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Approximation of Scale Parameter of Inverted Gamma Distribution by TOM Modified

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Abstract

The purpose of this research is to find an approximation to the scale parameter of inverted gamma distribution, where other parameter (shape parameter) is known, and compare it with several methods, where we found the preference of TOM by using Mean Square Error (MSE) to specify the best approximate to scale parameter in different samples size.

1. Introduction

The important subject in statistics is estimating parameters of such distribution. All the method that tries to estimating the parameter values gives approximating to their exact values. We try here to introduce method using the TOM (Term Omission Method) with some modifications to approximate the value of scale parameter of such distribution, and compare it with other methods as well as shown below and the results shown in table (1).

2. The Inverted Gamma Distribution

The inverted gamma is a special case of the generalized gamma distribution where the second shape parameter is equal to -1[3], and if $X \sim G(\alpha, \beta)$, with then Y has inverted gamma distribution or $Y \sim IG(\alpha, \beta)$.[5]

Definition 1: The X is a two-parameter inverted gamma random variable if it has the following function;

$$f(x;\alpha,\beta) = \frac{2\beta^{\alpha}}{\Gamma(\alpha)} x^{-2\alpha-1} e^{\left(\frac{-\beta}{x^2}\right)} \qquad \ldots (1)$$

with shape parameter α (α>0) and scale parameter β (β>0) and $x \in (0, \infty)$. [1]

3. Properties

The inverted gamma distribution of x, that denoted by $x \sim IG(\alpha,\beta)$, have the following properties: [2]

$$E(x) = \sqrt{\beta}\,\frac{\Gamma\left(\alpha - \frac{1}{2}\right)}{\Gamma(\alpha)}$$

$$Var(x) = \frac{\beta}{(\alpha - 1)} - \beta\left(\frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}\right)^2 \qquad \alpha > 1$$

and

$$M_X^{(j)}(t) = \beta^{\frac{j}{2}}\,\frac{\Gamma(\alpha - \frac{j}{2})}{\Gamma(\alpha)} \qquad j < 2\alpha$$

4. Some Different Methods of Estimation

4.1 *Maximum Likelihood Estimation (MLE)*

The probability density function of inverted gamma distribution

$$f(x;\alpha,\beta) = \frac{2\beta^{\alpha}}{\Gamma(\alpha)}\,x^{-2\alpha-1}\,e^{\left(\frac{-\beta}{x^2}\right)}$$

by taking the likelihood function we have with shape parameter α=1

$$L(\beta;\underset{\sim}{x}) = 2^n\beta^n\prod_{i=1}^{n} x_i^{-3}\,e^{-\sum_{i=1}^{n}\frac{\beta}{x_i^2}} \qquad \text{... (2)}$$

and by taking logarithm of the equation (2), we have,

$$Ln\,L(\beta;\underset{\sim}{x}) = n\ln 2 + n\ln\beta - 3\sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\frac{\beta}{x_i^2} \qquad \text{... (3)}$$

Therefore, we put the first derivative of equation (3) w.r.t. β equal to 0, then we get,

$$\frac{\partial Ln\,L(\beta;\underset{\sim}{x})}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\frac{1}{x_i^2} = 0$$

and hence, the estimating of scale parameter β is

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i^2}} \qquad \text{... (4)}$$

4.2 Ordinary Least Square (OLS):

In this method, firstly, we have to find the cumulative function of the gamma distribution (1) when shape parameter α=1, as follows:

$$F(x) = e^{-\frac{\beta}{x^2}} \qquad \text{... (5)}$$

by taking logarithm to both sides of the equation (5), we get,

$$\ln F(x) = -\frac{\beta}{x^2}$$

and hence, we can set the variable x depended on scale parameter β and distribution function $F(x)$, as follows,

$$x = \sqrt{\beta}\sqrt{-\frac{1}{\ln F(x)}} \qquad \text{... (6)}$$

equation (6) can be written as a linear regression function,

$$y^* = \beta^* x^* \qquad \text{...(7)}$$

where

$$y^* = x\,,$$

$$\beta^* = \sqrt{\beta}\,,$$

$$x^* = \sqrt{-\frac{1}{\ln F(x)}}$$

so, the estimating value of scale parameter β can be written as

$$\hat{\beta}^* = \frac{n\sum_{i=1}^{n} x_i^* y_i^* - \sum_{i=1}^{n} x_i^* \sum_{i=1}^{n} y_i^*}{n\sum_{i=1}^{n}\left(x_i^*\right)^2 - \left(\sum_{i=1}^{n} x_i^*\right)^2}$$

