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Coefficient Inequalities for a Novel Class of m-Fold Symmetric Bi-Univalent Functions Related to λ-Pseudo-Starlike and λ-Pseudo-Convex Functions

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ABSTRACT

This article addresses the problem of coefficient bounds for certain subclasses of normalized holomorphic and m-fold symmetric bi-univalent functions associated with ʎ -pseudo-starlike and λ-pseudo-convex structures in the open unit disk Ữ. Upper estimates for the initial Taylor–Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ are derived for two newly considered families $K_{\Sigma_m}(\varsigma, ʎ, \mu; \alpha)$ and $K^*_{\Sigma_m}(\varsigma, ʎ, \mu; \beta)$. Several previously known results follow as particular instances of the obtained estimates.

MSC..

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1. Introduction

Let $\mathcal{A}$ denote the collection of normalized analytic functions defined on the open unit disk Ữ $= \{Ȥ \in \mathbb{C} : |Ȥ| < 1\}$, where the normalization is given by $f(0) = f'(0) - 1 = 0$. Under these conditions, each function admits a Taylor series representation around the origin of the following:

$$f(Ȥ) = Ȥ + \sum_{k=2}^{\infty} a_k Ȥ^k \, . \tag{1.1}$$

Let S represent the subclass of $\mathcal{A}$ comprising functions that satisfy condition (1.1) and are univalent in the open unit disk.

A function $f \in \mathcal{A}$ is called starlike of order ς $(0 \le \varsigma < 1)$, if

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$$\mathrm{Re}\left\{\frac{Ꝣf'(Ꝣ)}{f(Ꝣ)}\right\} > ς, \quad (Ꝣ \in \tilde{Ʊ}).$$

Babalola [6] introduced the class $\mathcal{L}_{ʎ}(\gamma)$, consisting of $ʎ$-pseudo-starlike functions of order γ, defined for functions $f \in \mathcal{A}$ that satisfy the following condition.

$$\mathrm{Re}\left\{\frac{Ꝣ(f'(Ꝣ))^{ʎ}}{f(Ꝣ)}\right\} > ς,$$

where $0 \le ς < 1, ʎ \ge 1$ and $Ꝣ \in \tilde{Ʊ}$. It is observed that for $ʎ = 1$, we have the family of starlike functions.

By virtue of the Koebe one-quarter theorem (see [9]), each function belonging to the class S admits an inverse function f^{-1}.

$$f^{-1}(f(Ꝣ)) = Ꝣ, (Ꝣ \in \tilde{Ʊ})$$

and

$$f(f^{-1}(ω)) = ω, \qquad \left(|ω| < r_0(f), r_0(f) \ge \frac{1}{4}\right),$$

where

$$g(ω) = f^{-1}(ω) = ω - a_2ω^2 + (2a_2^2 - a_3)ω^3 - (5a_2^3 - 5a_2a_3 + a_4)ω^4 + \cdots . \qquad (1.2)$$

A function f, analytic in the unit disk $\tilde{Ʊ}$, is said to be bi-univalent if both f and its inverse f^{-1} are injective in $\tilde{Ʊ}$. This family of functions has been investigated by numerous authors and has attracted increasing interest in recent years, leading to the development of various subclasses within the framework of bi-univalent functions. The notation Σ is used to denote the class of m-fold symmetric univalent functions defined in the open unit disk $\tilde{Ʊ}$, where each function has the form given in equation (1.1). This class of functions has been studied by several researchers, and in recent years it has received renewed attention, leading to the introduction of various sub-classes within the family of bi-univalent functions and studied analogously by the many authors (see, for example, [1,3,5,10,11,13,17,23,28,29,30,31]).

$\forall\ f \in S$, $ɧ(Ꝣ) = \sqrt[ɱ]{f(Ꝣ^{ɱ})}$, $(Ꝣ \in \tilde{Ʊ}, ɱ \in \mathbb{N})$ is univalent and maps the unit disk $\tilde{Ʊ}$ into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [13]), if it has the following normalized form:

$$f(Ꝣ) = Ꝣ + \sum_{k=1}^{\infty} a_{ɱk+1} z^{ɱk+1}, \quad (Ꝣ \in \tilde{Ʊ}, ɱ \in \mathbb{N}) . \qquad (1.3)$$

We denote by $S_{ɱ}$ the class of m-fold symmetric univalent functions defined in the open unit disk $\tilde{Ʊ}$, whose normalized representations are given by the series expansion (1.3). It is worth noting that functions belonging to the class S correspond to the special case of one-fold symmetry.

