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Comparison between numerical solutions and analytical solutions of the fractional logistic biological equation in its linear form

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ABSTRACT

Microsoft Excel is commonly treated as a simple spreadsheet program that is mainly applied This study compares the analytical and numerical solutions of linear fractional logistic equation qualitatively. Findings differ from both numerical of fractional solution biological logistic equation and analytical solution of first order linear fractional logistic both have same behavior during ascent process but do present some difference.

MSC..

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1. Introduction

The concept of fractional calculus, although widely applied in the 20th and 21st centuries due to its effectiveness in improving mathematical models, dates back to 1695 in a letter from Leibniz, where he explored the generalization of differential operators. He anticipated its future significance, particularly in scientific modeling.

Fractional derivatives differ from classical derivatives because they are nonlocal operators associated with memory effects, meaning that their values depend on the entire history of a system rather than only its present state. This property makes fractional calculus especially suitable for modeling complex biological, physical, and engineering processes that inherently exhibit memory behavior. Additionally, fractional-order differential equations often reduce modeling errors compared to traditional integer-order models. In biological applications, the logistic equation has been extended using fractional derivatives to better describe growth phenomena, such as tumor development. Solutions to these equations can be obtained using the Laplace transform under certain assumptions. However, mathematical challenges arise because fractional-order logistic equations do not satisfy some classical properties, such as the chain rule, making verification of solutions more difficult. Therefore, the primary objective of

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this study is to compare the numerical solutions of the standard fractional-order logistic equation with those of its biological form. The paper also presents the necessary definitions of fractional derivatives and the Laplace transform techniques required to solve such equations.

2. Primer

2.1 The Gel'fand-Shilov function

Of particular importance to the definition of an arbitrary order Riemann-Liouville integral, the Gel'fand-Shilov function, is defined from the Taylor series coefficients, to $n \in N$,

$$\emptyset_n(x) = \begin{cases} \frac{x^{n-1}}{n-1)!}, x \geq 0 \\ 0, \quad x < 0 \end{cases}$$

Accordingly, in cases where:

$$Re(v) \in C - Z$$

$$\emptyset_v(x) = \begin{cases} \frac{x^{n-1}}{\Gamma(v)}, x \geq 0 \\ 0, \ x < 0 \end{cases} \quad (1)$$

2.1.1 The Laplace transform of the Gel'fand-Shilov function

Now, we can compute the Laplace transform of the Gel'fand-Shilov function under the property's conditions.

$x \geq 0$ and $Re(v) \in C - Z$

$$\mathcal{L}[\phi_v(t)] = \mathcal{L}\left[\frac{t^{v-1}}{\Gamma(v)}\right] = \int_{-\infty}^{+\infty} \frac{t^{v-1}}{\Gamma(v)} e^{-st} dt = \frac{1}{\Gamma(v)} \int_0^{+\infty} t^{v-1} e^{-st} dt =$$

$$= \frac{1}{\Gamma(v)} \int_0^{+\infty} \left(\frac{u}{s}\right)^{v-1} e^{-u} \frac{du}{s} = \frac{1}{\Gamma(v) s^v} \int_0^{+\infty} u^{v-1} e^{-u} du =$$

$$= \frac{1}{\Gamma(v) s^v} \Gamma(v) = \frac{1}{s^v} = s^{-v}.$$

2.2 Functions (Maytag-Levelr)

In (Ordinary Differential Equation), linear and invariant coefficients, the solution has the form of an exponential function, while in a differential equation with improper order, linear and invariant coefficients, the solution is given, in most cases, in terms of the so-called Mitag-Leffler functions, which generalize the exponential function, both by obtaining a multiple of it.

Definition 2.1 The Maytag-Leffler function

The function $E_\alpha(z)$ Complex because it depends on a parameter

$\alpha \in C$ with $Re(\alpha) \geq 0$

It is given through the following series:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

As pointed out before, we consider the Maytag-Leffler function as a generalized form of the exponential

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

function since α = 1 gives us the exponential function.:

Definition 2-2 A two-parameter Maytag-Leffler function A is a complex function $E_{\alpha,\beta}(z)$. It has two complex parameters α and β with

$\mathrm{Re}(\alpha) \geq 0$ and $\mathrm{Re}(\beta) \geq 0$

It is given through the following series:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}$$

Generalizing the one-parameter Maytag-Leffler function, just admit β = 1, and return the function with one parameter

$$E_{\alpha,1}(z) = E_{\alpha}(z)$$

2.2.1 Laplace transformation of Mitag-Leffler function.

We begin by calculating the Laplace transform of this function.

