha}(\Upsilon) = \dot{\beta}(\Upsilon)$;

$\dot{\alpha}(\Upsilon) = \dot{\beta}(\Upsilon) < \infty$ and $p(\Upsilon)$ or $q(\Upsilon)$ is finite leads to $p(\Upsilon) = q(\Upsilon)$.

Here, in this work, we use the notation $\mathcal{L}^{-1}(X)$ as the space

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$\mathcal{L}^{-1}(X) = \{\Upsilon \in \mathcal{L}(X);\ \dot{\alpha}(\Upsilon) = \dot{\beta}(\Upsilon) = 0\}$. The classes of all upper semi Fredholm operators $\Omega_U(X)$, all lower semi-Fredholm operators $\Omega_L(X)$, all semi Fredholm operators $\Omega_{UL}(X)$ and all Fredholm operators are defend in the following, respectively [3]

$$\Omega_U(X) = \{\Upsilon \in \mathcal{L}(X);\ \dot{\alpha}(\Upsilon) < \infty \text{ and } \mathcal{R}(\Upsilon) \text{ is a closed}\}$$

$$\Omega_L(X) = \{\Upsilon \in \mathcal{L}(X);\ \dot{\beta}(\Upsilon) < \infty\}$$

$$\Omega_{UL}(X) = \Omega_U(X) \cup \Omega_L(X)$$

$$\Omega_{UL}(X) = \{\Upsilon \in \mathcal{L}(X);\ \dot{\alpha}(\Upsilon) < \infty, \dot{\beta}(\Upsilon) < \infty \text{and } \mathcal{R}(\Upsilon) \text{ is a closed}\}$$

$$\Omega(X) = \Omega_U(X) \cap \Omega_L(X)$$

Additionally, the index of $\Upsilon \in \Omega_{UL}(X)$ is defined as $\widetilde{Ind}(\Upsilon) = \dot{\alpha}(\Upsilon) - \dot{\beta}(\Upsilon)$. Therefore, the classes $\Omega_U^-(X)$ and $\Omega_L^+(X)$ are defined as follows

$$\Omega_U^-(X) = \{\Upsilon \in \Omega_U(X): \widetilde{Ind} \leq 0\},\ \Omega_L^+(X) = \{\Upsilon \in \Omega_L(X): \widetilde{Ind}(\Upsilon) \geq 0\}.$$

On the other hand, let us consider the following classes associate with $\mathcal{B}$ −operator as follows [4]

$$\mathcal{B}\Omega_U(X) = \{\Upsilon, \Upsilon_{[m]} \in \Omega_U(X),\ m \in Z^+\},$$

$$\mathcal{B}\Omega_L(X) = \{\Upsilon, \Upsilon_{[m]} \in \Omega_L(X), m \in Z^+\}$$

$$\mathcal{B}\Omega(X) = \{\Upsilon, \Upsilon_{[m]} \in \Omega(X), \quad m \in Z^+\}$$

where, the notations $\Upsilon_{[m]}, \mathcal{B}\Omega_U(X), \mathcal{B}\Omega_L(X), \mathcal{B}\Omega_U^-(X)$ and $\mathcal{B}\Omega_L^+(X)$ represent $\Upsilon_{[m]} = \Upsilon^m$, *for some* $m \in Z^+$, upper semi $\mathcal{B}$-Fredholm Fredholm, lower semi $\mathcal{B}$-Fredholm Fredholm and $\mathcal{B}$-Fredholm operator, respectively. In the same context, we define the following classes

$$\mathcal{B}\Omega_U^-(X) = \{\Upsilon \in \mathcal{B}\Omega_U(X): \widetilde{Ind}(\Upsilon) \leq 0\}$$

$$\mathcal{B}\Omega_L^+(X) = \{\Upsilon \in \mathcal{B}\Omega_L(X): \widetilde{Ind}(\Upsilon) \geq 0\}$$

Clearly, $\widetilde{Ind}(\Upsilon_{[n]}) = \mathrm{Ind}(\Upsilon)$.