In [26], Srivastava et al. introduced the concept of m-fold symmetric bi-univalent functions as a natural generalization of m-fold symmetric univalent functions. Several fundamental properties were established, including the result that each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for every $ɱ \in \mathbb{N}$. Moreover, by considering the normalized form of f defined in (1.3), the authors derived the corresponding series expansion for the inverse function f^{-1} as follows:

$$g(ω) = ω - a_{ɱ+1}ω^{ɱ+1} + \left[(ɱ+1)a_{ɱ+1}^2 - a_{2ɱ+1}\right]ω^{2ɱ+1}$$
$$-\left[\frac{1}{2}(ɱ+1)(3ɱ+2)a_{ɱ+1}^3 - (3ɱ+2)a_{ɱ+1}a_{2ɱ+1} + a_{3ɱ+1}\right]ω^{3ɱ+1} + \cdots \qquad (1.4)$$

Let $f^{-1} = g$ denote the inverse of the function f. We define $\Sigma_{ɱ}$ as the class of m-fold symmetric bi-univalent functions that are analytic in the open unit disk $\tilde{Ʊ}$. It can be readily verified that, for $ɱ = 1$, equation (1.4) simplifies to equation (1.2), which characterizes the family Σ. Several illustrative examples of m-fold symmetric bi-univalent functions are presented as follows:

$$\left(\frac{Ꝣ^{ɱ}}{1-Ꝣ^{ɱ}}\right)^{\frac{1}{ɱ}}, \quad \left[\frac{1}{2}\log\left(\frac{1+Ꝣ^{ɱ}}{1-Ꝣ^{ɱ}}\right)\right]^{\frac{1}{ɱ}} \text{ and } \left[-\log(1-Ꝣ^{ɱ})\right]^{\frac{1}{ɱ}}$$

Are expressed, respectively, as

$$\left(\frac{\mathrm{w}^{\mathrm{m}}}{1+\mathrm{w}^{\mathrm{m}}}\right)^{\frac{1}{\mathrm{m}}},\ \left(\frac{e^{2\mathrm{w}^{\mathrm{m}}}-1}{e^{2\mathrm{w}^{\mathrm{m}}}+1}\right)^{\frac{1}{\mathrm{m}}}\ \text{and}\ \left(\frac{e^{\mathrm{w}^{\mathrm{m}}}-1}{e^{\mathrm{w}^{\mathrm{m}}}}\right)^{\frac{1}{\mathrm{m}}},$$

respectively.

In recent years, considerable attention has been devoted to deriving coefficient bounds for different subclasses of m-fold symmetric bi-univalent functions (see, for instance, [2,4,8,15,19–22,24,27,32,33]).

To establish our principal findings, the subsequent lemma is required.

Lemma 1.1 [9].
Let $\mathfrak{h} \in \mathcal{P}$ be given by the following series:

$$\mathfrak{h}(\zeta) = 1 + c_1\zeta + c_2\zeta^2 + \cdots, (\zeta \in \tilde{U}).$$

The sharp estimate is given by

$$|c_k| \leq 2, \text{where } n \in \mathbb{N}$$

holds true.

2. Coefficient Bounds for the Function Family $K_{\Sigma_{\mathrm{m}}}(\varsigma, \Lambda, \mu; \alpha)$

Definition 2.1. A function $f \in \Sigma_{\mathrm{m}}$ *defined by equation* (1.3) is said to belong to the class $K_{\Sigma_{\mathrm{m}}}(\varsigma, \Lambda, \mu; \alpha)$ $(0 < \alpha \leq 1, 0 \leq \varsigma \leq 1, \Lambda \geq 1, 0 \leq \mu \leq 1)$ if it satisfies for the following conditions:

$$\left|\arg\left(\left((1-\varsigma)\frac{\zeta(f'(\zeta))^{\Lambda}}{f(\zeta)} + \varsigma\left(\frac{\left((\zeta f'(\zeta))'\right)^{\Lambda}}{f'(\zeta)}\right)\right)^{\mu}\right)\right| < \frac{\alpha\pi}{2}, \quad (\zeta \in \tilde{U}) \tag{2.1}$$

and

$$\left|\arg\left(\left((1-\varsigma)\frac{\mathrm{w}(g'(\mathrm{w}))^{\Lambda}}{g(\mathrm{w})} + \varsigma\left(\frac{\left((\mathrm{w}g'(\mathrm{w}))'\right)^{\Lambda}}{g'(\mathrm{w})}\right)\right)^{\mu}\right)\right| < \frac{\alpha\pi}{2}. \quad (\mathrm{w} \in \tilde{U}) \tag{2.2}$$

With the inverse mapping g= f^{-1} specified by (1.4).

