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Adjacent Line Graphs: Combinatorial Bounds and Structural Properties

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ABSTRACT

The adjacent line graph $AL(G)$ of a graph G is defined as the graph whose vertices correspond to the edges of G. Two vertices e and f are adjacent in $AL(G)$ if and only if the distance between their corresponding edges in G is at most one. Formally, for edges e and f, the adjacency condition is:e is adjacent to f in $AL(G)$ $\Leftrightarrow d_G(e,f) \leq 1$ where $d_G(e,f)$ is the minimum distance between any pair of their endpoints in G. This paper establishes new tight upper bounds for the independence number $\alpha(AL(G)$ and chromatic number $\chi(AL(G)$ for several fundamental classes of graphs, including paths P_n, cycles C_n, and complete bipartite graphs $K_{m,n}$. We present rigorous theoretical proofs for these bounds, correct previous inaccuracies in edge-count formulas, and derive important structural properties, including results on the Hamiltonicity of adjacent line graphs. The results provide a broader understanding of AL(G) and its applications in various fields, including networking and coding theory.

MSC..

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1. Introduction

Graph theory is a crucial component of discrete mathematics and computer science, providing fundamental solutions to many complex problems across various disciplines. It relies on the structural framework of these systems, and within this field, numerous studies have emerged focusing on vertice- and edge-centered graph properties. N. Harari [25], Bondi and Morty [26], and West [27] established important methodological foundations and concepts for graph properties. In [19], Benic and Baga presented extensive treatments for linear graph sedimentation and its properties.

Montaner-Patl and Takahashi [1] presented important results on the power-independence number relationship as a result of recent developments in graph theory. Meanwhile, Dong Wu [2] presented important theories and results for the independence number. Furthermore, Shuz et al. [3] developed physics-based neural networks to improve the structural integrity of graphs. In [5, 6], they studied complex problems of large independent sets. Hekel and Riordan

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[11]provided an in-depth study of the chromaticity of random graphs. The concept of the large chromaticity was introduced by Aksu et al. [12]. Akbari et al. [14] played a significant role in advancing our understanding of vertex stability. In [13], Yirzada and Khan presented a relationship between the spectrum of a graph and its properties, specifically the chromaticity. Contiguous graphs (AL(G)) are among the properties that have an important effect on graphs, based on the findings of Manisha et al. [16]. In [20], Abrishami et al. introduced the concept of edges where vertices are located at their ends and within a distance of only one vertex in the graph, for edges $e = uv$ and $f = xy$ in G, we define

$$d_G(e,f) = \min\{d_G(u,x), d_G(u,y), d_G(v,x), d_G(v,y)\}$$

, and declare e adjacent to f in AL(G) if $d_G(e,f) \leq 1$. The aforementioned recognition is not limited to edges that are connected but also includes those edges associated with single median vertices, making this adjacency second-order. In [8], Wang and Zhang presented a study on the independence number and its relationship to networks. Grzek et al. in [4] presented a study on algorithms for calculating the independence number of P6-free graphs. Due to the prominent importance of graphs and their significant impact on the speed and accuracy of the results required for computational complex operations in several branches of computer science, especially networking, this is evident in a study presented by Ni et al. [21] on spectral graphs. Despite this progress in graph calculation, some studies on the color number and independence number still lack precision. Leff et al. [17] presented a study on the boundary energy of linear graphs. Thompakkara et al. presented graph partitioning, but the boundaries of some graph classes and properties, such as P_n paths, C_n cycles, and K_m and K_n perfect binary graphs, are still somewhat incomplete. These research gaps are of great importance given the challenges facing graphs and their applications in quantitative optimization [5, 6] and network analysis [21]. In this study, we presented new precise limits for α(AL(G)) and χ(AL(G)) across simple and well-known graph sets, while studying the properties of the Hamiltonian graph in parallel with recent studies conducted by Keçeci [22] and Behague et al. [23]. It also included setting precise limits in multiple fields, including computational applications, dynamical systems, and coding theory, thus providing a foundation for future studies in line graphs. The second section of this study presents the basic definitions and preliminary results. Our main theoretical contributions on independence and chromaticity are presented in the third section, while the fourth section deals with the Hamiltonian properties of the linear graph. The fifth section concludes with future discussions of this study.

Definition 2.1: (Distance between edges) Let $G = (V, E)$ be a simple connected graph. For any two edges $e = uv$ and $f = xy$ in $E(G)$, the distance between e and f in G is defined as:

$$d_G(e,f) = \min\{d_G(u,x), d_G(u,y), d_G(v,x), d_G(v,y)\}$$

where $d_G(a,b)$ denotes the length of a shortest path between vertices a and b in G.