Below, we provide a concise overview of certain definitions and notations that will be used in this article [5]

- Weyl operators class: $\mathcal{w}(\Upsilon) = \{\Upsilon \in \Omega(X): \widetilde{Ind}(\Upsilon) = 0\}$.
- $\mathcal{B}$ − Weyl operators class: $\mathcal{B}\mathcal{w}(\Upsilon) = \{\Upsilon \in \mathcal{B}\Omega(X): \widetilde{Ind}(\Upsilon) = 0\}$.
- Spectrum: $\sigma(\Upsilon) = \{\psi \in \mathbb{C}: \psi I - \Upsilon \text{ is not invertable}\}$.
- Defect spectrum: $\sigma_{sur}(\Upsilon) = \{\psi \in \mathbb{C}: \psi I - \Upsilon \text{ is not surjective}\}$.
- Approximate point spectrum: $\sigma_a(\Upsilon) = \{\psi \in \mathbb{C}: \psi I - \Upsilon \text{ is not bounded below}\}$.
- Weyl spectrum: $\sigma_{\mathcal{w}}(\Upsilon) = \{\psi \in \mathbb{C}: \psi I - \Upsilon \notin \mathcal{w}(\Upsilon)\}$.
- $\mathcal{B}$ −Weyl spectrum: $\sigma_{\mathcal{B}\mathcal{w}}(\Upsilon) = \{\psi \in \mathbb{C}: \psi I - \Upsilon \notin \mathcal{B}\mathcal{w}(\Upsilon)\}$.
- Weyl essential approximate point spectrum: $\sigma_{\Omega_U^-}(\Upsilon) = \{\psi \in \mathbb{C}: \psi I - \Upsilon \notin \Omega_U^-(X)\}$.
- Weyl essential subjectivity spectrum: $\sigma_{\Omega_L^+}(\Upsilon) = \{\psi \in \mathbb{C}: \psi I - \Upsilon \notin \Omega_L^+(X)\}$.
- $\mathcal{B}$ −Weyl essential approximate point spectrum: $\sigma_{\mathcal{B}\Omega_U^-}(\Upsilon) = \{\psi \in \mathbb{C}: \psi I - \Upsilon \notin \mathcal{B}\Omega_U^-(X)\}$
- $\mathcal{B}$ −Weyl essential subjectivity spectrum: $\sigma_{\mathcal{B}\Omega_L^+}(\Upsilon) = \{\psi \in \mathbb{C}: \psi I - \Upsilon \notin \mathcal{B}\Omega_L^+(X)\}$.
- Set of all upper semi Browder operators: $b_U(X) = \{\Upsilon \in \Omega_U(X): \wp(\mathcal{S}) < \infty\}$.

- Set of all lower semi Browder operators: $b_L(X) = \{\Upsilon \in \Omega_L(X): q(\mathcal{S}) < \infty\}$.
- Set of all Browder operators: $b(X) = b_U(X) \cap b_L(X)$.
- Browder spectrum: $\sigma_b(\Upsilon) = \{\psi \in \mathbb{C}: \psi I - \Upsilon \notin b(X)\}$.
- $\mathcal{B}_a(X) = \{\Upsilon \in \mathcal{L}(X): \Upsilon$ *is* bounded below operator$\}$
- $\mathcal{SU}(X) = \{\Upsilon \in \mathcal{L}(X): \Upsilon$ *is* surjective operator$\}$
- Class of left Drazin invertible: $LD(X) = \{\Upsilon \in \mathcal{L}(X): p(\Upsilon) < \infty \text{ and } \mathcal{R}(\Upsilon^{p(\Upsilon)+1}) \text{ is closed}\}$ [6].
- left Drazin invertible spectrum: $\sigma_{LD}(\Upsilon) = \{\psi \in \mathbb{C}: \psi I - \Upsilon \notin LD(X)\}$ [1].

