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A Chronological Analysis of the Development of the Fourth Hankel Determinant in Analytic Function Classes

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A B S T R A C T

In the geometric study of analytic functions with a single derivative, the fourth Hankel determinant, $H_4(1)$, is essential. This article provides a systematic overview of the historical and current evolution of research on $H_4(1)$, beginning with basic studies, moving on to more complex methods involving symmetry and subordination, and ending with current applications utilising special polynomials like Chebyshev within extended subclasses.

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1. Introduction: Hankel Determinants' Function

A fundamental analytical tool for examining the behavior of coefficient sequences obtained from analytic functions in the unit disc is the Hankel determinant.

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$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & . \\ . & & . & . \\ . & & & . \\ . & & . & \\ a_{n+q-1} & \cdots & a_{n+2q-2} & \end{vmatrix}$$

Where $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$, $q \geqslant 1$.

The determinant $H_4(1)$, which is built from the Taylor coefficients, sheds light on a number of geometric features, including growth, distortion, and symmetry, given a function written as follows:

$f(z) = z + a_n z^2 + a_n z^3 + a_n z4 + \ldots$

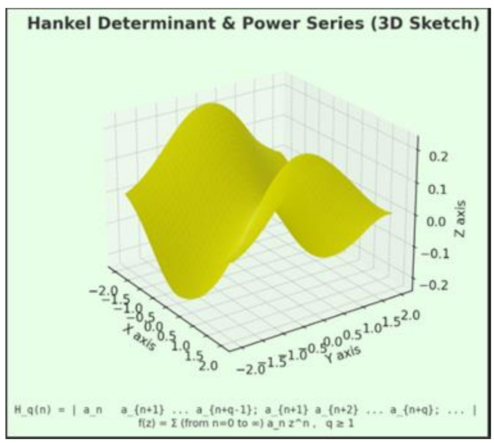


Fig. 1 .Three-Dimensional Visualization of a Hankel Determinant Surface with Power Series.

2. Historical Bases

A formal framework for studying Hankel determinants was presented by Noonan and Thomas in 1976 [1]. Suppose $f \in S_p$, Then as, $n \to \infty$,

$$H_2(n) = a_n a_{n+2} - a_{n+1}^2 = \begin{cases} o(1)n^{-1}, & 0 < p < 1/4 \\ o(1)n^{2p-3/2}, & 1/4 < p < 5/4 \\ O(1)n^{4p-4}, & p > 5/4. \end{cases} \quad (1)$$

They particularly concentrated on $H_2(1)$ in connection with the Fekete–Szegö inequality. A turning point was reached later in 1985[2] when de Branges proved the Bieberbach conjecture, reviving interest in higher-order Hankel determinants and establishing tight coefficient bounds.

$$\sigma_n(t) + \frac{t\sigma_n'(t)}{n} = \sigma_{n+1}(t) - \frac{t\sigma_{n+1}'(t)}{n+1} \quad (2)$$

Where the family space assume $\boldsymbol{\sigma_{o(t)}}, \boldsymbol{t \geqslant 1}$, Then the inequality

$$\left\| \log \frac{B(z)}{zB'(0)} + f(B(z)) \right\|_{\sigma_{\sigma(a)}}^2 \leqslant \|f(z)\|_{\sigma_{\sigma(b)}}^2 + 4\sum_{n=1}^{\infty} \frac{\sigma_n(a) - \sigma_n(b)}{n} \quad (3)$$

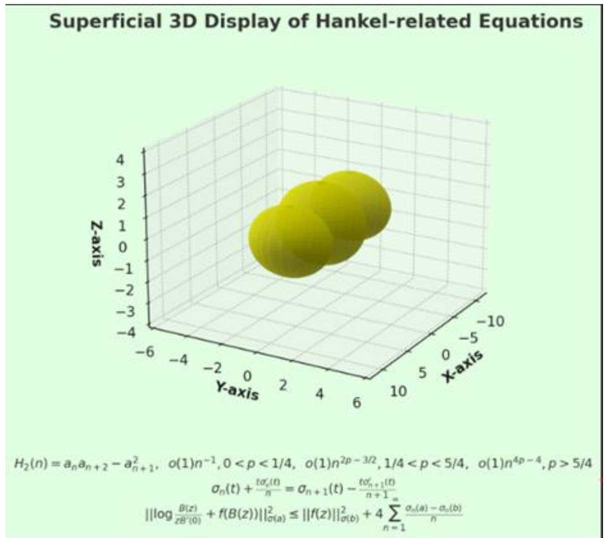


Fig. 2. 3D Visualization of Hankel Determinant Equations and Related Inequalities