$t^{\beta-1} E_{\alpha,\beta}(at^{\alpha})$

$\alpha \in C$, $\mathrm{Re}(s) \geq 0$, $\alpha \geq 0$ and $\beta \geq 0$

$$\mathcal{L}[t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})] = \int_0^{\infty} e^{-st} t^{\beta-1} \sum_{n=0}^{\infty} \frac{(at^{\alpha})^n}{\Gamma(\alpha n+\beta)} = \sum_{n=0}^{\infty} \frac{a^n}{\Gamma(\alpha n+\beta)} \int_0^{\infty} e^{-st} t^{\alpha n+\beta-1} dt =$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{\Gamma(\alpha n+\beta)} \int_0^{\infty} e^{-u} \left(\frac{u}{s}\right)^{\alpha n+\beta-1} \frac{du}{s} = \frac{1}{s^{\beta}} \sum_{n=0}^{\infty} \frac{a^n}{\Gamma(\alpha n+\beta)} \frac{1}{s^{\alpha n}}$$

$$\int_0^{\infty} e^{-u} u^{\alpha n+\beta-1} du = \frac{1}{s^{\beta}} \sum_{n=0}^{\infty} \frac{a^n}{\Gamma(\alpha n+\beta)} \frac{1}{s^{\alpha n}} \Gamma(\alpha n+\beta) = \frac{1}{s^{\beta}} \sum_{n=0}^{\infty} \left(\frac{a}{s^{\alpha}}\right)^n$$

that issue

$\left|\frac{a}{s^{\alpha}}\right| < 1$

We have a close geometric progression. then:

$$\mathcal{L}[t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})] = \frac{1}{s^{\beta}} \frac{1}{1-\frac{a}{s^{\alpha}}} = \frac{s^{\alpha-\beta}}{s^{\alpha}-a}$$

2.3 Partial integration

The systematic integral is the distorted form of both the Gel'fand-Shilov function and the function being integrated.

$$I^n f(t) = \phi_n(t) * f(t) = \int_0^t \phi_n(t-\tau) f(\tau) d\tau = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} f(\tau) d\tau$$

2.4 a capito derivative

Let f(t) be a function whose nth derivative is integrable. Let α be a complex number with Re(α) > 0 and α is defined. The Caputo derivative is defined as follows.

$$D^\alpha f(t) = I^{n-\alpha} D^n f(t) \tag{4}$$

for this reason

$$n - 1 < \mathrm{Re}(\alpha) \leq n$$

Let's get into a specific case of

$f(t) = t^0 = 1$:

$$D^\alpha f(t) = I^{n-\alpha} D^n 1 = I^{n-\alpha} 0 = \phi_{n-\alpha} * 0 = 0.$$

Therefore, as we can observe in the case of the capito derivative, when we admit f(t) as a constant, the result of the derivative is zero, as in the classical case.

2.4.1 Laplace transform of the Caputo derivative

Next we derive the Laplace transform of Caputo's derivative.

$$\mathcal{L}[D^\alpha f(t)] = \mathcal{L}[I^{n-\alpha} D^n f(t)] = \mathcal{L}[\phi_{n-\alpha} * D^n f(t)] = \mathcal{L}[\phi_{n-\alpha}(t)]\mathcal{L}[D^n f(t)] =$$

$$= s^{\alpha-n}\left[s^n F(s) - \sum_{j=0}^{n-1} s^j \left[\left(\frac{d^{n-1-j}}{dt^{n-1-j}}(0)\right)\right]\right] =$$

$$= s^\alpha F(s) - \left[\sum_{j=0}^{n-1} s^{\alpha-n-j} \left[\left(\frac{d^{n-1-j}}{dt^{n-1-j}}(0)\right)\right]\right].$$

3. Classical logistic equation

Malthus' model (1798) is of paramount importance for all subsequent studies of modeling population dynamics. The English economist was the first to deal with population growth, taking into account some basic considerations, as food is necessary for human sustenance and his biological sex in the species is permanent, so he assumed that the ability of human reproduction exceeds food production. In this way, the population grows in proportion to itself. Thus we have in equation (5) a Malthus model for the order of integers (BASSANEZI, 2014).

$$\frac{d\,P(t)}{dt} = k\,P(t) \tag{5}$$

In 1838 Verhulst proposed a biological equation incorporating demographic data into the model which improving the Malthusian theory. Other matters, such as the spread of diseases, could be taken into account as it entails factors that may limit the proliferation of viruses in the population. The order of Verhulst equation is correct because of.