Now, the notations $\pi(\Upsilon)$ and $\pi^0(\Upsilon)$ represent the set of all poles of the resolvent of Υ and the set of all poles of the resolvent of Υ of finite rank, respectively [7]. A $\psi \in \sigma_a(\Upsilon)$ is called left pole of Υ, if satisfy $\psi I - \Upsilon$ is left Drazin invertible and that $\psi \in \sigma_a(\Upsilon)$ is a left pole of T of finite rank if ψ is a left pole of Υ and $\dot{\alpha}(\psi I - \Upsilon) < \infty$. The notations $\pi_a(\Upsilon)$ represents set of all left poles of Υ, and $\pi_a^0(\Upsilon)$ represents set of all left poles of Υ of finite rank[8]. For each $\Upsilon \subset \mathbb{C}$, $iso(\Upsilon)$ is the set of isolated points of Υ. $E(\Upsilon)$ is the set of all isolated eigenvalues of Ţ, the sets $E^0(\Upsilon)$ and E_a^0 are defined, respectively $E^0(\Upsilon) = \{\psi \in iso(\sigma(\Upsilon)): 0 < \dot{\alpha}(\psi I - \Upsilon) < \infty\}$ and $E_a^0(\Upsilon) = \{\psi \in iso(\sigma_a): 0 < \dot{\alpha}(\psi I - \Upsilon) < \infty\}$ [1]. With regard to the literature review and previous studies, the key can be finding can be summarized as follows: An operator $\Upsilon \in \mathcal{L}(X)$ satisfies the following theorems and properties under assumptions

Source	**Theorem (property)**	**Associated condition**
[9]	Weyl's theorem	$\sigma(\Upsilon) \setminus \sigma_{\mathcal{W}}(\Upsilon) = E^0(\Upsilon)$
[9]	a-Weyl's theorem	$\sigma_a(\Upsilon) \setminus \sigma_{\Omega_U^-}(\Upsilon) = E_a^0(\Upsilon)$.
[9]	Generalized Weyl's Theorem	$\sigma_a(\Upsilon) \setminus \sigma_{\mathcal{B}\Omega_U^-}(\Upsilon) = E_a(\Upsilon)$
[10]	Browder's theorem	$\sigma(\Upsilon) \setminus \sigma_{\mathcal{W}}(\Upsilon) = \pi^0(\Upsilon)$
[10]	a- Browder's theorem	$\sigma_a(\Upsilon) \setminus \sigma_{\Omega_U^-}(\Upsilon) = \pi_a^0(\Upsilon)$
[10]	Generalized Browder's theorem	$\sigma(\Upsilon) \setminus \sigma_{\mathcal{B}\mathcal{W}}(\Upsilon) = \pi(\Upsilon)$
[10]	Generalized a- Browder's Theorem	$\sigma_a(\Upsilon) \setminus \sigma_{\mathcal{B}\Omega_U^-}(\Upsilon) = \pi_a(\Upsilon)$
[11]	Property (w)	$\sigma_a(\Upsilon) \setminus \sigma_{\Omega_U^-}(\Upsilon) = E^0(\Upsilon)$
[11]	Property (gw)	$\sigma(\Upsilon) \setminus \sigma_{\mathcal{B}\Omega_U^-}(\Upsilon) = E(\Upsilon)$
[1]	Property (aw)	$\sigma(\Upsilon) \setminus \sigma_{\mathcal{W}}(\Upsilon) = E_a^0(\Upsilon)$
[1]	Property (gaw)	$\sigma(\Upsilon) \setminus \sigma_{\mathcal{B}\mathcal{W}}(\Upsilon) = E_a(\Upsilon)$

The contributions of of this work is to establish a new generalization of Weyl's theorem by utilizing the properties of pseudo-operators. Throughout this research, the proposed generalization is thoroughly discussed, and several new properties and extensions within the framework of Weyl's theory are introduced. This approach not only broadens the theoretical foundation of Weyl's theorem but also provides a deeper analytical perspective on the spectral behavior of linear operators.