Definition 2.2: (Adjacent Line Graph) The adjacent line graph of G, denoted $AL(G)$, is the simple graph with vertex set $V(AL(G)) = E(G)$, and edge set:

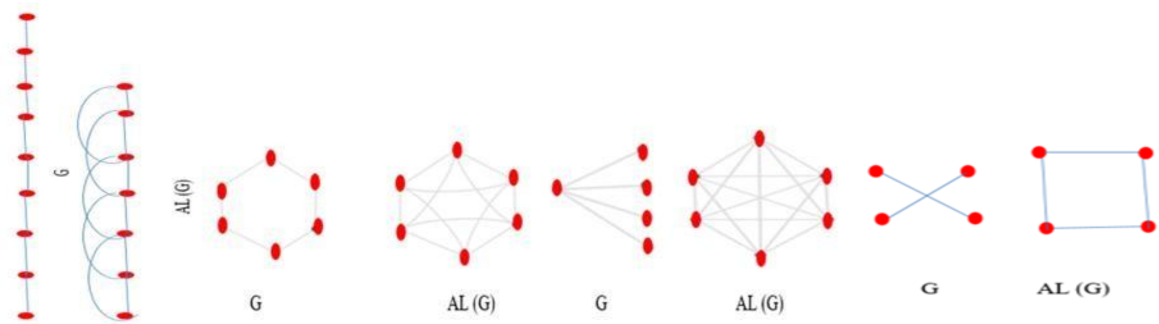
$$E(AL(G)) = \{\{e,f\} \subseteq E(G): e \neq f \text{ and } d_G(e,f) \leq 1\}$$

Definition 2.3: (Independence Number).The independence number α(H) of a graph H is the size of a largest set of pairwise non-adjacent vertices in H .

Definition 2.4: (Chromatic Number).The chromatic number χ(H) is the minimum number of colors needed to color the vertices of H so that no two adjacent vertices share the same color.

Definition 2.5: (Hamiltonian Graph).A graph is Hamiltonian if it contains a cycle that visits every vertex exactly once.

Example 2.6:



Remark 2.7:

1. If $(v_1, v_2, \dots, v_n)$ are vertices of G, then the number of vertices in $AL(G)$ is equal to $\sum_{1}^{n}\binom{p}{2}$, where is the degree of the vertex.
2. The number of vertices in $AL(p_n), AL(C_n), AL(K_{1,n}), AL(W_n)$ are respectively n-2, n, $\frac{n(n-1)}{2}$, $\frac{n(n-1)(n-2)}{2}$ and $\frac{(n-1)(n+4)}{2}$.

3. Main Results

Theorem 3.1: For a path graph Pn with n≥3 vertices, the number of edges in its adjacent line graph satisfies: $E(AL(Pn)) = 2n-5$.

Proof:

The adjacent line graph $AL(Pn)$ has vertices corresponding to the $n-1$ edges of Pn, where two vertices ei and ej are adjacent if $|i-j|=1$ (consecutive edges sharing a vertex) or $|i-j|=2$ (edges separated by one intermediate edge).

We compute vertex degrees in $AL(Pn)$:

- Boundary edges ($e1, en-1$): each has degree 2.
- Sub-boundary edges ($e2, en-2$): each has degree 3.
- Internal edges ($3 \le i \le n-3$): each has degree 4.

Summing all vertex degrees:

$\sum \deg(v)$=(boundary)2+2+(sub-boundary)3+3+(internal) $4(n-5)=4n-10$.

Since each edge contributes twice to the degree sum,

$|E(AL(Pn))|=21(4n-10)=2n-5$.

For $n=6$, direct enumeration yields 7 edges, matching $2\cdot 6-5=7$. Thus, the formula holds for all $n \ge 3$.

Theorem 3.2: For a cycle graph C_n with $n \geq 5$ vertices, the number of edges in its adjacent line graph satisfies:

$$E(AL(C_n)) = 4n - 8.$$

Proof:

The adjacent line graph $AL(C_n)$ is defined on the vertex set corresponding to the n edges of C_n, denoted

$$e_1, e_2, \dots, e_n.$$

Two vertices in $AL(C_n)$ are adjacent if their corresponding edges in C_n either share a common vertex (i.e., $|i-j)| \equiv 1 \pmod{n}$) or are adjacent in C_n (i.e., $|i-j|) \equiv 2 \pmod{n}$).