$$\frac{d\,y(t)}{dt} = r\,y(t)\left(1 - \frac{y(t)}{y_{max}}\right)$$

where y(t) is the population at time r, t is the growth rate and y(max) is the maximum value of the population. We can look at the equation

$$\frac{d\,y(t)}{dt} = k\,y(t)(1 - y(t))$$

4. Partial generalization

We will perform a fractional generalization of equation (6) for this purpose, we will use the fractional Capito derivative instead of the (Riemann-Liouville) derivative in the models, and we will also do the solution using the method of integral transformations. When we calculate the Laplace transform of a (Riemann-Liouville) derivative, we generate a result that depends on the integral differential operator of a function that does not have an integer order, while the Laplace transform of a Capito derivative (as mentioned earlier) depends on the derivatives of an integer order of f(0) (CAMARGO , 2015).

Equation (6) is a nonlinear and separable ordinary differential equation. Although the Verhulst equation can be easily solved because it is separable, we cannot say the same about the improper order version. In order to make the equation linear so that the fractional version can be solved using the methodology of integral transformations, we will make the following change in the variable, bearing in mind that the equation is of

Bernoulli type:
$$y(t) = z(t)^{-1} \quad \Longrightarrow \quad \frac{d\,y(t)}{dt} = \frac{-1}{z^2}\frac{d\,z(t)}{dt}$$

Then we have the following ODE:

$$\frac{d\,z(t)}{dt} = k(1 - z(t)) \qquad (7)$$

It is a linear and separable ODE:

$$\frac{d\,z(t)}{dt} = k(1 - z(t)) \leftrightarrow \frac{d\,z(t)}{1 - z(t)} = k\,dt \implies z(t) = 1 + \frac{1}{c}e^{-kt}$$

example

$$z(0) = \frac{1}{c} + 1, \text{ i.e, } \frac{1}{c} = \frac{1}{y(0)} - 1$$

Then we have a solution to equation (6):

$$y(t) = \frac{1}{1 + \left[\frac{1}{y(0)} - 1\right]e^{-kt}} \qquad (8)$$

4.1 Analytical fractional solution

When we pass equation (7) into its fractional generalization, taking into account the unit of measure of the constant: we get the following equation:

$$\frac{d^{\alpha} z(t)}{dt} = k^{\alpha}(1 - z(t)) \tag{9}$$

We will now solve equation (9) analytically, but we must point out that the chain rule arising from classical calculus is not valid in invalid calculus (Varalta; Gomez; Camargo, 2014), that is:

$$\frac{d^{\alpha} y(t)}{dt^{\alpha}} = \frac{d^{\alpha}(1/z(t))}{dt^{\alpha}} \neq -z(t)^{-2}\frac{d^{\alpha}}{dt^{\alpha}}z(t)$$

Even knowing this fact, we decided to perform the calculations to see if the solution was similar to the numerical solution presented in an article (Al-Saqqa, 2007)

In order to deal with (9), we utilise the integral transform method once again. Keeping in mind nonlocal and hereditary thermofluidal processes, we use the fractional derivative in the Caputo sense. Next, we apply the Laplace transform. More specifically, we apply it to (9).

$$s^{\alpha}F(s) - s^{\alpha-1}z(0) = \frac{k^{\alpha}}{s} - k^{\alpha}F(s) \implies F(s)[s^{\alpha} + k^{\alpha}] = s^{\alpha-1}z(0) + \frac{k^{\alpha}}{s}$$

$$F(s) = \frac{s^{\alpha-1}z(0)}{s^{\alpha}+k^{\alpha}} + \frac{k^{\alpha}}{s(s^{\alpha}+k^{\alpha})} = \frac{s^{\alpha-1}z(0)}{s^{\alpha}+k^{\alpha}} + k^{\alpha}\left[\frac{s^{-1}}{s^{\alpha}+k^{\alpha}}\right].$$

Calculate the Laplace transformation of the following function.