2. Theory and formula

The pseudo inverse or relatively regular operator plays a significant role in the study of finite or infinite dimensional spaces, particularly when traditional operator inverses do not exist. The pseudo inverse is widely applied in control theory [12], Markov chains, and the analysis of dynamical systems [13]. This section outlines the key definitions, properties, and conditions associated with the pseudo inverse, establishing the theoretical groundwork necessary for the subsequent development of the proposed results.

An operator $\Upsilon \in \mathcal{L}(X)$ is said to be pseudo invertible or relatively regular if the exists $\Upsilon^{\dagger} \in \mathcal{L}(X)$, which it satisfies $\Upsilon^{\dagger}\,\Upsilon\, y = y$ for all $y \in X/\mathcal{N}(\Upsilon)$ and $\Upsilon^{\dagger} y = 0$ for all $y \in X/\mathcal{R}(\Upsilon)$ [14]. In the context, $\Upsilon \in \mathcal{L}(X)$ is said to be a $\mathcal{B}-$relatively regular when , there exists $r \in Z_0 = Z^+ \cup \{0\}$, such that $\Upsilon^r\Upsilon^{\dagger} = \Upsilon^{\dagger}\Upsilon^r$ and $\Upsilon^r\Upsilon^{\dagger}\,\Upsilon^r = \Upsilon^r$.

Proposition 1. [15] Let $\Upsilon \in \Omega_{UL}(X)$. Then

- $\Upsilon \in \mathcal{W}(\Upsilon) \Leftrightarrow \Upsilon^{\dagger} \in \mathcal{L}^{-1}(X)$ and $\Upsilon\Upsilon^{\dagger}\,\Upsilon = \Upsilon$.
- $\Upsilon \in \Omega_L^{+}(X) \Leftrightarrow \Upsilon^{\dagger}$ is a bounded below and $\Upsilon\Upsilon^{\dagger}\,\Upsilon = \Upsilon$.
- $\Upsilon \in \Omega_U^{-}(X) \Leftrightarrow \Upsilon^{\dagger}$ is a surjective and $\Upsilon\Upsilon^{\dagger}\,\Upsilon = \Upsilon$.

Proposition 2. [15] Consider $\Upsilon \in \mathcal{L}(X)$, then $p(\Upsilon) = q(\Upsilon) < \infty$ if and only if there exists $\Upsilon^{\dagger} \in \mathcal{L}^{-1}(X)$ and $r \in Z_0$ satisfies $\Upsilon^r\Upsilon^{\dagger}\,\Upsilon^r = \Upsilon^r$ and $\Upsilon^r\Upsilon^{\dagger} = \Upsilon^{\dagger}\Upsilon^r$, $\Upsilon^{\dagger} \in \mathcal{L}^{-1}(X)$.

The following notations are defined as follows:

- $\mathcal{R}_{se}(X) = \{\Upsilon \in \mathcal{L}(X) : \Upsilon \text{ is relatively regular}\}$.
- $\mathcal{R}_{se}^{\Omega_{UL}}(X) = \Omega_{UL}(X) \cap \mathcal{R}_{se}(X)$.
- $\mathcal{RB}_{se}(X) = \{\Upsilon \in \mathcal{L}(X) : \Upsilon \text{ is } \mathcal{B}-\text{relatively regular}\}$.
- $\mathcal{R}_{se}^{sur}(X) = \{\Upsilon \in \mathcal{R}_{se}(X) : \Upsilon\Upsilon^{\dagger}\,\Upsilon = \Upsilon\,;\ \Upsilon^{\dagger} \in \mathcal{SU}(X)\}$.
- $\mathcal{RB}_{se}^{sur}(X) = \{\Upsilon \in \mathcal{RB}_{se}(X) :\ \Upsilon^{\dagger} \in \mathcal{SU}(X)\}$
- $\sigma_{\mathcal{R}_{se}^{sur}}(\Upsilon) = \{\psi \in \mathbb{C} : \psi I - \Upsilon \notin \mathcal{R}_{se}^{sur}(X)\}$
- $\sigma_{\mathcal{RB}_{se}^{sur}}(\Upsilon) = \{\psi \in \mathbb{C} : \psi I - \Upsilon \notin \mathcal{RB}_{se}^{sur}(X)\}$
- $\mathcal{R}_{se}^{o}(X) = \{\Upsilon \in \mathcal{R}_{se}(X) : \Upsilon\Upsilon^{\dagger}\,\Upsilon = \Upsilon\,,\ \Upsilon^{\dagger} \in \mathcal{L}^{-1}(X)\}$
- $\sigma_{\mathcal{R}_{se}^{o}}(\Upsilon) = \{\psi \in \mathbb{C} : \psi I - \Upsilon \notin \mathcal{R}_{se}^{o}(X)\}$
- $\mathcal{RB}_{se}^{o}(X) = \{\Upsilon \in \mathcal{RB}_{se}(X) : \Upsilon\Upsilon^{\dagger}\,\Upsilon = \Upsilon\,,\ \Upsilon^{\dagger} \in \mathcal{L}^{-1}(X)\}$
- $\sigma_{\mathcal{RB}_{se}^{o}}(\Upsilon) = \{\psi \in \mathbb{C} : \psi I - \Upsilon \notin \mathcal{RB}_{se}^{o}(X)\}$

Corollary 3. [15] Consider X is a Banach space. Then,

- $\mathcal{W}(X) = int\left(\mathcal{R}_{se}^{o}(X)\right)$
- $\Omega_U^{-}(X) = int\left(\mathcal{R}_{se}^{sur}(X)\right)$
- $\mathcal{BW}(X) = int\left(\mathcal{RB}_{se}^{o}(X)\right)$

In this paper, a necessary and critical assumption is required to ensure the validity of the result. This assumption is defined as the space X has property $\mathcal{SFSE}$ if, $\mathcal{R}_{se}(X) = \mathcal{R}_{se}^{\Omega_{UL}}(X)$.

3. Result discussions

This section is dedicated to introducing a novel spectral properties associated with Wyle's -type operators. Several fundamental characteristics of these properties are analyzed and derived, leading to meaningful generalizations of Wyle's theorem. Moreover, we investigate sufficient conditions under which these properties are satisfied.

Definition 1. [8] An operator $\Upsilon \in \mathcal{L}(\Upsilon)$, is considered to have the property (aw), provided it satisfies $\sigma(\Upsilon)\backslash\sigma_{\mathcal{w}}(\Upsilon) = E_a^0(\Upsilon)$, and considered to have property $(g$aw) provided it satisfies $\sigma(\Upsilon)\backslash\sigma_{\mathcal{B}\mathcal{w}}(\Upsilon) = E_a(\Upsilon)$.

Definition 2. An operator $\Upsilon \in \mathcal{L}(X)$, is said to have property $(\mathcal{R}$aw), if

$\sigma(\Upsilon)\backslash\sigma_{\mathcal{R}_{se}^o}(\Upsilon) = E_a^0(\Upsilon)$, and have property $(g\mathcal{R}$aw) if

$\sigma(\Upsilon)\backslash\sigma_{\mathcal{R}\mathcal{B}_{se}^o}(\Upsilon) = E_a(\Upsilon)$.

Lemma 4. Let X be a Banach space which it has property $\mathcal{SFSe}$. If operator

$\Upsilon \in \mathcal{L}(X)$ is a belong to class $\mathcal{RB}_{se}^o(X)$, *and* $\dot{\alpha}(\Upsilon)$ *is finite then* Υ *belong to the class* $\mathcal{R}_{se}^o(X)$.