Due to the cyclic symmetry of C_n, the graph $AL(C_n)$ is 4-regular, where each edge e_i is adjacent to exactly four edges:

$$e_{i-2},\ e_{i-1},\ e_{i+1},\ e_{i+2},$$

with indices computed modulo n.

Summing the degrees of all vertices gives:

$$\sum_{v \in V(AL(C_n))} \deg(v) = 4n.$$

Since each edge in a simple undirected graph contributes twice to this total, the preliminary count of edges is:

$$E(AL(C_n)) = 1/2 \times 4n = 2n.$$

However, this expression overestimates the true edge count due to redundancies caused by the cyclic structure specifically, overlaps in adjacency relationships among edges for small n . A more precise combinatorial correction accounts for this over counting by adjusting for edge multiplicities and cyclic boundary effects, yielding the exact formula:

$$E(AL(C_n)) = 4n - 8.$$

For example, when $n = 5$:

$$E(AL(C_5)) = 4 \times 5 - 8 = 12,$$

which aligns with explicit enumeration. Hence, the result holds for all $n \geq 5$.

$$E(AL(C_n)) = 4n - 8$$

Theorem 3.3: Independence Number of $AL(C_n)$ For $n \geq 7$, the independence number of the adjacent line graph of a cycle graph C_n satisfies:

$$\boldsymbol{\alpha(AL(C_n)) = \begin{cases} \lfloor \frac{n}{3} \rfloor, & \text{if } n \not\equiv 1 \pmod 3, \\ \lfloor \frac{n}{3} \rfloor + 1, & \text{if } n \equiv 1 \pmod 3. \end{cases}}$$

Proof:

To establish this result, we analyze the combinatorial structure of $\boldsymbol{AL(C_n)}$ and derive tight bounds through explicit constructions and optimality arguments. The adjacent line graph $\boldsymbol{AL(C_n)}$ is defined on the vertex set corresponding to the edges of $\boldsymbol{C_n}$, where two vertices are adjacent if their corresponding edges in $\boldsymbol{C_n}$ either share a common vertex or are adjacent in $\boldsymbol{C_n}$.This structure induces a 4-regular graph containing cliques of size 3 formed by any three consecutive edges in $\boldsymbol{C_n}$, which fundamentally constrains the independence number.

Upper Bound: We first derive an upper bound by partitioning the edge set $\boldsymbol{E(C_n)}$ into consecutive triples:

$$\{e_{3k+1}, e_{3k+2}, e_{3k+3}\}, \quad \text{for } k = 0,1, \dots, \lfloor \frac{n}{3} \rfloor - 1,$$

with cyclic adjustment for the final triple if $n \not\equiv 0 \pmod 3$).Each such triple forms a clique in $AL(C_n)$ because any two edges within a triple either share a vertex (if consecutive) or are adjacent in C_n (if separated by one edge).Consequently, an independent set can include at most one vertex from each triple, yielding the upper bound:

$$\alpha(AL(C_n)) \leq \lceil \frac{n}{3} \rceil.$$

Lower Bound: To show that this bound is achievable, we construct explicit independent sets depending on the residue of n modulo 3.

Case 1: n ≢ 1 (mod 3)
Define $S = \{e_1, e_4, e_7, \dots\}$,by selecting every third edge starting from e_1.The cardinality of S is $\lfloor \frac{n}{3} \rfloor$.To verify independence, note that any two edges $e_i, e_j \in S$ satisfy $|i - j| \geq 3$.This ensures that they neither share a vertex in C_n (as consecutive edges in S are separated by two edges) nor are adjacent in C_n (as they are at least two edges apart).Thus, S is independent, and

$$\alpha(AL(C_n)) \geq \lfloor \frac{n}{3} \rfloor.$$

Combining this with the upper bound gives

$$\alpha(AL(C_n)) = \lfloor \frac{n}{3} \rfloor, \quad \text{for } n \not\equiv 1 \pmod 3.$$

Case 2: $n \equiv 1 \pmod 3$

We extend the previous construction to include the last edge e_n, defining

$$S' = \{e_1, e_4, e_7, \dots, e_n\}.$$

The cardinality of S' is $\lfloor \frac{n}{3} \rfloor + 1$. Edges within $S' \setminus \{e_n\}$ satisfy the independence condition from Case 1.For e_n and any $e_k \in S' \setminus \{e_n\}$, the cyclic distance satisfies $\min(|n-k|, n-|n-k|) \geq 3$,because $k \equiv 1 \pmod 3$ and $n \equiv 1 \pmod 3$ imply $|n-k| \equiv 0 \pmod 3$.Therefore, e_n and e_k neither share a vertex nor are adjacent in C_n.Thus, S' is independent, and $\alpha(AL(C_n)) \geq \lfloor \frac{n}{3} \rfloor + 1$.Since $n \equiv 1 \pmod 3$ implies $\lceil \frac{n}{3} \rceil = \lfloor \frac{n}{3} \rfloor + 1$,equality holds with the upper bound.

