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Semi Annihilator Large Small Submodules

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ABSTRACT

We will outline the idea of semi annihilator large small submodules. It was one of generalizations of annihilator small submodules. an $\mathfrak{R}$-module $\mathfrak{H}$ with $(0) \neq S \leq \mathfrak{H}$ is named semi-annihilator large small submodule of $\mathfrak{H}$ $if\ S + \aleph = \mathfrak{H}$ for $\aleph$ is submodule in $\mathfrak{H}$ imply $\boldsymbol{\Lambda}_{\mathfrak{R}}(\aleph)$ is large small in $\mathfrak{R}$, where $\boldsymbol{\Lambda}_{\mathfrak{R}}(\aleph) = \{r \in \mathfrak{R}; r\aleph = 0\}$. In addition, investigate properties and characterizations of semi annihilator large small submodules. Additionally, we introduce many the notion which serve our concept. These concepts extend existing ideas and offer new perspectives on the relationships between different submodule and module types within $\mathfrak{R}$-modules.

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Introduction

All modules in this research is an $\mathfrak{R}$-module and are unitary left modules and $\mathfrak{R}$ is a commutative ring with identity. A submodule $\mathfrak{I}$ of an $\mathfrak{R}$-module $\mathfrak{H}$ will be identified as $(\mathfrak{I} \sqsubseteq \mathfrak{H})$. A proper submodule $\mathfrak{I}$ of an $\mathfrak{R}$-module $\mathfrak{H}$ will be identified as Large (essential) and termed $(\mathfrak{I} \sqsubseteq^{l} \mathfrak{H})$ if for $0 \neq \aleph \sqsubseteq \mathfrak{H}$ with $\mathfrak{I} \cap \aleph \neq (0)$ [1]. Let $\mathfrak{H}$ be an $\mathfrak{R}$-module and $0 \neq \mathfrak{I} \sqsubseteq \mathfrak{H}$. A submodule $\mathfrak{I}$ of an $\mathfrak{R}$-module $\mathfrak{H}$ is termed small and indicated by $(\sqsubseteq^{s})$ if $\mathfrak{I} + \aleph = \mathfrak{H}$ for $\aleph \sqsubseteq \mathfrak{H}$ implies $\aleph = \mathfrak{H}$ [1]. Also A submodule $\mathfrak{I}$ of an $\mathfrak{R}$-module $\mathfrak{H}$ is named large small (briefly, LS-submodule) if $\mathfrak{I} + \aleph = \mathfrak{H}$ for $\aleph \sqsubseteq^{l} \mathfrak{H}$ means $\aleph = \mathfrak{H}$ this symbolled by $\mathfrak{I} \sqsubseteq^{l.s} \mathfrak{H}$ [2]. A symbol $\boldsymbol{\Lambda}_{\mathfrak{R}}(\mathfrak{H})$ is referred to be annihilator submodule where $\boldsymbol{\Lambda}_{\mathfrak{R}}(\mathfrak{H}) = \{r \in \mathfrak{R}, rm = 0, for\ all\ m \in \mathfrak{H}\}$ [3]. If $\boldsymbol{\Lambda}_{\mathfrak{R}}(\mathfrak{H}) = 0$ so $\mathfrak{H}$ is named faithful. The set $Z(\mathfrak{H}) = \{m \in \mathfrak{H}: \boldsymbol{\Lambda}(m) \sqsubseteq^{l} \mathfrak{R}\}$, if $Z(\mathfrak{H}) = \mathfrak{H}$, if $Z(\mathfrak{H}) = (0)$ then $\mathfrak{H}$ is referred to be non-singular module see [4]. If R is an integral domain then an $\mathfrak{R}$-module $\mathfrak{H}$ is named a torsion free $\mathfrak{R}$-module if $\Lambda(x) = 0$, for $0 \neq x \in \mathfrak{H}$ [4]. The generalized Jacobson radical $GJ(\mathfrak{H}) = \{\sum \aleph_i , \aleph_i \sqsubseteq^{l.s} \mathfrak{H}\}$ [2].