Proof: If $\Upsilon \in \mathcal{RB}_{se}^o(X)$, then $\Upsilon \in \mathcal{RB}_{se}(X)$, $\Upsilon\Upsilon^\dagger\Upsilon = \Upsilon$, $\Upsilon^\dagger \in \mathcal{L}^{-1}(X)$. But the space X satisfies the property $\mathcal{SF}$Se, from Corollary 2.3 and (Lemma 3.2. [1]), the desired result can be concluded.

Theorem 5. 3.2 For any $\Upsilon \in \mathcal{L}(X)$ that satisfies property $(g\mathcal{R}$aw), and X satisfies $\mathcal{SFSe}$ property, it necessarily satisfies property $(\mathcal{R}$aw).

Proof: Suppose that the operator Υ has $(g\mathcal{R}$aw) property, this means from Definition 1. $\sigma(\Upsilon)\backslash\sigma_{\mathcal{RB}_{se}^o}(\Upsilon) = E_a(\Upsilon)$. Let $\psi \in \sigma(\Upsilon)\backslash\sigma_{\mathcal{R}_{se}^o}(\Upsilon)$,then $\psi \in \sigma(\Upsilon)\backslash\sigma_{\mathcal{RB}_{se}^o}(\Upsilon)$, and ψ is an eigenvalue of Υ isolated in $\sigma_a(\Upsilon)$. By using the assumption that the space X possesses the property $\mathcal{SFSe}$, from Corollary 3. , we obtain that $\psi I - \Upsilon$ is a Wyle operator and $\dot{\alpha}(\psi I - \Upsilon) < \infty$, therefore,

$\psi \in E_a^0(\Upsilon)$.

Conversely, let $\psi \in E_a^0(\Upsilon)$, then ψ is an eigenvalue of Υ isolated in $\sigma_a(\Upsilon)$ and $\dot{\alpha}(\psi I - \Upsilon) < \infty$, but Υ satisfies property $(g\mathcal{R}$aw) then $\psi \in \sigma(\Upsilon)\backslash\sigma_{\mathcal{RB}_{se}^o}(\Upsilon)$ and $\psi I - \Upsilon \in \mathcal{B}\Omega_U(X)$. Since $\dot{\alpha}(\psi I - \Upsilon) < \infty$, this leads from Lemma 4. $\psi I - \Upsilon \in \Omega_U(X)$. On the other hand,since $\widetilde{\text{Ind}}(\psi I - \Upsilon) = 0$, then $\psi I - \Upsilon \in \mathcal{w}(\Upsilon - \psi I)$ and $\psi \in \sigma(\Upsilon)\backslash\sigma_{\mathcal{w}}(\Upsilon)$. Finally, we have $\sigma(\Upsilon)\backslash\sigma_{\mathcal{R}_{se}^o}(\Upsilon) = E_a^0(\Upsilon)$.

In general, the converse of the above theorem does not hold as shown by the following examples.

Example 1.

Let us define an operator Υ on a Banach space $X = l^2(N) \oplus l^2(N)$ as follows $\Upsilon(v_1, v_2, v_3, \dots) = 0 \oplus \left(0, \frac{v_1}{2}, \frac{v_2}{2}, \frac{v_3}{2}, \dots\right)$, then the space X has the property $\mathcal{SFSe}$ and the bounded linear operator Υ satisfies $\sigma(\Upsilon) = \{0\}$, $\sigma_{\mathcal{R}_{se}^o}(\Upsilon) = \{0\}$, $\sigma_{\mathcal{RB}_{se}^o}(\Upsilon) = \{0\}$, $E_a^0(\Upsilon) = \emptyset$ and $E_a(\Upsilon) = \{0\}$, *therefore,* $\sigma(\Upsilon)\backslash\sigma_{\mathcal{R}_{se}^o}(\Upsilon) = E_a^0(\Upsilon)$ *but* $\sigma(\Upsilon)\backslash\sigma_{\mathcal{RB}_{se}^o}(\Upsilon) \neq E_a(\Upsilon)$. *Then,* Υ *satisfies property* $(g\mathcal{R}$aw) *but does not satisfies property* $(\mathcal{R}$aw).