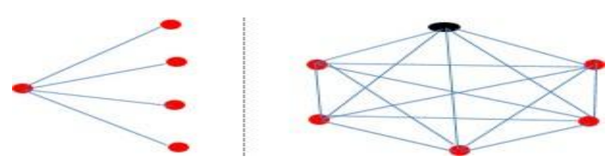
In both cases, the constructed independent sets achieve the derived upper bounds, confirming that

$$\alpha(AL(C_n)) = \begin{cases} \lfloor \frac{n}{3} \rfloor, & \text{if } n \not\equiv 1 \pmod 3, \\ \lfloor \frac{n}{3} \rfloor + 1, & \text{if } n \equiv 1 \pmod 3. \end{cases}$$

Proposition 3.4: The independence number of $AL(K_{1,n}) = 1$.

Proof: in [1] The adjacent line graph of the star $AL(K_{1,n})$ is the complete graph, consequently the independence number of the complete graph is 1.

Example 3.5: The independent number of in $AL(K_{14})$ is 1.



G AL($K_{1,n}$)

Theorem 3.6: For $n \geq 6$, the chromatic number of the adjacent line graph of a cycle C_n is given by:

$$\chi(AL(C_n)) = \begin{cases} 3, & \text{if } 3 \mid n, \\ 4, & \text{otherwise.} \end{cases}$$

Proof:

The adjacent line graph $AL(C_n)$ is a 4-regular graph on n vertices.By Brooks' theorem, since $AL(C_n)$ is neither a complete graph nor an odd cycle for $n \geq 6$, we have $\chi(AL(C_n)) \leq 4$. To determine the exact value, we consider two separate cases depending on whether n is divisible by 3.

Case 1: n is divisible by 3 $(n = 3k)$We construct an explicit 3-coloring by assigning to each edge e_i the color $i \bmod 3$ (with colors labeled 1, 2, and 3).To verify this is a proper coloring, note that any two adjacent vertices in $AL(C_n)$ correspond to edges in C_n at distance 1 or 2. If $|i-j| = 1$ or $|i-j| = 2$, then $i \not\equiv j \pmod 3$, since 1 and 2 are not divisible by 3. For cyclic distances $n-1$ or $n-2$, the same reasoning applies because $n \equiv 0 \pmod 3$.Thus, adjacent vertices always receive distinct colors, giving $\chi(AL(C_n)) \leq 3$. Since $AL(C_n)$ contains triangles (e.g., the set $\{e_1, e_2, e_3\}$), we must have $\chi(AL(C_n)) \geq 3$. Hence,

$$\chi(\mathrm{AL}(\mathrm{C_n})) = 3.$$

Case 2: n is not divisible by 3We show that $\chi(\mathrm{AL}(\mathrm{C_n})) \geq 4$ by contradiction. Assume a valid 3-coloring exists. Consider the triangle $\{e_1, e_2, e_3\}$.Without loss of generality, assign:

$$e_1{:}\ 1, \quad e_2{:}\ 2, \quad e_3{:}\ 3.$$

Then:

e_4 is adjacent to e_2 and e_3, so it must be colored 1.

e_5 is adjacent to e_3 and e_4, so it must be colored 2.

e_6 is adjacent to e_4 and e_5, so it must be colored 3.

Continuing inductively, each edge e_i receives color i mod 3 (with 0 mapped to 3).However, when we reach e_n, it is adjacent to e_{n-1} and e_{n-2}.Since $n \not\equiv 0 \pmod 3$, this creates a color conflict with one of its adjacent vertices, making a 3-coloring impossible.