Specifically, for one-fold symmetric bi-univalent functions, we introduce the family $K_{\Sigma_1}(\varsigma, \Lambda, \mu; \alpha) = K_{\Sigma}(\varsigma, \Lambda, \mu; \alpha)$.

Remark 2.1. The family $K_{\Sigma}(\varsigma, \Lambda, \mu; \alpha)$is a generalization annmber of well-established sub classes in vestigated in earlier works,
(1) when $\varsigma = 0$ and $\mu = 1$, the family$K_{\Sigma}(\varsigma, \Lambda, \mu; \alpha)$ leads to the family $\mathcal{L}B_{\Sigma}^{\lambda}(\alpha)$ in troduced by Joshi et al. [12];
(2) For $\Lambda = 1$ and $\mu = 1$, the family $K_{\Sigma}(\varsigma, \Lambda, \mu; \alpha)$ leads to the family $M_{\Sigma}(\alpha, \delta)$ studied in Liu and Wang [16];
(3) setting $\varsigma = 0$ and $\Lambda = \mu = 1$, the family $K_{\Sigma}(\varsigma, \Lambda, \mu; \alpha)$ leads to the family $S_{\Sigma}^{*}(\alpha)$ discussed by Brannan and Taha [7].

Theorem 2.1. Suppose that $f \in k_{\Sigma_{\mathrm{m}}}(\varsigma, \Lambda, \mu; \alpha)$ $(0 < \alpha \leq 1, 0 \leq \varsigma \leq 1, \Lambda \geq 1, 0 \leq \mu \leq 1, \mathrm{m} \in \mathbb{N})$ be given by (1.3). Then

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{\left|\begin{array}{c}\alpha\mu\left[2\left\{1-\varsigma-\frac{1}{2}(m+1)(1-\varsigma)[2-(\kappa-1)(m+1)]+\frac{1}{2}(\varsigma(m+1)^2(\kappa(m+1)[(\kappa-1)(m+1)-2]+2)\right\}\right. \\ \left.+(\mu-1)(\kappa(m+1)-1)^2(\varsigma m+1)^2+(\kappa(2m+1)-1)(2\varsigma m+1)(m+1)\right]+(1-\alpha)\mu^2[(\kappa(m+1)-1)(\varsigma m+1)]^2\end{array}\right|}}, \tag{2.3}$$

and

$$|a_{2m+1}| \leq \frac{4(m+1)\alpha^2\mu^2}{[(\kappa(m+1)-1)(\varsigma m+1)]^2}+\frac{2\alpha}{\mu(\kappa(2m+1)-1)(2\varsigma m+1)}. \tag{2.4}$$

Proof. Relations (2.1) and (2.2) imply that

$$\left((1-\varsigma)\frac{z(f'(z))^{\kappa}}{f(z)}+\varsigma\left(\frac{((zf'(z))')^{\kappa}}{f'(z)}\right)\right)^{\mu}=[p(z)]^{\alpha}, \tag{2.5}$$

and

$$\left((1-\varsigma)\frac{w(g'(w))^{\kappa}}{g(w)}+\varsigma\left(\frac{((wg'(w))')^{\kappa}}{g'(w)}\right)\right)^{\mu}=[q(w)]^{\alpha}, \tag{2.6}$$

here $g = f^{-1}$, *while* $p, q \in P$ admit the representations

$$p(z)=1+p_m z^m+p_{2m}z^{2m}+p_{3m}z^{3m}+\cdots, \tag{2.7}$$

and

$$q(w)=1+q_m w^m+q_{2m}w^{2m}+q_{3m}w^{3m}+\cdots. \tag{2.8}$$

By equating the coefficients of (2.5) and (2.6) we obtain

$$\mu[(\kappa(m+1)-1)(\varsigma m+1)]\, a_{m+1}=\alpha p_m, \tag{2.9}$$

$$\mu\left[[\kappa(2m+1)-1)(2\varsigma m+1)]a_{2m+1}+\left\{1-\varsigma-\frac{1}{2}(m+1)(1-\varsigma)[2-(\kappa-1)(m+1)]+\frac{1}{2}(\varsigma(m+1)^2(\kappa(m+1)[(\kappa-1)(m+1)-2]+2)+(\mu-1)(\kappa(m+1)-1)^2(\varsigma m+1)^2)\right\}\right] a_{m+1}^2=\alpha p_{2m}+\frac{\alpha(\alpha-1)}{2}p_m^2, \tag{2.10}$$