As we know, many authors have addressed many concepts similar to ours. In 2013, T. Amouzegar and D. Keskin introduced the definition of small annihilator as if $\mathfrak{H}$ is an $\mathfrak{R}$-module, then $\mathfrak{I} \sqsubseteq \mathfrak{H}$ is defined as small annihilator if, $\mathfrak{H} = \mathfrak{I} + \aleph$, for $\aleph \sqsubseteq \mathfrak{H}$, implies $\boldsymbol{\Lambda}_{S}(\aleph) = 0$, where $S = End(\mathfrak{H})$ [5]. In 2017, Hamdi and AL-Bahraany introduced the concept of $\mathfrak{R}$-annihilator small as, a submodule $\mathfrak{I} \sqsubseteq \mathfrak{H}$ is a $\mathfrak{R}$-annihilator-small ($\mathfrak{R}$-a-small) if $\mathfrak{I} + \aleph = \mathfrak{H}$, for $\aleph \sqsubseteq \mathfrak{H}$, means $\boldsymbol{\Lambda}_{s}(\aleph) = 0$ [6]. In 2016 S. M. Yaseen introduced the idea of semi annihilator small submodule as, if for every submodule $\mathfrak{I}$ of an $\mathfrak{R}$-module $\mathfrak{H}$, $\mathfrak{H} = \mathfrak{I} + \aleph, \aleph \sqsubseteq \mathfrak{H}$, implies $\boldsymbol{\Lambda}_{\mathfrak{R}}(\aleph)$ is small in $\mathfrak{H}$ [7].

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In this paper we will introduce a generalization of concept of semi annihilator small submodule that was introduced by M. Yaseen in 2016 [7] as a new class of semi annihilator modules. The main objective of this manuscript is to highlight the generalization of these concepts, discuss their implications, and define the behavior of submodules under the influence of this concept.

This paper divided in to two main sections, in first section we have establish the notation of semi annihilator large small submodules with many examples and investigate properties throughout, various properties, characterizations, and examples of these concepts are explored to deepen the understanding of the structure of submodules in $\mathfrak{R}$-modules $\mathfrak{H}$. The second section is containing main results; and discusses some important relations between our concepts and other related concepts in module theory.

2. Semi Annihilator Large Small submodules

We will introduce our main definition of semi annihilator Large small submodules with examples and interesting consequences.

Definition 2.1.

Assume we have an $\mathfrak{R}$-module $\mathfrak{H}$ with $(0) \neq S \sqsubseteq \mathfrak{H}$ is named semi-annihilator large small submodule of $\mathfrak{H}$ (briefly SALS submodule) indicated by $S \sqsubseteq^{s.a.l.s} \mathfrak{H}$ if $S + \aleph = \mathfrak{H}$ for $\aleph$ is submodule in $\mathfrak{H}$ imply $\mathbf{\Lambda}_{\mathfrak{R}}(\aleph)$ is large small in $\mathfrak{R}$, denoted $(\mathbf{\Lambda}_{\mathfrak{R}}(\aleph) \sqsubseteq^{l.s} \mathfrak{R})$ where $\mathbf{\Lambda}_{\mathfrak{R}}(\aleph) = \{r \in \mathfrak{R}; r\aleph = 0\}$. An $\mathfrak{R}$-module $\mathfrak{H}$ is termed semi-annihilator large small module if all its submodules are semi-annihilator large small submodules. Also an ideal I of ring $\mathfrak{R}$ is called semi-annihilator large small ideal for $\mathfrak{R}$ as an $\mathfrak{R}$-module.

Examples 2.2.

1. If $\mathfrak{H} = \mathbb{Z}$ as a $\mathbb{Z}$-module then all its submodules are SALS in $\mathfrak{H}$ since if trivially, for $\mathfrak{H} = n\mathbb{Z} + m\mathbb{Z}$ and $\Lambda_{\mathfrak{R}}(m\mathbb{Z}) = 0 \sqsubseteq^{l.s} \mathbb{Z}$.

2. If $\mathfrak{H} = \mathbb{Q}$ be $\mathbb{Z}$-module, thus $\mathbb{Z}$ is SALS of $\mathbb{Q}$, since $\mathbb{Z} \sqsubseteq \mathbb{Q}$ and $\mathbb{Q} = \mathbb{Q} + \mathbb{Z}$, $\Lambda_{\mathfrak{R}}(\mathbb{Z}) = 0 \sqsubseteq^{l.s} \mathbb{Z}$. while if $\mathfrak{H} = \mathbb{Z}$ as $\mathbb{Z}$-module, thus $\mathbb{Z}$ is not SALS of $\mathbb{Z}$, since $\mathbb{Z} = \mathbb{Z} + 0$ and $\Lambda_{\mathfrak{R}}(0) = \mathbb{Z}$ which is not SALS of $\mathbb{Z}$.