$$\mathcal{L}^{-1}[F(s)] = z(0)\mathcal{L}^{-1}\left[\frac{s^{\alpha-1}}{s^{\alpha}+k^{\alpha}}\right] + k^{\alpha}\mathcal{L}^{-1}\left[\frac{s^{-1}}{s^{\alpha}+k^{\alpha}}\right] =$$

$$z(0)E_{\alpha}(-(kt)^{\alpha}) + k^{\alpha}t^{\alpha}E_{\alpha,\alpha+1}(-(kt)^{\alpha}) =$$

$$z(0)E_{\alpha}(-(kt)^{\alpha}) + 1 - E_{\alpha}(-(kt)^{\alpha}) =$$

$$1 + [z(0) - 1]E_{\alpha}(-(kt)^{\alpha}),$$

This means that the solution for z(t) is:

$$z(t) = 1 + [z(0) - 1]E_{\alpha}(-(kt)^{\alpha})$$

Knowing that

$$y(t) = 1/z(t)$$

Then we have the solution to the logistic equation:

$$y(t) = \frac{1}{1 + \left[\frac{1}{y(0)} - 1\right]E_{\alpha}(-(kt)^{\alpha})} \tag{11}$$

If we do not correct the unit of measure for the constant k: the solution to y(t) becomes:

$$y(t) = \frac{1}{1 + \left[\frac{1}{y(0)} - 1\right]E_{\alpha}(-kt^{\alpha})} \tag{12}$$

4.2 Graphics

As mentioned earlier

$$\frac{d^\alpha y(t)}{dt} \neq \frac{d^\alpha (1/z(t))}{dt}$$

We made a comparison between the analytical solution, despite the lack of equivalence between the classical chain rule and the numerical solution presented by El-Sayed, El-Messery and El-Saqqa (2007), and thus the following graphs were produced:

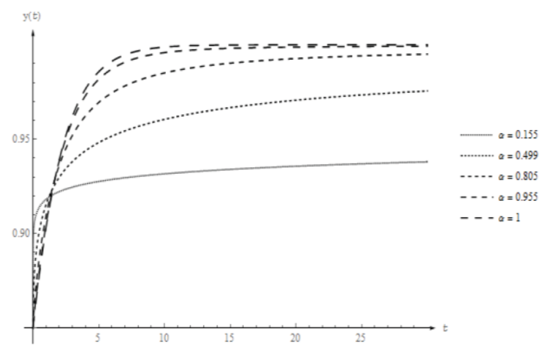

Figure 1: Built using MATLB to solve equation (11).

Taking y(0) = 0.85, the values of α are equal to those given by(El-Sayed, El-Messery, and El-Sakka (2007)) and k = 0.5.

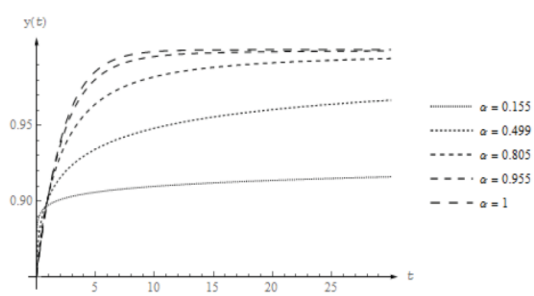

Figure 2: Built using MATLB to solve equation (12).

Taking y(0) = 0.85, the values of α are equal to those given by El-Sayed, El-Messery, and El-Sakka (2007). and k = 0.

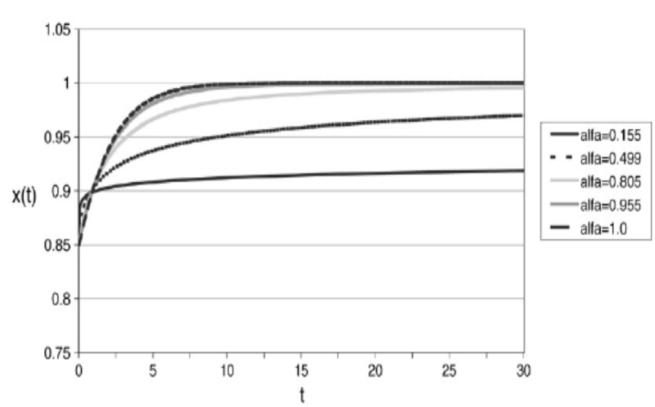

Figure 3: An article by Al Sakka (2017) produced the following result with y(0)=0.85 and k=0.5.

5. Conclusion

In these analyzed cases, correcting the measured unit we did not observe a very significant difference in the graphical analysis of solutions (11) and (12). As mentioned earlier, the chain rule is not valid for CF, so we compared the analytical solution with the numerical solution given by Sakka (2007). We use the same coefficients found in the article in question. The numerical method used by El-Sayed, El-Messery, and El-Saqqa (2007) is called the Adams method. When we draw the solution (11), we notice that it is more similar to the numerical solution than using (12**).**

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