$$-\mu[\kappa(m+1)-1)(\varsigma m+1)]a_{m+1}=\alpha q_m, \tag{2.11}$$

and

$$\mu\left[[\kappa(2m+1)-1)(2\varsigma m+1)]\left((m+1)a_{m+1}^2-a_{2m+1}\right)+\left\{1-\varsigma-\frac{1}{2}(m+1)(1-\varsigma)[2-(\kappa-1)(m+1)]+\frac{1}{2}(\varsigma(m+1)^2(\kappa(m+1)[(\kappa-1)(m+1)-2]+2)+(\mu-1)(\kappa(m+1)-1)^2(\varsigma m+1)^2)\right\}\right]a_{m+1}^2=\alpha q_{2m}+\frac{\alpha(\alpha-1)}{2}q_m^2. \tag{2.12}$$

By (2.9) and (2.11), we derive

$$p_m=-q_m, \tag{2.13}$$

and

$$2\mu^2[\mathcal{K}(\mathfrak{m}+1)-1)(\varsigma\mathfrak{m}+1)]^2 a_{\mathfrak{m}+1}^2=\alpha^2\left(p_{\mathfrak{m}}^2+q_{\mathfrak{m}}^2\right). \tag{2.14}$$

Also, from (2.10), (2.12) and (2.14), we find that

$$\mu\left(2\left\{1-\varsigma-\frac{1}{2}(\mathfrak{m}+1)(1-\varsigma)[2-(\mathcal{K}-1)(\mathfrak{m}+1)]+\frac{1}{2}(\varsigma(\mathfrak{m}+1)^2(\mathcal{K}(\mathfrak{m}+1)[(\mathcal{K}-1)(\mathfrak{m}+1)-2]+2)\right.\right.$$
$$\left.\left.+(\mu-1)(\mathcal{K}(\mathfrak{m}+1)-1)^2(\varsigma\mathfrak{m}+1)^2)\right\}+\mathcal{K}(2\mathfrak{m}+1)-1)(2\varsigma\mathfrak{m}+1)(\mathfrak{m}+1)\right)a_{\mathfrak{m}+1}^2$$
$$=\alpha\left(p_{2\mathfrak{m}}+q_{2\mathfrak{m}}\right)+\frac{\alpha(\alpha-1)}{2}\left(p_{\mathfrak{m}}^2+q_{\mathfrak{m}}^2\right)$$
$$=\alpha\left(p_{2\mathfrak{m}}+q_{2\mathfrak{m}}\right)+\frac{(\alpha-1)\mu^2[\mathcal{K}(\mathfrak{m}+1)-1)(\varsigma\mathfrak{m}+1)]^2}{\alpha}a_{\mathfrak{m}+1}^2 .$$

Hence, we have

$$a_{\mathfrak{m}+1}^2=\frac{\alpha^2\left(p_{2\mathfrak{m}}+q_{2\mathfrak{m}}\right)}{\begin{array}{c}\alpha\mu\left[2\left\{1-\varsigma-\frac{1}{2}(\mathfrak{m}+1)(1-\varsigma)[2-(\mathcal{K}-1)(\mathfrak{m}+1)]+\frac{1}{2}(\varsigma(\mathfrak{m}+1)^2(\mathcal{K}(\mathfrak{m}+1)[(\mathcal{K}-1)(\mathfrak{m}+1)-2]+2)\right\}\right.\\ \left.+(\mu-1)(\mathcal{K}(\mathfrak{m}+1)-1)^2(\varsigma\mathfrak{m}+1)^2+(\mathcal{K}(2\mathfrak{m}+1)-1)(2\varsigma\mathfrak{m}+1)(\mathfrak{m}+1)\right]+(1-\alpha)\mu^2[(\mathcal{K}(\mathfrak{m}+1)-1)(\varsigma\mathfrak{m}+1)]^2\end{array}}. \tag{2.15}$$

By Lemma 1.1, to the absolute value of (2.15), one can derive for the coefficients $p_{2\mathfrak{m}}$ and $q_{2\mathfrak{m}}$, yields

$$\left|a_{\mathfrak{m}+1}\right|\le\frac{2\alpha}{\sqrt{\left|\begin{array}{c}\alpha\mu\left[2\left\{1-\varsigma-\frac{1}{2}(\mathfrak{m}+1)(1-\varsigma)[2-(\mathcal{K}-1)(\mathfrak{m}+1)]+\frac{1}{2}(\varsigma(\mathfrak{m}+1)^2(\mathcal{K}(\mathfrak{m}+1)[(\mathcal{K}-1)(\mathfrak{m}+1)-2]+2)\right\}\right.\\ \left.+(\mu-1)(\mathcal{K}(\mathfrak{m}+1)-1)^2(\varsigma\mathfrak{m}+1)^2)+(\mathcal{K}(2\mathfrak{m}+1)-1)(2\varsigma\mathfrak{m}+1)(\mathfrak{m}+1)\right]+(1-\alpha)\mu^2[(\mathcal{K}(\mathfrak{m}+1)-1)(\varsigma\mathfrak{m}+1)]^2\end{array}\right|}}.$$

Hence estimate for $\left|a_{\mathfrak{m}+1}\right|$ in (2.3) is obtained.