Remark 2.3.

Suppose that $\mathfrak{H}$ is an $\mathfrak{R}$-module. If $\mathfrak{I} \sqsubseteq \mathfrak{H}$ and $\mathfrak{H}$ is SALS. Hence, $\Lambda_{\mathfrak{R}}(\mathfrak{I}) \sqsubseteq^{l.s} GJ(\mathfrak{R})$. Since $\mathfrak{H} = \mathfrak{H} + \mathfrak{I}$, so $\Lambda_{\mathfrak{R}}(\mathfrak{I}) \sqsubseteq^{l.s} GJ(\mathfrak{R})$ and since $GJ(\mathfrak{R}) = \{\sum_{G_i \sqsubseteq^{l.s},} G_i \sqsubseteq \mathfrak{H}\}$. Thus, $\Lambda_{\mathfrak{R}}(\mathfrak{I}) \sqsubseteq^{l.s} GJ(\mathfrak{R})$.

Proposition 2.4.

If the ring R is an integral domain and $\mathfrak{H}$ is a torsion free $\mathfrak{R}$-module then every submodule of $\mathfrak{H}$ is SALS.

Proof.

Assume $\mathfrak{H} = \mathfrak{I} + \aleph$, for $\mathfrak{I}, \aleph \sqsubseteq \mathfrak{H}$. So $\Lambda_{\mathfrak{R}}(\aleph) = 0 \sqsubseteq^{l.s} \mathfrak{R}$. Since $\mathfrak{R}$ is an integral domain and $\mathfrak{H}$ is a torsion-free $\mathfrak{R}$-module. thus $\mathfrak{I} \sqsubseteq^{s.a.l.s} \mathfrak{H}$.

Proposition 2.5.

Suppose that $\aleph \sqsubseteq \mathfrak{H}$, and $\mathfrak{H}$ is faithful $\mathfrak{R}$-module $\mathfrak{H}$ which has Large annihilator in $\mathfrak{R}$ is SALS of $\mathfrak{H}$.

Proof.

Assume $\mathfrak{H} = \mathfrak{I} + \aleph$, where $\mathfrak{I}, \aleph \sqsubseteq \mathfrak{H}$. As, $\mathfrak{H}$ is faithful thus $0 = \mathbf{\Lambda}_{\mathfrak{R}}(\mathfrak{H}) = \mathbf{\Lambda}_{\mathfrak{R}}(\mathfrak{I} + \aleph) = \mathbf{\Lambda}_{\mathfrak{R}}(\mathfrak{I}) \cap \mathbf{\Lambda}_{\mathfrak{R}}(\aleph)$. Since $\mathbf{\Lambda}_{\mathfrak{R}}(\mathfrak{I})$ is large in $\mathfrak{R}$ thus $\mathbf{\Lambda}_{\mathfrak{R}}(\aleph) = 0$. Thus $\mathfrak{I} \sqsubseteq^{s.a.l.s} \mathfrak{H}$.

Lemma 2.6.

If $\mathfrak{H}$ is an $\mathfrak{R}$-module and $\mathfrak{I}, \mathcal{L} \sqsubseteq \mathfrak{H}$, So

1. $\Lambda_{\mathfrak{R}}(\mathfrak{I}) \sqsubseteq \Lambda_{\mathfrak{R}}(\mathfrak{I} + \mathcal{L})$ and $\Lambda_{\mathfrak{R}}(\mathcal{L}) \sqsubseteq \Lambda_{\mathfrak{R}}(\mathfrak{I} + \mathcal{L})$.

2. $\Lambda_{\mathfrak{R}}(\mathfrak{I}) \sqsubseteq \Lambda_{\mathfrak{R}}(\mathfrak{I} \cap \mathcal{L})$ and $\Lambda_{\mathfrak{R}}(\mathfrak{I}) \sqsubseteq \Lambda_{\mathfrak{R}}(\mathfrak{I} \cap \mathcal{L})$.

3. If $\eta: \mathfrak{I} \longrightarrow \mathcal{L}$ is homomorphism. Then $\Lambda_{\mathfrak{R}}(\mathfrak{I}) \sqsubseteq \Lambda_{\mathfrak{R}}(\eta(\mathfrak{I}))$.