3. First Attempts on $H_4(1)$ From 2015 to 2019

During this time, research focused on classes such as bi-univalent and starlike functions. Important work was done on $H_2(2)$ and $H_3(1)$, which prepared the way for the study of $H_4(1)$. Numerous articles examining these preliminary estimates surfaced in specialised journals [3].
the inverse function

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \tag{4}$$

Defined the class $\mathcal{J}_\beta^q(h)(\beta \geq 0)$ if the following quasi subordination holds

$$\left[\frac{zf'(z)}{f(z)}\right]\left[\frac{f(z)}{z}\right]^\beta - 1 \prec_q (h(z) - 1) \text{ and } \left[\frac{wg'(w)}{g(w)}\right]\left[\frac{g(w)}{w}\right]^\beta - 1 \prec_q (h(w) - 1) \tag{5}$$

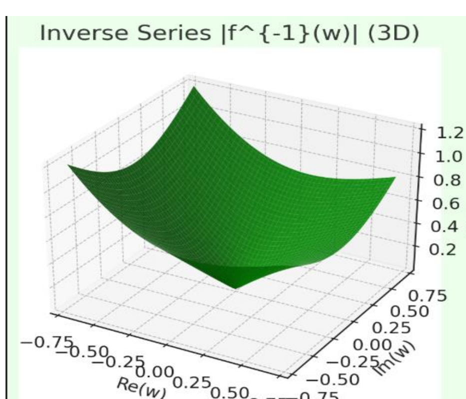


Fig. 3. Visualization of the Inverse Function f^{-1} (w) Real and Complex Domains

4. Contributions in Advance from 2020 to 2023

Cho and associates investigated star-like functions related to exponential and sine functions in 2020 [4] .

$$|a_n| \leq 1 + \lambda\sqrt{n-1}\sqrt{\sum_{k=0}^{n-2} \lambda^{2k}}(0 < \lambda \leq 1, n \geq 2) \tag{6}$$

where $n \geq 5$.

Proved that $|a_5| \leq \frac{1}{2}(2 + 2\lambda + 5\lambda^2 + 6\lambda^3 + 4\lambda^4)$. Chebyshev polynomials of the first kind were used by Arif and associates in 2021 [5] to give sharp bounds for $H_4(1)$. A generalised class W(δ, β, λ, t) was introduced by Rahman et al. the following year, utilising second-kind Chebyshev polynomials to improve estimations. If $f \in \mathcal{S}^*$ then

$$|H_{4,1}(f)| \leq \frac{457571}{129600} = 3.5306 \ldots \tag{7}$$

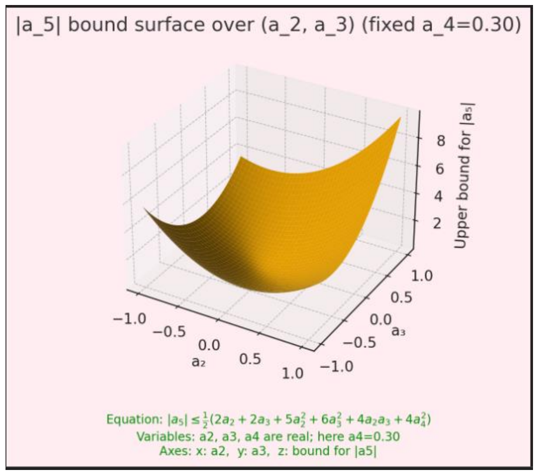


Fig 4. Surface Representation of the Upper Bound of $|a_n|$ in Terms of n and λ

5. Current Events (2023-2025)

Salman and Atshan proposed the function class $\mathcal{F}(\delta, z, t)$ in 2023 [6] .
consider the function $\mathcal{N}(t, z) = \frac{1}{1-2tz+z^2}, t \in \left(\frac{1}{2}, 1\right), z \in U$

$$\text{if } t = \cos\alpha, \alpha \in \left(0, \frac{\pi}{3}\right), \text{then}$$

$$\mathcal{N}(t, z) = 1 + \sum_{n=1}^{\infty} \frac{\sin[(n+1)\alpha]}{\sin\alpha} z^n = 1 + 2\cos\alpha z + (3\cos^2\alpha - \sin^2\alpha)z^2 + (8\cos^3\alpha - 4\cos\alpha)z^3 + \cdots, z \in U \tag{8}$$