To derive an upper bound for $\left|a_{2\mathfrak{m}+1}\right|$, by subtracting (2.12) from (2.10), we have

$$\mu[\mathcal{K}(2\mathfrak{m}+1)-1)(2\varsigma\mathfrak{m}+1)]\left(2a_{2\mathfrak{m}+1}-(\mathfrak{m}+1)a_{\mathfrak{m}+1}^2\right)$$
$$=\alpha\left(p_{2\mathfrak{m}}-q_{2\mathfrak{m}}\right)+\tfrac{\alpha(\alpha-1)}{2}\left(p_{\mathfrak{m}}^2-q_{\mathfrak{m}}^2\right). \tag{2.16}$$

From (2.13), (2.14) and (2.16) that

$$a_{2\mathfrak{m}+1}=\frac{(\mathfrak{m}+1)\alpha^2\mu^2\left(p_{\mathfrak{m}}^2+q_{\mathfrak{m}}^2\right)}{2[\mathcal{K}(\mathfrak{m}+1)-1)(\varsigma\mathfrak{m}+1)]^2}+\frac{\alpha\left(p_{2\mathfrak{m}}-q_{2\mathfrak{m}}\right)}{2\mu[\mathcal{K}(2\mathfrak{m}+1)-1)(2\varsigma\mathfrak{m}+1)]}. \tag{2.17}$$

By considering the modulus of (2.17) to gether with arepeated application of Lemma 1.1 to coefficients p_m, p_{2m}, q_m and q_{2m} , yields

$$\left|a_{2\mathfrak{m}+1}\right|\le\frac{4\mu^2(\mathfrak{m}+1)\alpha^2}{[\mathcal{K}(\mathfrak{m}+1)-1)(\varsigma\mathfrak{m}+1)]^2}+\frac{2\alpha}{\mu(\mathcal{K}(2\mathfrak{m}+1)-1)(2\varsigma\mathfrak{m}+1)}.$$

Restricting Theorem 2.1 to one-fold symmetric bi-univalent functions, leads to subsequent

corollary:

Corollary 2.1. Assume $\mathrm{f}\in K_{\Sigma}(\varsigma,\mathcal{K},\mu;\alpha)$ $(0<\alpha\le 1, 0\le\varsigma\le 1, \mathcal{K}\ge 1, 0\le\mu\le 1)$asgivenin (1.1). Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{\left|\begin{array}{c}\alpha\mu\left[2\left\{1-\varsigma-\Lambda(1-\varsigma)[2-2(\Lambda-1)]+\frac{1}{2}(4\varsigma(2\Lambda[2(\Lambda-1)-2]+2)\right\}\right. \\ \left.+(\mu-1)(2\Lambda-1)^2(\varsigma+1)^2)+2(3\Lambda-1)(2\varsigma+1)\right]+(1-\alpha)[(2\Lambda-1)\mu^2(\varsigma+1)]^2\end{array}\right|}},$$

and

$$|a_3| \leq \frac{8\alpha^2\mu^2}{[(2\Lambda-1)(\varsigma+1)]^2}+\frac{2\alpha}{\mu(3\Lambda-1)(2\varsigma+1)}.$$

Remark 2.2. In Corollary 2.1, by taking

(1) $\varsigma=0$and $\mu=1$, the resulting estimates coincide with those established by Joshi et al. [12, Theorem 1];
(2) $\Lambda=\mu=1$, the results coincide with those of Liu and Wang [16, Theorem2.2];
(3) $\varsigma=0$ and $\Lambda=\mu=1$, the corresponding then results of Murugusundaramoorthy et al. [18, Corollary 6].