Proof.

1. Let $r \in \Lambda_{\mathfrak{R}}(\mathfrak{I})$, so $r \in \Lambda_{\mathfrak{R}}(\mathfrak{I}) \cap \Lambda_{\mathfrak{R}}(\mathcal{L}) = \Lambda_{\mathfrak{R}}(\mathfrak{I} + \mathcal{L})$. Thus $\Lambda_{\mathfrak{R}}(\mathfrak{I}) \sqsubseteq \Lambda_{\mathfrak{R}}(\mathfrak{I} + \mathcal{L})$. The other part by the same way.

2. Let $r \in \Lambda_{\mathfrak{R}}(\mathfrak{I})$, so $r\mathfrak{I} = 0$, thus $r(\mathfrak{I} \cap \mathcal{L}) = r\mathfrak{I} \cap r\mathcal{L} = 0$. Hence, $r \in \Lambda_{\mathfrak{R}}(\mathfrak{I} \cap \mathcal{L})$. The other part by the same way.

3. Let $r \in \Lambda_{\mathfrak{R}}(\mathfrak{I})$, so $r\mathfrak{I} = 0$, thus $\eta(r.\mathfrak{I}) = \eta(0)$. Since η is homomorphism thus $r.\eta(\mathfrak{I}) = 0$. This imply $r \in \Lambda_{\mathfrak{R}}(\eta(\mathfrak{I}))$. Hence, $\Lambda_{\mathfrak{R}}(\mathfrak{I}) \sqsubseteq \Lambda_{\mathfrak{R}}(\eta(\mathfrak{I}))$.

Proposition 2.7.
If $\mathfrak{H}$ is a faithful $\mathfrak{R}$-module with R is integral domain then every submodule in $\mathfrak{H}$ has nonzero annihilator is SALS.
Proof.
Let $\mathfrak{H} = \mathfrak{I} + \aleph$, for $\mathfrak{I}, \aleph \sqsubseteq \mathfrak{H}$ and $\Lambda_{\mathfrak{R}}(\mathfrak{I}) \neq 0$. Since $\mathfrak{H}$ is faithful thus $0 = \Lambda_{\mathfrak{R}}(\mathfrak{H}) = \Lambda_{\mathfrak{R}}(\mathfrak{I} + \aleph) = \Lambda_{\mathfrak{R}}(\mathfrak{I}) \cap \Lambda_{\mathfrak{R}}(\aleph)$. Since $\mathfrak{R}$ is integral domain and $\Lambda_{\mathfrak{R}}(\mathfrak{I}) \neq 0$ thus $\Lambda_{\mathfrak{R}}(\aleph) = 0$. Thus $\mathfrak{I} \sqsubseteq^{s.a.l.s} \mathfrak{H}$.

Proposition 2.8.

If $\mathfrak{H}$ is faithful $\mathfrak{R}$-module and $\mathfrak{I}$ is LS-submodule of a $\mathfrak{H}$ then $\mathfrak{I}$ is SALS.

Proof.

Assume that $\mathfrak{I} \sqsubseteq^{l.s} \mathfrak{H}$. Now let $\mathfrak{I} + \aleph = \mathfrak{H}$ with $\aleph \sqsubseteq^{l} \mathfrak{H}$. Since $\mathfrak{I}$ is LS-submodule in $\mathfrak{H}$, thus $\aleph = \mathfrak{H}$. Hence $\Lambda_{\mathfrak{R}}(\aleph) = \Lambda_{\mathfrak{R}}(\mathfrak{H}) = 0 \sqsubseteq^{l.s} \mathfrak{R}$. Thus $\mathfrak{I} \sqsubseteq^{s.a.l.s} \mathfrak{H}$.

Proposition 2.9.

For a submodule $\mathfrak{I}$ of faithful $\mathfrak{R}$-module $\mathfrak{H}$, with $\mathbf{\Lambda}_{\mathfrak{R}}(\mathfrak{I}) \sqsubseteq^{l} \mathfrak{R}$, then $r_{\mathfrak{H}}.\mathbf{\Lambda}_{\mathfrak{R}}(\mathfrak{I}) \sqsubseteq^{s.a.l.s} \mathfrak{H}$.

Proof.