Theorem 3.3 6. Let X be a Banach space which satisfies $\mathcal{SFSe}$ property. Then, for each $\Upsilon \in \mathcal{L}(X)$, satisfies property $(g\mathcal{R}$aw), if and only if it satisfies generalized Browder's theorem and $\pi(\Upsilon) = \pi_a(\Upsilon) = E_a(\Upsilon)$.

Proof. Assume that the property $(g\mathcal{R}$ab) of Υ is satisfied, then

$\sigma(\Upsilon)\backslash\sigma_{\mathcal{RB}_{se}^o}(\Upsilon) = E_a(\Upsilon)$, if we take $\psi \in \sigma(\Upsilon)\backslash\sigma_{\mathcal{RB}_{se}^o}(\Upsilon)$ this gives , $\psi \in iso(\sigma_a(\Upsilon))$ and $\psi I - \Upsilon \in \mathcal{RB}_{se}^o(\Upsilon)$ but Giving that the space X possesses the property $\mathcal{SFSe}$, from Corollary 3. immediately yields the result $\psi I - \Upsilon \in \mathcal{B}\Omega_L^+(X)$, hence from (Theorem 2.8.[16]) we have $\psi \in \pi_a(\Upsilon)$ and from (Corollary 2.7. [1]) $\pi(\Upsilon) = \pi_a(\Upsilon)$. This leads to $\sigma(\Upsilon)\backslash\sigma_{\mathcal{RB}_{se}^o}(\Upsilon) \subseteq \pi_a(\Upsilon)$. Now, Let $\psi \in \pi_a(\Upsilon)$, but, $\pi_a(\Upsilon) \subseteq E_a(\Upsilon)$ is always hold, then $\psi \in E_a(\Upsilon)$ and Υ satisfies generalized Browder's theorem and $\pi(\Upsilon) = \pi_a(\Upsilon) = E_a(\Upsilon)$.

Conversely, Assume that Υ satisfies generalized Browder's theorem and $\pi(\Upsilon) = \pi_a(\Upsilon) = E_a(\Upsilon)$, then $\sigma(\Upsilon)\setminus\sigma_{\mathcal{B}w}(\Upsilon) = \pi_a(\Upsilon)$. Therefore, $\sigma(\Upsilon)\setminus\sigma_{\mathcal{B}w}(\Upsilon) = E_a(\Upsilon)$ and Υ satisfies property $(g\mathcal{R}aw)$.

Similarly to Theorem 6., we have the following result concerning property $(\mathcal{R}aw)$, which it is gave without proof.

Theorem 7. 3.4 If $\Upsilon \in \mathcal{L}(X)$, and X has meets property $\mathcal{SFS}e$ then, Υ satisfies meets property $(\mathcal{R}aw)$, if and only if Browder's theorem and $\pi^0(\Upsilon) = \pi_a^0(\Upsilon) = E_a^0(\Upsilon)$.

Theorem 8. 3.5 Let $\Upsilon \in \mathcal{L}(X)$ and X has property $\mathcal{SFS}e$. Then, the following statements are equivalent:

i. Υ meets property$(g\mathcal{R}aw)$

ii. Υ meets property $(\mathcal{R}aw)$ and $E_a(\Upsilon) = E(\Upsilon)$.