or

$$\hat{\beta} = \left(\frac{n\sum_{i=1}^{n} x_i^* y_i^* - \sum_{i=1}^{n} x_i^* \sum_{i=1}^{n} y_i^*}{n\sum_{i=1}^{n}\left(x_i^*\right)^2 - \left(\sum_{i=1}^{n} x_i^*\right)^2}\right)^2 \qquad \text{... (8)}$$

4.3 Relative least Square (RLS):

This method can found from the linear regression equation,

$$p = a + bq \qquad \text{... (9)}$$

so, if we take the square of the differences between both sides of equation (9) and take the minimum of the summation of relative difference square, then we have

$$R = Min.\sum_{i=1}^{n}\left(\frac{p_i - a - bq_i}{p_i}\right)^2 \qquad \dots (10)$$

or, with substitute y_i^* with p_i and x_i^* with q_i, equation (10) become

$$R = Min.\sum_{i=1}^{n}\left(1 - a\frac{1}{y_i^*} - b\frac{x_i^*}{y_i^*}\right)^2 \qquad \dots (11)$$

where y_i^* and x_i^* is defined in equation (7).

with $z_i = \frac{1}{y_i^*}$ and $w_i = \frac{x_i^*}{y_i^*}$, the equation (11) becomes

$$R = Min.\sum_{i=1}^{n}(1 - az_i - bw_i)^2 \qquad \dots (12)$$

and the estimating value of scale parameter β can be written as,

$$\hat{\beta} = \frac{\sum_{i=1}^{n} z_i^2 w_i - \sum_{i=1}^{n} z_i w_i \sum_{i=1}^{n} z_i}{\sum_{i=1}^{n} z_i^2 w_i^2 - \left(\sum_{i=1}^{n} z_i w_i\right)^2} \qquad \dots (13)$$

4.4 Moments

From the correlation between moments of society and the moment of the distribution we can find the estimation of the scale parameter β, that is
the first moment of distribution is

$$M_1^1 = E(x) = \sqrt{\pi\beta} \qquad \dots (14)$$

the first moment of society is

$$M_1^1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{X} \qquad \dots (15)$$

and by equality the equations (14, 15) we get the following relation

$$\sqrt{\pi\beta} = \overline{X}$$

so,

$$\hat{\beta} = \frac{\overline{X}^2}{\pi}$$

4.5 Term Omission Method (TOM)

Without generalization and of this distribution per se, we can use the TOM for the inverted gamma distribution to get the value of the scale parameter β instead of its value estimation: [4]

The p.d.f. of inverted gamma distribution with shape parameter α=1

$$f(x;\beta) = 2\beta x^{-3} e^{\left(\frac{-\beta}{x^2}\right)}$$

By using TOM for two values of x we have

x_i	y_i
x_1	$y_1 = 2\beta x_1^{-3} e^{\left(\frac{-\beta}{x_1^2}\right)}$
x_2	$y_2 = 2\beta x_2^{-3} e^{\left(\frac{-\beta}{x_2^2}\right)}$

by divided values of y_1 and y_2 on x^{-3} , where x_1 , $x_2 \neq 0$, we have

x_1	$y_1 = 2\beta e^{\left(\frac{-\beta}{x_1^2}\right)}$
x_2	$y_2 = 2\beta e^{\left(\frac{-\beta}{x_2^2}\right)}$

and by taking the logarithm of values y_1 and y_2 , where x_1 , x_2 >0, we find:

x_1	$y_1 = Ln2 + Ln\beta - \frac{\beta}{x_1^2}$
x_2	$y_2 = Ln2 + Ln\beta - \frac{\beta}{x_2^2}$

and by subtracting the values of y_2 from the value of y_1 we obtain:

$$\beta\left(\frac{1}{x_1^2} - \frac{1}{x_2^2}\right)$$

Hence, when we divide the final result on $\left(\frac{1}{x_1^2} - \frac{1}{x_2^2}\right)$, where x_1 , $x_2 \neq 0$, we get the value of β. Hence $\beta = \frac{Ln\left(\frac{y_i}{x_i^{-3}}\right) - Ln\left(\frac{y_j}{x_j^{-3}}\right)}{x_j^2 - x_i^2}(x_i^2 x_j^2)$ where $1 \leq i < j \leq n$; i j