Assume $r.\mathbf{\Lambda}_{\mathfrak{R}}(\mathfrak{I}) + \aleph = \mathfrak{H}$ for $\aleph \sqsubseteq \mathfrak{H}$. So $0 = \mathbf{\Lambda}_{\mathfrak{R}}(\mathfrak{H}) = \mathbf{\Lambda}_{\mathfrak{R}}(r_{\mathfrak{H}})\mathbf{\Lambda}_{\mathfrak{R}}(\mathfrak{I}) \cap \mathbf{\Lambda}_{\mathfrak{R}}(\aleph) = \mathbf{\Lambda}_{\mathfrak{R}}(\mathfrak{I}) \cap \mathbf{\Lambda}_{\mathfrak{R}}(\aleph) \sqsubseteq^{l.s} \mathfrak{R}$, and since $\mathbf{\Lambda}_{\mathfrak{R}}(\aleph) \sqsubseteq \mathbf{\Lambda}_{\mathfrak{R}}(\mathfrak{I}) \cap \Lambda_{\mathfrak{R}}(\aleph) \sqsubseteq^{l.s} \mathfrak{R}$ thus $\Lambda_{\mathfrak{R}}(\aleph) \sqsubseteq^{l.s} \mathfrak{R}$ by lemma 2.6(2) and [8, Proposition 2.5(1(a))].

Proposition 2.10.

Let $\mathfrak{I}, \mathcal{L} \sqsubseteq \mathfrak{H}$ and $\mathfrak{H}$ is an $\mathfrak{R}$-module, then

1. If $\mathfrak{I} \sqsubseteq \mathcal{L} \sqsubseteq \mathfrak{H}$ and $\mathcal{L} \sqsubseteq^{s.a.l.s} \mathfrak{H}$, then $\mathfrak{I} \sqsubseteq^{s.a.l.s} \mathfrak{H}$.

2. If $\mathfrak{I} \sqsubseteq \mathcal{L} \sqsubseteq \mathfrak{H}$ and $\mathfrak{I} \sqsubseteq^{s.a.l.s} \mathcal{L}$, then $\mathfrak{I} \sqsubseteq^{s.a.l.s} \mathfrak{H}$.

Proof.

1. Let $\mathfrak{I} + \aleph = \mathfrak{H}$, for $\aleph \sqsubseteq \mathfrak{H}$. hence $\mathfrak{H} = \mathfrak{H} + \mathcal{L} = \mathfrak{I} + \aleph + \mathcal{L}$ since $\mathcal{L} \sqsubseteq^{s.a.l.s} \mathfrak{H}$ so by lemma 2.6(1), $\Lambda_{\mathfrak{R}}(\aleph) \sqsubseteq \Lambda_{\mathfrak{R}}(\mathfrak{I} + \aleph) \sqsubseteq^{l.s} \mathfrak{R}$ implies $\Lambda_{\mathfrak{R}}(\aleph) \sqsubseteq^{l.s} \mathfrak{R}$ by [8, Proposition 2.5(1(a))], thus $\mathfrak{I} \sqsubseteq^{s.a.l.s} \mathfrak{H}$.

2. Let $\mathfrak{I} + \aleph = \mathfrak{H}$, for $\aleph \sqsubseteq \mathfrak{H}$. Now $\mathcal{L} = \mathcal{L} \cap \mathfrak{H} = \mathcal{L} \cap (\mathfrak{I} + \aleph) = \mathfrak{I} + (\mathcal{L} \cap \aleph)$ by modular low. Since $\mathfrak{I} \sqsubseteq^{s.a.l.s} \mathcal{L}$ thus $\Lambda_{\mathfrak{R}}(\mathcal{L} \cap \aleph) \sqsubseteq^{l.s} \mathfrak{R}$. Now, by Lemma 2.6(2) $\Lambda_{\mathfrak{R}}(\aleph) \sqsubseteq \Lambda_{\mathfrak{R}}(\mathcal{L} \cap \aleph)$ thus $\Lambda_{\mathfrak{R}}(\aleph) \sqsubseteq^{l.s} \mathfrak{R}$ by [8, Proposition 2.5(1(a))], so $\mathfrak{I} \sqsubseteq^{s.a.l.s} \mathfrak{H}$.

3. Main Results

In this section we will introduce many interesting results and discusses some relations between our concept and other concepts related with modules.