That is

$$\mathcal{N}(t, z) = 1 + U_1(t)z + U_2(t)z^2 + U_3(t)z^3 + \cdots t \in \left(\frac{1}{2}, 1\right), z \in U \tag{9}$$

Defined the class $\mathcal{F}(\delta, z, t)$

$$\frac{z}{2\delta}[f''(z) + zf'''] \prec \mathcal{L}(t, z), \tag{10}$$

$$\text{where } \mathcal{L}(t, z) = 1 + U_1(t)z + U_2(t)z^2 + U_3(t)z^3 + \cdots t \in \left(\frac{1}{2}, 1\right), z \in U, \delta \geq 0. \tag{11}$$

Then

$$\frac{z}{2\delta}[f''(z) + zf'''] = \mathcal{L}(t, \mathcal{E}(z)). \tag{12}$$

It is defined by the subordination z / (2δ)(f''(z) + z f''(z)) ≺ L(t, z).

They arrived at the following bound using Chebyshev polynomial expansions: $H_4(1) \le 1230227 / 1875\ \delta^4 + 1067929 / 45000\ \delta^3$

This framework was extended to bounded turning functions connected with sine-based kernels in 2024.

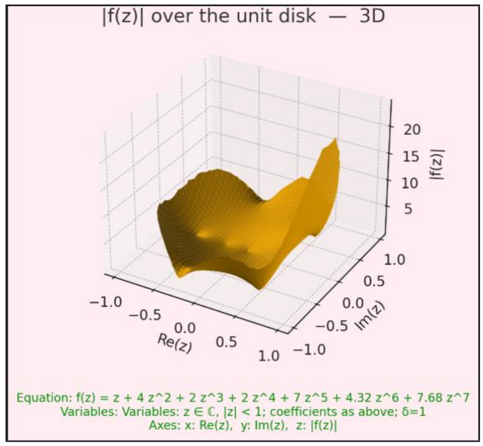


Fig 5. 3D Surface Visualization of $|N(t, z)|$ over the Unit Disk (for $= t0.80$)

6. Summary of Key Methods

Comparative bounds for $H_4(1)$

$$H_{4,1}(f) = (a_5 - a_3^2)(a_3 a_7 - a_5^2) \tag{13}$$

7. Examples to Show

Example 1 :

Assume that f(z) $\in \mathcal{F}(\delta, z, t)$ with δ = 1 and coefficients estimated as follows: $a_2 = 4$, $a_3 = 2$, $a_4 = 2$, $a_5 = 2.7$, $a_6 = 4.32$, and $a_7 = 7.68$.

We estimate that $H_4(1) \approx 679.85$ using the specified formula.

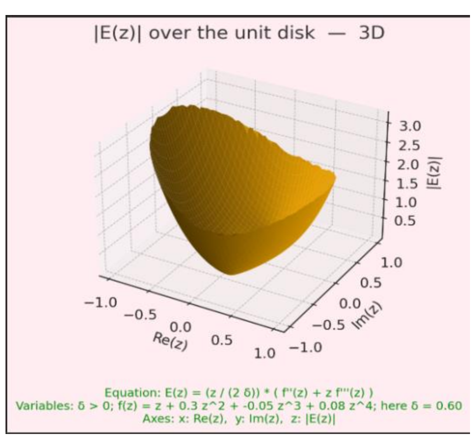
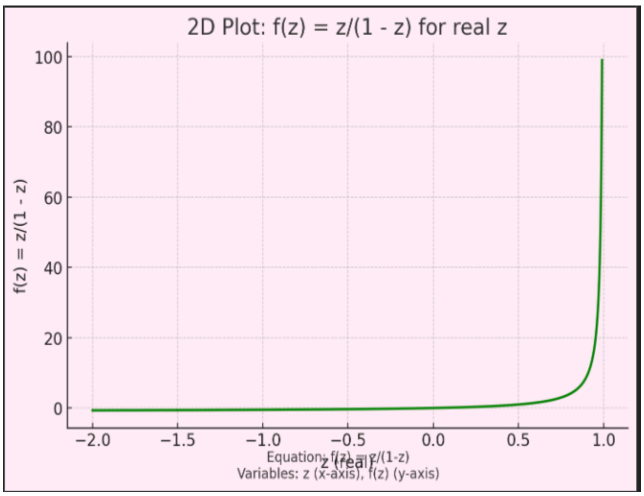


Fig 6. 3D Visualization of the Complex Function.