3. Coefficient Estimates for the Function Family $\boldsymbol{K}^*_{\Sigma_m}(\boldsymbol{\varsigma},\Lambda,\boldsymbol{\mu};\boldsymbol{\beta})$

Definition 3.1. Let $f\in\Sigma_m$ be defined asin (1.3). Then f is amember of the family $K^*_{\Sigma_m}(\varsigma,\Lambda,\mu;\beta)$

$(0\leq\beta<1, 0\leq\varsigma\leq1, \Lambda\geq1, 0\leq\mu\leq1)$ provided that:

$$Re\left\{\left((1-\varsigma)\frac{z(f'(z))^{\Lambda}}{f(z)}+\varsigma\left(\frac{\left((zf'(z))'\right)^{\Lambda}}{f'(z)}\right)\right)^{\mu}\right\}>\beta, \tag{3.1}$$

and

$$Re\left\{\left((1-\varsigma)\frac{w(g'(w))^{\Lambda}}{g(w)}+\varsigma\left(\frac{\left((wg'(w))'\right)^{\Lambda}}{g'(w)}\right)\right)^{\mu}\right\}>\beta. \tag{3.2}$$

With g denoting the invers of f, as specified in (1.4).

Specifically, in the class of one-fold symmetric bi-univalent functions, the corresponding family is denoted $K^*_{\Sigma_1}(\varsigma,\Lambda,\mu;\beta)=K^*_{\Sigma}(\varsigma,\Lambda,\mu;\beta)$.

Remark 3.1. It is worth noting that the class $K^*_{\Sigma}(\varsigma,\Lambda,\mu;\beta)$. In particular, the following sub class are obtained:

(1) Setting $\varsigma=0$and $\mu=1$, the class $K^*_{\Sigma}(\varsigma,\Lambda,\mu;\beta)$ coincides with the family $\mathcal{L}B_{\Sigma}(\lambda,\beta)$ introduced by Joshi et al. [12];
(2) For the parameter value $\Lambda=1$and $\mu=1$, the class $K^*_{\Sigma}(\varsigma,\Lambda,\mu;\beta)$ becomes the class $B_{\Sigma}(\beta,\delta)$ examined by Liu and Wang [16];
(3) In the special case $\varsigma=0$ and $\Lambda=\mu=1$, the class $K^*_{\Sigma}(\varsigma,\Lambda,\mu;\beta)$ coincides with the family $S^*_{\Sigma}(\beta)$ analyzed by Brannan and Taha [7].

Theorem 3.1. Assume $f\in K^*_{\Sigma_m}(\varsigma,\Lambda,\mu;\beta)$ $(0\leq\beta<1, 0\leq\varsigma\leq1, \Lambda\geq1, 0\leq\mu\leq1, m\in\mathbb{N})$, be given by (1.3). Then

$$|a_{m+1}| \leq \sqrt{\frac{2(1-\beta)}{\left|\begin{array}{c}\mu\left[2\left\{1-\varsigma-\frac{1}{2}(m+1)(1-\varsigma)[2-(\Lambda-1)(m+1)]+\frac{1}{2}(\varsigma(m+1)^2(\Lambda(m+1)[(\Lambda-1)(m+1)-2]+2)\right\}\right. \\ \left.+(\mu-1)(\Lambda(m+1)-1)^2(\varsigma m+1)^2)+(\Lambda(2m+1)-1)(2\varsigma m+1)(m+1)\right]\end{array}\right|}},$$

(3.3)

and

$$\left|a_{2\mathfrak{m}+1}\right| \leq \frac{4\mu^2(1-\beta)^2}{[(\Lambda(\mathfrak{m}+1)-1)(\varsigma\mathfrak{m}+1)]^2}+\frac{4(1-\beta)}{\mu(\Lambda(2\mathfrak{m}+1)-1)(2\varsigma\mathfrak{m}+1)}. \quad (3.4)$$

Proof. Based on assumption (3.1) and (3.2), one can assert the existence of $p, q \in \mathfrak{p}$ for which

$$\left((1-\varsigma)\frac{\mathcal{Z}\left(f'(\mathcal{Z})\right)^{\Lambda}}{f(\mathcal{Z})}+\varsigma\left(\frac{\left(\left(\mathcal{Z}f'(\mathcal{Z})\right)'\right)^{\Lambda}}{f'(\mathcal{Z})}\right)\right)^{\mu}=\beta+(1-\beta)p(\mathcal{Z}), \quad (3.5)$$

and

$$\left((1-\varsigma)\frac{\mathrm{w}\left(g'(\mathrm{w})\right)^{\Lambda}}{g(\mathrm{w})}+\varsigma\left(\frac{\left(\left(\mathrm{w}g'(\mathrm{w})\right)'\right)^{\Lambda}}{g'(\mathrm{w})}\right)\right)^{\mu}=\beta+(1-\beta)q(\mathrm{w}), \quad (3.6)$$