Recall [9] for an $\mathfrak{R}$-module we have $Z_s(\mathfrak{H}) = \{m \in \mathfrak{H}; \Lambda_s(m) = \Lambda_s(m\mathfrak{R}) \sqsubseteq^{l} S\}$.

Proposition 3.1.

Let $\mathfrak{I}$ is SALS submodule in finitely generated $\mathfrak{R}$-module $\mathfrak{H}$, then $\mathfrak{I} + GJ(\mathfrak{H}) + Z_s(\mathfrak{H})$ is also SALS of $\mathfrak{H}$.

Proof.

Let $(\mathfrak{I} + GJ(\mathfrak{H}) + Z_s(\mathfrak{H})) + \aleph = \mathfrak{H}$ where $\aleph \sqsubseteq \mathfrak{H}$. Since $\mathfrak{H}$ is finitely generated thus by [10, Lemma 5.4] $GJ(\mathfrak{H}) \sqsubseteq^{l.s} \mathfrak{H}$ thus $\mathfrak{I} + Z_s(\mathfrak{H}) + \aleph = \mathfrak{H}$. Suppose that $\mathfrak{H}_{\mathfrak{R}} = \sum_{i=1}^{n} \propto_i \mathfrak{R}$. If $\propto_i = h_i + z_i + a_i$ where $h_i \in \aleph, z_i \in Z_s(\mathfrak{H})$ and $a_i \in \mathfrak{I}$. Thus $\mathfrak{I} + \sum_{i=1}^{n} z_i\mathfrak{R} + \aleph = \mathfrak{H}$. Since $\mathfrak{I} \sqsubseteq^{s.a.l.s} \mathfrak{H}$ thus $\Lambda_{\mathfrak{R}}(\aleph) \sqsubseteq \cap_{i=1}^{n} \Lambda_{\mathfrak{R}}(z_i\mathfrak{R}) \cap \Lambda_{\mathfrak{R}}(\aleph) = \Lambda_{\mathfrak{R}}(\sum_{i=1}^{n} z_i\mathfrak{R} + \aleph) \sqsubseteq^{l.s} \mathfrak{R}$. thus Therefore, $\Lambda_{\mathfrak{R}}(\aleph) \sqsubseteq^{l.s} \mathfrak{R}$ by lemma (2.6)(1) and [8, Proposition 2.5(1(a))]. Thus $\mathfrak{I} + GJ(\mathfrak{H}) + Z_s(\mathfrak{H})$ is also SALS of $\mathfrak{H}$.

Proposition 3.2.

Let K, L are two $\mathfrak{R}$-modules such that $\eta: K \longrightarrow L$ is epimorphism. If $\mathfrak{I}$ is SALS of L, then $\eta^{-1}(\mathfrak{I})$ is SALS of K.

Proof.

Let $K = \eta^{-1}(\mathfrak{I}) + \aleph$ for $\aleph \sqsubseteq K$. So $\eta(K) = \eta(\eta^{-1}(\mathfrak{I})) + \eta(\aleph)$. Since η is epimorphism. Thus $L = \mathfrak{I} + \eta(\aleph)$. Since $\mathfrak{I}$ is SALSs of L, so $\Lambda_{\mathfrak{R}}(\eta(\aleph)) \sqsubseteq^{l.s} \mathfrak{R}$. $\Lambda_{\mathfrak{R}}(\aleph) \sqsubseteq \Lambda_{\mathfrak{R}}(\eta(\aleph))$, from lemma 2.6(3). Thus $\Lambda_{\mathfrak{R}}(\aleph) \sqsubseteq^{l.s} \mathfrak{R}$. Therefore, $\eta^{-1}(\mathfrak{I})$ is SALS in K.

Proposition 3.3.

Let $\mathfrak{I} \sqsubseteq \mathcal{L} \sqsubseteq \mathfrak{H}$. If $\mathcal{L}/_{\mathfrak{I}} \sqsubseteq^{s.a.l.s} \mathfrak{H}/_{\mathfrak{I}}$ then $\mathcal{L} \sqsubseteq^{s.a.l.s} \mathfrak{H}$.

Proof.