5. Comparison between Methods

By using simulation or Random Number Generator to inverted gamma distribution from equation (6) we can generate a sample space that distributed inverted gamma distribution by Visual Basic language for different size of samples (n= 20, 60, 100, 1000) as shown in Table (1) and compare methods by using MSE with equation below:

$$MSE = \frac{\sum_{i=1}^{n}\left[F(x_i;\beta) - F(x_i;\hat{\beta})\right]^2}{n}$$

where $F(x_i;\beta), F(x_i;\hat{\beta})$ as shown in equation (5).

6. Conclusions

From the Table (1) if we compare between the all methods in this research for IG distribution we can see preference to TOM of any sample size firstly and OLS method secondly with large sample size and thirdly RLS with other methods by using the MSE in all samples. Also in TOM modified we can see that to find approximation to the scale parameter β we need just two points, and that will avoid us a difficult calculation on a large sample size.

Table (1): Comparison between estimated scale parameters methods using MSE

No.	β	S. Sizes	MLE MSE	OLS MSE	RLS MSE	M MSE	TOM(*) MSE	Best
1	0.2	20	0.2132 0.00311	0.1999 0.00343	0.4472 0.00646	0.2524 0.00076	0.2 0.00000	TOM
		60	0.2008 0.04002	0.2 0.00000	0.4472 0.04314	0.3467 0.00804	0.2 0.00000	TOM OLS
		100	0.2021 0.00003	0.2 0.00000	0.4472 0.02282	0.2935 0.00031	0.2 0.00000	TOM OLS
		1000	0.19757 0.01307	0.2 0.00000	0.316228 0.00244	0.091559 0.00010	0.2 0.00000	TOM OLS
2	1	20	1.0658 0.00382	1 0.00000	1 0.00000	0.5645 0.00365	1 0.00000	TOM OLS RLS
		60	1.0043 0.00014	1 0.00000	1 0.00000	0.7754 0.00532	1 0.00000	TOM OLS RLS
		100	1.0104 0.00003	1 0.00000	1 0.00000	0.6562 0.00421	1 0.00000	TOM OLS RLS
		1000	0.961368 0.01603	1 0.00000	1 0.00000	1.090739 0.00004	1 0.00000	TOM OLS RLS
3	2.5	20	2.6646 0.00279	2.4999 0.00242	1.5811 0.01521	0.8925 0.00271	2.5 0.00000	TOM
		60	2.5108 0.00033	2.5 0.00000	1.5811 0.01561	1.2259 0.04641	2.5 0.00000	TOM OLS
		100	2.5261 0.00013	2.4999 0.00242	1.5811 0.00917	1.0376 0.00001	2.5 0.00000	TOM OLS
		1000	2.50838 0.01481	2.5 0.00000	1.581139 0.00091	2.368275 0.00003	2.5 0.00000	TOM OLS
4	3.5	20	3.7304 0.00170	3.4999 0.00138	1.8708 0.02816	1.0560 0.00148	3.5 0.00000	TOM OLS
		60	3.5152 0.00065	3.5000 0.00000	1.8708 0.02842	1.4506 0.00509	3.5 0.00000	TOM OLS
		100	3.5365 0.00013	3.5 0.00000	1.8708 0.01952	1.2277 0.00170	3.5 0.00000	TOM OLS
		1000	3.609333 0.01463	3.5 0.00000	1.870829 0.00163	3.785415 0.00003	3.5 0.00000	TOM OLS
5	5	20	5.3292 0.00746	4.9999 0.00639	2.2361 0.04632	1.2622 0.00624	5 0.00000	TOM OLS
		60	5.0216 0.00162	5 0.00000	2.2361 0.05032	1.7337 0.00100	5 0.00000	TOM OLS
		100	5.0521 0.00002	5 0.00000	2.2361 0.02776	1.4674 0.00088	5 0.00000	TOM OLS
		1000	5.228518 0.01468	5 0.00000	2.236068 0.00263	5.587202 0.00004	5 0.00000	TOM OLS

(*) the value calculated between any two values of x.

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