where $p(\mathcal{Z})$ and $q(\mathrm{w})$ have the forms (2.7) and (2.8), respectively. Equating coefficients (3.5) and (3.6) yields

$$\mu[(\Lambda(\mathfrak{m}+1)-1)(\varsigma\mathfrak{m}+1)]\, a_{\mathfrak{m}+1}=(1-\beta)p_{\mathfrak{m}}, \quad (3.7)$$

$$\mu\left[[\Lambda(2\mathfrak{m}+1)-1)(2\varsigma\mathfrak{m}+1)]a_{2\mathfrak{m}+1}+\left\{1-\varsigma-\tfrac{1}{2}(\mathfrak{m}+1)(1-\varsigma)[2-(\Lambda-1)(\mathfrak{m}+1)]+\tfrac{1}{2}(\varsigma(\mathfrak{m}+1)^2(\Lambda(\mathfrak{m}+1)[(\Lambda-1)(\mathfrak{m}+1)-2]+2)+(\mu-1)(\Lambda(\mathfrak{m}+1)-1)^2(\varsigma\mathfrak{m}+1)^2)\right\}\right] a_{\mathfrak{m}+1}^2=(1-\beta)p_{2\mathfrak{m}}, \quad (3.8)$$

$$-\mu[(\Lambda(\mathfrak{m}+1)-1)(\varsigma\mathfrak{m}+1)]a_{\mathfrak{m}+1}=(1-\beta)q_{\mathfrak{m}}, \quad (3.9)$$

and

$$\mu\left[[\Lambda(2\mathfrak{m}+1)-1)(2\varsigma\mathfrak{m}+1)]\left((\mathfrak{m}+1)a_{\mathfrak{m}+1}^2-a_{2\mathfrak{m}+1}\right)+\left\{1-\varsigma-\frac{1}{2}(\mathfrak{m}+1)(1-\varsigma)[2-(\Lambda-1)(\mathfrak{m}+1)]+\frac{1}{2}(\varsigma(\mathfrak{m}+1)^2(\Lambda(\mathfrak{m}+1)[(\Lambda-1)(\mathfrak{m}+1)-2]+2)+(\mu-1)(\Lambda(\mathfrak{m}+1)-1)^2(\varsigma\mathfrak{m}+1)^2)\right\}\right] a_{\mathfrak{m}+1}^2 =(1-\beta)q_{2\mathfrak{m}}. \quad (3.10)$$

From (3.7) and (3.9), we get

$$p_{\mathfrak{m}}=-q_{\mathfrak{m}}, \quad (3.11)$$

and

$$2\mu^2[(\Lambda(\mathfrak{m}+1)-1)(\varsigma\mathfrak{m}+1)]^2a_{m+1}^2=(1-\beta)^2\left(p_{\mathfrak{m}}^2+q_{\mathfrak{m}}^2\right). \quad (3.12)$$

By summing (3.8) and (3.10), we derive

$$\mu\Bigg(2\Big\{1-\varsigma-\frac{1}{2}(ɱ+1)(1-\varsigma)[2-(\kappa-1)(ɱ+1)]+\frac{1}{2}(\varsigma(ɱ+1)^2(\kappa(ɱ+1)[(\kappa-1)(ɱ+1)-2]+2)$$
$$+(\mu-1)(\kappa(ɱ+1)-1)^2(\varsigma ɱ+1)^2)\Big\}+\kappa(2ɱ+1)-1)(2\varsigma ɱ+1)(ɱ+1)\Bigg)a_{ɱ+1}^2$$
$$=(1-\beta)\left(p_{ɱ}^2+q_{ɱ}^2\right). \qquad (3.13)$$

This implies that

$$a_{ɱ+1}^2=\frac{(1-\beta)\left(p_{2ɱ}+q_{2ɱ}\right)}{\mu\left[2\left\{1-\varsigma-\frac{1}{2}(ɱ+1)(1-\varsigma)[2-(\kappa-1)(ɱ+1)]+\frac{1}{2}(\varsigma(ɱ+1)^2(\kappa(ɱ+1)[(\kappa-1)(ɱ+1)-2]+2)\right\}\atop +(\mu-1)(\kappa(ɱ+1)-1)^2(\varsigma ɱ+1)^2+(\kappa(2ɱ+1)-1)(2\varsigma ɱ+1)(ɱ+1)\right]}.$$

An application of Lemma 1.1 to the coefficients $p_{2ɱ}$ and $q_{2ɱ}$, yields

$$|a_{ɱ+1}|\le\sqrt{\frac{2(1-\beta)}{\left|\mu\left[2\left\{1-\varsigma-\frac{1}{2}(ɱ+1)(1-\varsigma)[2-(\kappa-1)(ɱ+1)]+\frac{1}{2}(\varsigma(ɱ+1)^2(\kappa(ɱ+1)[(\kappa-1)(ɱ+1)-2]+2)\right\}\atop +(\mu-1)(\kappa(ɱ+1)-1)^2(\varsigma ɱ+1)^2+(\kappa(2ɱ+1)-1)(2\varsigma ɱ+1)(ɱ+1)\right]\right|}}.$$

This gives the desired estimate for $|a_{ɱ+1}|$ as asserted in (3.3).