Let $\eta: \mathfrak{H} \longrightarrow \mathfrak{H}/_{\mathfrak{I}}$ be natural epimorphism and $\mathcal{L}/_{\mathfrak{I}} \sqsubseteq \mathfrak{H}/_{\mathfrak{I}}$. From Proposition 3.2 we get that $\eta^{-1}(\mathcal{L}/_{\mathfrak{I}}) \sqsubseteq^{s.a.l.s} \mathfrak{H}$. Since $\eta^{-1}\left(\mathcal{L}/_{\mathfrak{I}}\right) = \mathcal{L}$, hence $\mathcal{L} \sqsubseteq^{s.a.l.s} \mathfrak{H}$.

Proposition 3.4.

Let $\mathfrak{H}$ be an $\mathfrak{R}$-module with $\mathfrak{I} \sqsubseteq \mathcal{L} \sqsubseteq C \sqsubseteq \mathfrak{H}$. If $C/_{\mathcal{L}} \sqsubseteq^{s.a.l.s} \mathfrak{H}/_{\mathcal{L}}$ then $C/_{\mathfrak{I}} \sqsubseteq^{s.a.l.s} \mathfrak{H}/_{\mathfrak{I}}$.

Proof.

Let $\eta: \mathfrak{H}/_{\mathfrak{I}} \longrightarrow \mathfrak{H}/_{\mathcal{L}}$ be defined as $\eta(v + \mathfrak{I}) = v + \mathcal{L}, v \in \mathfrak{H}$. It's obvious that η is epimorphism. Since $C/_{\mathcal{L}} \sqsubseteq \mathfrak{H}/_{\mathcal{L}}$ thus by proposition 3.2, $C/_{\mathfrak{I}} = \eta^{-1}(C/_{\mathcal{L}}) \sqsubseteq^{s.a.l.s} \mathfrak{H}/_{\mathfrak{I}}$.

If K, L are two $\mathfrak{R}$-modules such that $\eta: K \longrightarrow L$ is epimorphism and $\mathfrak{I}$ is SALS of K, then $\eta(\mathfrak{I})$ is not need to be SALS of L. As we will show that in the following instance.

Example 3.5.

If we have $\mathbb{Z}, \mathbb{Z}_6$ as $\mathbb{Z}$-modules and $\eta: \mathbb{Z} \longrightarrow \mathbb{Z}_6$ be natural epimorphism map; we see that $(0_{\mathbb{Z}})$ is SALS in $\mathbb{Z}$, but $\eta((0_{\mathbb{Z}})) = 0_{\mathbb{Z}_6}$ is not SALS in $\mathbb{Z}_6$ since $\mathbb{Z}_6 = 0 + \mathbb{Z}_6$ and $\Lambda_{\mathbb{Z}}(\mathbb{Z}_6) = 6\mathbb{Z}$ which is not LS-submodule in $\mathbb{Z}$.

The sum of any two SALSs need not be SALS as we will show that in the following instance.

Example 3.6.

In $\mathbb{Z}$ as $\mathbb{Z}$-module we have both $2\mathbb{Z}$ and $3\mathbb{Z}$ are SALS in $\mathbb{Z}$ since $2\mathbb{Z}+3\mathbb{Z}=\mathbb{Z}$ and $\Lambda_{\mathbb{Z}}(2\mathbb{Z})=\Lambda_{\mathbb{Z}}(3\mathbb{Z})=0\sqsubseteq^{l.s}\mathbb{Z}$ but $\mathbb{Z}=\mathbb{Z}+0$ and $\Lambda_{\mathbb{Z}}(0)=\mathbb{Z}$ is not LS-submodule of $\mathbb{Z}$. Thus, $\mathbb{Z}$ is not SALS in $\mathbb{Z}$.

Proposition 3.7.

Let $\mathfrak{H}$ and $\mathcal{H}$ be two $\mathfrak{R}$-modules, if $\mathfrak{I}$ is SALS in $\mathfrak{H}$ and $\mathcal{L}$ is SALS in $\mathcal{H}$. Then $\mathfrak{I}\oplus\mathcal{L}$ is SALS in $\mathfrak{H}\oplus\mathcal{H}$.

Proof.