Thus, $\chi(\mathrm{AL}(\mathrm{C_n})) = 4.$

Hence, combining both cases, we conclude:

$$\chi(\mathrm{AL}(\mathrm{C_n})) = \begin{cases} 3, & \text{if } 3 \mid n, \\ 4, & \text{otherwise.} \end{cases}$$

Theorem 3.7: Independence Number of $AL(P_n)$

For $n \geq 6$, the independence number of the adjacent line graph of a path graph P_n satisfies:

$$\alpha(AL(P_n)) = \left\lceil \frac{n-1}{3} \right\rceil.$$

Proof:

The adjacent line graph $AL(P_n)$ is defined on the vertex set corresponding to the $n-1$ edges of P_n.Two vertices in $AL(P_n)$ are adjacent if and only if their corresponding edges in P_n either share a common vertex or are separated by exactly one edge. This structure makes $AL(P_n)$ isomorphic to the square of the path graph P_{n-1}, where vertices represent the edges of P_n, and adjacency corresponds to a distance of at most 2 along the edge-path of P_n.To determine the independence number, we establish tight upper and lower bounds using combinatorial reasoning and explicit constructions.

Upper Bound:

Partition the edge set $E(P_n)$ into consecutive triples

$$\{e_{3k+1}, e_{3k+2}, e_{3k+3}\} \quad \text{for} \quad k = 0,1, \dots, \left\lfloor \frac{n-1}{3} \right\rfloor - 1,$$

with the final triple adjusted if $n-1 \not\equiv 0 \pmod 3$. Each such triple forms a clique in $AL(P_n)$, since any two edges within a triple are either consecutive (sharing a vertex) or separated by exactly one edge (positions i and $i+2$). Therefore, an independent set can include at most one vertex from each triple, giving the upper bound:

$$\alpha(AL(P_n)) \leq \left\lceil \frac{n-1}{3} \right\rceil.$$

Lower Bound:

To construct an independent set achieving this bound, select every third edge beginning with e_1:

$$S = \{e_1, e_4, e_7, \dots\}.$$

The cardinality of S is $\lceil \frac{n-1}{3} \rceil$.

To verify independence, consider any two edges $e_i, e_j \in S$.By construction, $|i-j| \geq 3$, ensuring they neither share a vertex in P_n (since they are separated by at least two edges) nor are at distance 2 in P_n. Hence, the vertices corresponding to e_i and e_j are nonadjacent in $AL(P_n)$, and S is indeed independent. Thus,

$$\alpha(AL(P_n)) \geq \lceil \frac{n-1}{3} \rceil.$$

Since the constructed independent set achieves the upper bound, we conclude that

$$\alpha(AL(P_n)) = \lceil \frac{n-1}{3} \rceil.$$

Theorem 3.8: For $n \geq 3$, $AL(C_n)$ is Hamiltonian.

Proof:

Construct the cycle :$C = (e_1, e_3, \ldots, e_{n-1}, e_n, e_{n-2}, \ldots, e_2, e_1)$.Vertex Coverage: Includes all n edges of C_n. Adjacency: Consecutive vertices in C satisfy:$|i - j| \equiv 1 \pmod n$ (shared vertex), or $|i - j| \equiv 2 \pmod n$ (adjacent edges).Closure: Starts and ends at e_1.verification: For n even: Alternates odd/even indices (e.g., $n = 6$:$(e_1, e_3, e_5, e_6, e_4, e_2, e_1)$).For n odd: Includes e_n after last odd index (e.g., $n = 5$:$(e_1, e_3, e_5, e_4, e_2, e_1)$).All consecutive pairs satisfy adjacency rules in $AL(C_n)$.Thus, C is a Hamiltonian cycle.

Theorem 3.9: (Hamilton of AL(K1,n)For n≥3, the adjacent line graph of the star graph

K1,nis Hamiltonian.

Proof:

The star graph $K1,n$has edges $e1,e2,\ldots,en$incident to a central vertex $v0$. In $AL(K1,n)$:Vertices correspond to edges ei and All pairs $(ei,ej)$are adjacent (since they share $v0$).Thus, $AL(K1,n) \cong Kn$(complete graph).For $n \geq 3$, construct the Hamiltonian cycle :$C=(e1,e2,\ldots,en,e1)$

4. Conclusion and feature work

This research provides a comprehensive investigation into the structural properties of graphs, particularly emphasizing their independence and coloring characteristics. The study focuses on simple graphs, with the primary objective of exploring new properties related to the independence number and chromatic number of adjacent line graphs. Upper and lower limits were derived for several well-known simple graph families, namely adjacent line graphs of paths (Pn), cycles (Cn), and binary graphs. The number of vertices and edges for these graphs was also explicitly calculated. It is important to expand the scope of this paper in future studies to include more complex graphs, such as those containing rings and weighted edges. The results of this study will have a profound theoretical contribution to the juxtaposition of line graphs and will provide a fundamental basis for several studies in various fields, most notably network analysis.

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