Example 2:

Let $\boldsymbol{f(z) = z/(1-z) \Rightarrow an = 1}$ for all n ≥ 1 and assume that $\boldsymbol{f(z)}$ ∈ S* with $\boldsymbol{Re(z\, f'(z)/f(z)) > 0}$.

Because of matrix singularity, $H_4(1) = 0$.



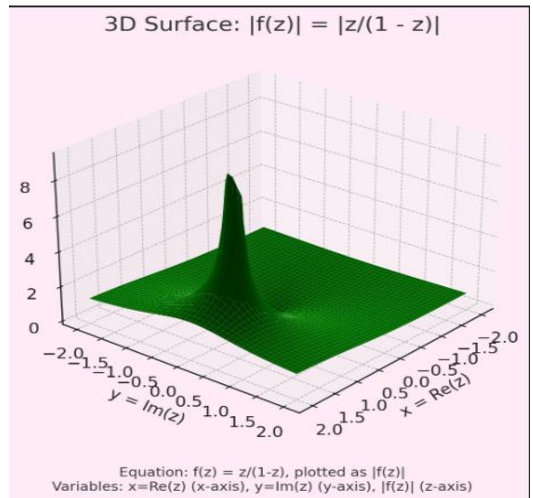


Fig 7. 2D Plot (Flat)

- **The graph shows**

$$\boldsymbol{f(z) = \frac{z}{1-z}}$$

along the real axis.

- As z approaches 1 from the left, the function blows up to infinity (a vertical asymptote at $z = 1$).
- The x-axis represents z (real values), and the y-axis represents $f(z)$.

Fig 8 . 3D Plot (Surface)

- **The graph shows the magnitude of the function:**

$$\boldsymbol{|f(z)| = \left|\frac{z}{1-z}\right|}$$

over the complex plane.

- $x = \Re(z)$ (real part of z)
- $y = \Im(z)$ (imaginary part of z)
- z-axis $= |f(z)|$
- There is a clear "spike" (pole) near the point $z = 1$, where the function tends to infinity.
- Everywhere else, the surface remains smooth and shows how $|f(z)|$ changes over the plane.

8. Conclusion and Prospects for the Future

Orthogonal polynomial tools and subordination principles are dynamically integrated in ongoing research on the fourth Hankel determinant. Future studies might concentrate on fractional derivatives, higher-order Hankel determinants, and using machine learning to estimate bounds predictively.

10. Other Hankel-Related Function Examples

Example 1:

Let $f(z) = \frac{z}{(1-z)}$ which grows to $f(z) = z + z^2 + z^3 + \ldots$ (14)

The fourth Hankel determinant, $H_4(1) = 0$, is a classic example of a function with constant coefficients, $a_0 = 1$.

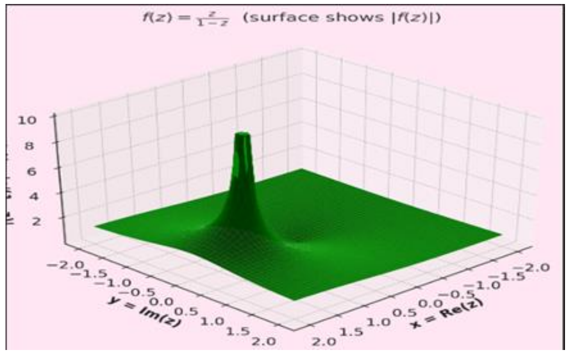


Fig 9 . 3D Visualization of the Complex Function $f(z)=\frac{z}{(1-z)}$

Example 2:

Examine the following function: $f(z) = z + 2z^2 + 3z^3 + 4z^4 + 5z^5 + 6z^6 + 7z^7$

The coefficients of this function, a_1 = n, increase linearly. One can construct its corresponding Hankel matrix and perform a numerical computation of $H_4(1)$.

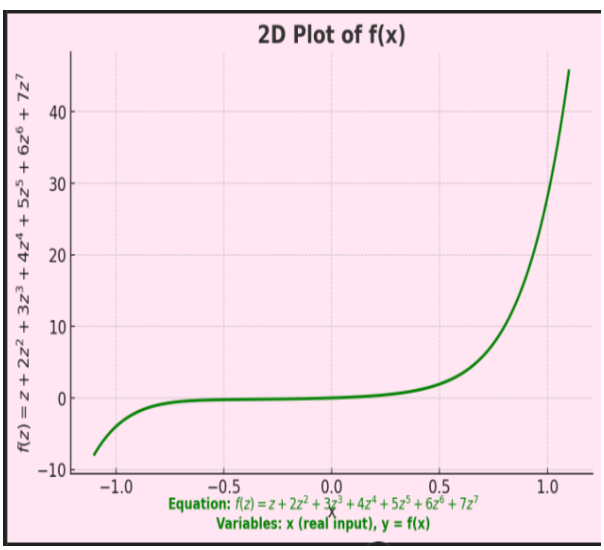


Fig 10 . 2D Plot (Flat)
A green curve representing the function $f(x)$ along the x-axis