Subtracting (3.10) from (3.8), allows us to derive abound for $|a_{2ɱ+1}|$, as follows:

$$\mu[(\kappa(2ɱ+1)-1)(2\varsigma ɱ+1)]\left((ɱ+1)a_{ɱ+1}^2-2a_{2ɱ+1}\right)=(1-\beta)\left(p_{2ɱ}-q_{2ɱ}\right).$$

Which is equivalent to

$$a_{2ɱ+1}=(ɱ+1)a_{ɱ+1}^2+\frac{(1-\beta)\left(p_{2ɱ}-q_{2ɱ}\right)}{2\mu[(\kappa(2ɱ+1)-1)(2\varsigma ɱ+1)]}.$$

Upon substituting the value of $a_{ɱ+1}^2$ from (3.12), hence

$$a_{2ɱ+1}=\frac{(1-\beta)^2\mu^2\left(p_{ɱ}^2+q_{ɱ}^2\right)}{2[(\kappa(ɱ+1)-1)(2\varsigma ɱ+1)]^2}+\frac{(1-\beta)\left(p_{2ɱ}-q_{2ɱ}\right)}{2\mu(\kappa(2ɱ+1)-1)(2\varsigma ɱ+1)}.$$

In voking Lemma 1.1 again for the coefficients $p_{ɱ}, p_{2ɱ}, q_m$ and $q_{2ɱ}$, yields

$$|a_{2ɱ+1}|\le\frac{4\mu^2(1-\beta)^2}{[(\kappa(ɱ+1)-1)(\varsigma ɱ+1)]^2}+\frac{4(1-\beta)}{\mu(\kappa(2ɱ+1)-1)(2\varsigma ɱ+1)},$$

Hence,the proof is concluded of Theorem 3.1.

As an application of Theorem 3.1 to one-fold symmetric bi-univalent functions, leads to for corollary:

Corollary 3.1. Suppose that $f\in K_\Sigma^*(\varsigma,\kappa,\mu;\beta)$ $(0\le\beta<1, 0\le\varsigma\le 1, \kappa\ge 1, 0\le\mu\le 1)$, expressed in (1.1). Then

$$|a_2|\le\sqrt{\frac{2(1-\beta)}{\left|\mu\left[2\left\{1-\varsigma-\kappa(1-\varsigma)[2-2(\kappa-1)]+\frac{1}{2}(4\varsigma(2\kappa[2(\kappa-1)-2]+2)\right\}\atop +(\mu-1)(2\kappa-1)^2(\varsigma+1)^2)+2(3\kappa-1)(2\varsigma+1)\right]\right|}},$$

and

$$|a_3|\le\frac{4\mu^2(1-\beta)^2}{[(2\kappa-1)(\varsigma+1)]^2}+\frac{4(1-\beta)}{\mu(3\kappa-1)(2\varsigma+1)}.$$

Remark 3.2. By specifying particular choices of the parameters in Corollary 3.1, several known results can be recovered as special cases.

(1) Specifically, when $\varsigma = 0$and$\mu = 1$, the obtained estimates coincide with those established by Joshi et al [12, Theorem 2];

(2) Furthermore, for $\lambda = \mu = 1$, our results reduce to the corresponding bounds derived by Liu and Wang [16, Theorem3.2];

(3) In addition, taking $\varsigma = 0$ together with $\lambda = \mu = 1$ yields the obtained findings by Murugusundaramoorthy et al. [18, Corollary 7].

4. Conclusion

This paper develops previously unexamined subclasses of normalized holomorphic and m-fold symmetric bi-univalent functions linked to the families $K_{\Sigma_m}(\varsigma, \lambda, \mu; \alpha)$ and $K^*_{\Sigma_m}(\varsigma, \lambda, \mu; \beta)$. These subclasses arise from the analytical structure of λ-pseudo-starlike and λ-pseudo-convex functions. Precise upper bounds for the initial Taylor–Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ are derived for functions within each subclass, contributing new coefficient estimates beyond those previously reported.

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