Proof: Consider that Υ meets property $(g\mathcal{R}aw)$, then from Theorem 3.1 and Theorem 3.3 Υ satisfies property $(\mathcal{R}aw)$ and $E_a(\Upsilon) = E(\Upsilon)$. Conversely, consider that Υ meets property $(\mathcal{R}aw)$ and $E_a(\Upsilon) = E(\Upsilon)$. From Theorem 7., Υ meets Browder's theorem, but Browder's theorem is equivalent to the generalized Browder's theorem [17], it follows that Υ satisfies the generalized Browder's theorem. Therefore, $\sigma(\Upsilon)\setminus\sigma_{\mathcal{B}w}(\Upsilon) = \pi(\Upsilon)$, but the space X possesses the property $\mathcal{SFS}e$, applying Corollary 3. and Theorem 7.we have $\sigma(\Upsilon)\setminus\sigma_{\mathcal{RB}_{se}^{o}}(\Upsilon) = E(\Upsilon) = E_a(\Upsilon)$ this implies to Υ meets property$(g\mathcal{R}aw)$

Theorem 9. Assume that is a bounded linear operator defined on a space Υ endowed with the $\mathcal{SF}$Se property. Then, Υ satisfies the $(\mathcal{R}aw)$ property, if and only if it possesses the (aw) property.

Proof: The satisfaction of $(\mathcal{R}aw)$ property by the operator $\Upsilon \in \mathcal{L}(\Upsilon)$ implies, in a straightforward manner, that $\sigma(\Upsilon)\setminus\sigma_{\mathcal{R}_{se}^{o}}(\Upsilon) = E_a^0(\Upsilon)$. Nevertheless, the operator Υ is defined on a space Υ endowed with the $\mathcal{SF}$Se property. Hence, by applying *Corollary 2.3., we obtain* $\sigma_{\mathcal{R}_{se}^{o}} = \sigma_{w}$, this leads to $\sigma(\Upsilon)\setminus\sigma_{w}(\Upsilon) = E_a^0(\Upsilon)$. Therefore, from Definition 3.1 Υ satisfies (aw) property. Conversely, if the operator Υ satisfies (aw) property, then it necessarily fulfills the condition $\sigma(\Upsilon)\setminus\sigma_{w}(\Upsilon) = \pi_a^0(\Upsilon)$. however, the space Υ on which the operator Υ is defined possesses $\mathcal{SF}$Se property, this gives $\mathcal{R}_{se}(\Upsilon) = \mathcal{R}_{se}^{\Phi UL}(\Upsilon)$ and from *Corollary 2.3., we get* $\sigma_{\mathcal{R}_{se}^{o}} = \sigma_{w}$ this means that $\sigma(\Upsilon)\setminus\sigma_{\mathcal{R}_{se}^{o}}(\Upsilon) = E_a^0(\Upsilon)$ and $(\mathcal{R}aw)$ property is hold.

A proof of the next theorem follows by applying reasoning analogous to that used in the proof of the foregoing result.

Theorem 10. Assume that Υ is a bounded linear operator defined on a space Υ endowed with the $\mathcal{SF}$Se property. Then, Υ satisfies the $(g\mathcal{R}aw)$ property, if and only if it possesses the $(g$aw) property.

4. Conclusion

Weyl's theorem and its generalizations play a significant and essential role in extending classical spectral results to broader and more comprehensive classes of bounded linear operators, particularly in infinite-dimensional spaces. In fact, these generalizations provide a deeper understanding of both the essential and point spectra and establish stronger connections among spectral properties.

In this study, new generalizations of Weyl's theorem as well as properties $(\mathcal{R}aw)$ and $(g\mathcal{R}aw)$are revealed using a class of operators known as pseudo invertible. By exploiting the characteristics of these operators, we establish the generalized results and prove the necessary and sufficient conditions to obtain new spectral findings. Additionally, the study of such type of operators is considered a very important topic and the focus of attention of many researchers in this field. It is of course a modern topic, investigated new properties of bounded linear operators by exploring the association between the relatively regular and generalizations of Wyle theorem.

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