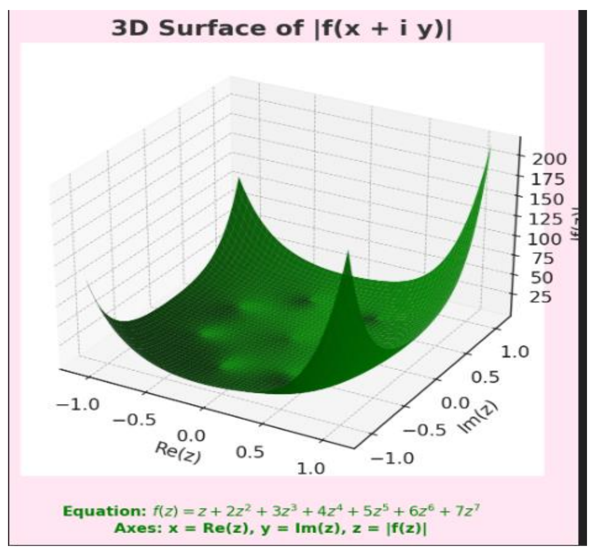


Fig 11. 3D Plot (Surface)
A three-dimensional surface showing $|f(x+iy)|$ over the complex plane (with the axes Re(z), Im(z), and the height $|f(z)|$].

Example 3:

Let $f(z) = z\,e^{\frac{z}{1-z}}$ This function has rich coefficient behavior and is closely related to subclasses of star like functions. Its Taylor expansion can be used to estimate $H_4(1)$.

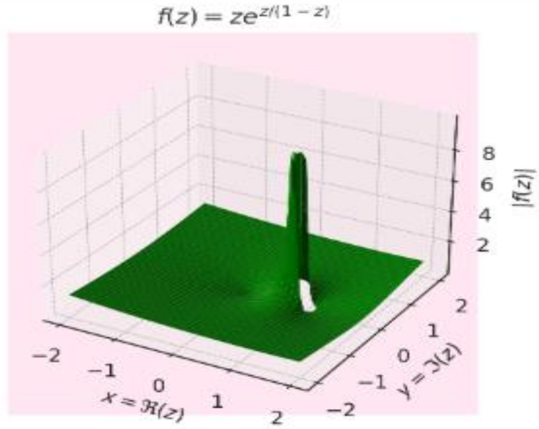


Fig 12. 3D Visualization of the Complex Function $f(z) = z\, e^{\frac{z}{1-z}}$

References

[1] J. W. Noonan and D. K. Thomas, "On the second Hankel determinant of areally mean p-valent functions," *Trans. Am. Math. Soc.*, vol. 223, pp. 337–346, 1976.

[2] L. De Branges, "A proof of the Bieberbach conjecture," *Acta Math.*, vol. 154, no. 1, pp. 137–152, 1985.

[3] J. M. Jahangiri and S. G. Hamidi, "Coefficient Estimates for Certain Classes of Bi-Univalent Functions," *Int. J. Math. Math. Sci.*, vol. 2013, no. 1, p. 190560, 2013.

[4] N. E. Cho and V. Kumar, "Initial coefficients and fourth Hankel determinant for certain analytic functions," *Miskolc Math. Notes*, vol. 21, no. 2, pp. 763–779, 2020.

[5] M. Arif, L. Rani, M. Raza, and P. Zaprawa, "Fourth Hankel Determinant for the Set of Star-Like Functions," *Math. Probl. Eng.*, vol. 2021, no. 1, p. 6674010, 2021.

[6] F. O. Salman and W. G. Atshan, "Results on Fourth Hankel Determinant of a Certain Subclass of Analytic Functions," *J. Al-Qadisiyah Comput. Sci. Math.*, vol. 15, no. 2, p. Page-132, 2023.

[7] I. A. R. Rahman, W. G. Atshan, and G. I. Oros, "New concept on fourth Hankel determinant of a certain subclass of analytic functions," *Afrika Mat.*, vol. 33, no. 1, p. 7, 2022.