Suppose that $\eta:\mathfrak{H}\oplus\mathcal{H}\longrightarrow\mathfrak{H}$ is projection map. Since $\mathfrak{I}\sqsubseteq^{s.a.l.s}\mathfrak{H}$ and $\mathcal{L}\sqsubseteq^{s.a.l.s}\mathcal{H}$ thus by proposition 3.2 we get $\mathfrak{I}\oplus\mathcal{H}=\eta^{-1}(\mathfrak{I})\sqsubseteq^{s.a.l.s}\mathfrak{H}\oplus\mathcal{H}$ also $\mathfrak{H}\oplus\mathcal{L}=\eta^{-1}(\mathcal{L})\sqsubseteq^{s.a.l.s}\mathfrak{H}\oplus\mathcal{H}$. Hence $(\mathfrak{I}\oplus\mathcal{H})\cap(\mathfrak{H}\oplus\mathcal{L})=(\mathfrak{I}\oplus\mathcal{L})\sqsubseteq^{s.a.l.s}\mathfrak{H}\oplus\mathcal{H}$.

Proposition 3.8.

Assume $\mathfrak{H}$ is an $\mathfrak{R}$-module with $\mathfrak{I}_1\sqsubseteq\mathcal{L}_1\sqsubseteq\mathfrak{H}$ and $\mathfrak{I}_2\sqsubseteq\mathcal{L}_2\sqsubseteq\mathfrak{H}$. If $(\mathcal{L}_1+\mathcal{L}_2)/(\mathfrak{I}_1+\mathfrak{I}_2)\sqsubseteq^{s.a.l.s}\mathfrak{H}/(\mathfrak{I}_1+\mathfrak{I}_2)$ then;

1. $(\mathcal{L}_1+\mathfrak{I}_2)/\mathfrak{I}_1\sqsubseteq^{s.a.l.s}\mathfrak{H}/\mathfrak{I}_1$.

2. $(\mathfrak{I}_1+\mathcal{L}_2)/\mathfrak{I}_2\sqsubseteq^{s.a.l.s}\mathfrak{H}/\mathfrak{I}_2$.

3. $\mathcal{L}_1/\mathfrak{I}_1\oplus\mathcal{L}_2/\mathfrak{I}_2\sqsubseteq^{s.a.l.s}\mathfrak{H}/\mathfrak{I}_1\oplus\mathfrak{H}/\mathfrak{I}_2$.

Proof.

1. Suppose that $\eta_1:\mathfrak{H}/\mathfrak{I}_1\longrightarrow\mathfrak{H}/(\mathfrak{I}_1+\mathfrak{I}_2)$ defined as $\eta_1(m_1+\mathfrak{I}_1)=m_1+\mathfrak{I}_1+\mathfrak{I}_2$ where $m_1\in\mathfrak{H}$ and by the same way we assume $\eta_2:\mathfrak{H}/\mathfrak{I}_2\longrightarrow\mathfrak{H}/(\mathfrak{I}_1+\mathfrak{I}_2)$ defined as $\eta_2(m_2+\mathfrak{I}_2)=m_2+\mathfrak{I}_1+\mathfrak{I}_2$ where $m_2\in\mathfrak{H}$. It's clear that η_1,η_2 are epimorphism. Since $(\mathcal{L}_1+\mathfrak{I}_2)/(\mathfrak{I}_1+\mathfrak{I}_2)\sqsubseteq(\mathcal{L}_1+\mathcal{L}_2)/(\mathfrak{I}_1+\mathfrak{I}_2)$ and $(\mathcal{L}_1+\mathcal{L}_2)/(\mathfrak{I}_1+\mathfrak{I}_2)\sqsubseteq^{s.a.l.s}\mathfrak{H}/(\mathfrak{I}_1+\mathfrak{I}_2)$. From proposition 3.4 we conclude $(\mathcal{L}_1+\mathfrak{I}_2)/(\mathfrak{I}_1+\mathfrak{I}_2)\sqsubseteq^{s.a.l.s}\mathfrak{H}/(\mathfrak{I}_1+\mathfrak{I}_2)$. By the same way we get $(\mathcal{L}_1+\mathfrak{I}_2)/\mathfrak{I}_1={\eta_2}^{-1}(\mathcal{L}_1+\mathfrak{I}_2/\mathfrak{I}_1+\mathfrak{I}_2)\sqsubseteq^{s.a.l.s}\mathfrak{H}/\mathfrak{I}_1$.

2. Now since $(\mathfrak{I}_1+\mathcal{L}_2)/(\mathfrak{I}_1+\mathfrak{I}_2)\sqsubseteq(\mathcal{L}_1+\mathcal{L}_2)/(\mathfrak{I}_1+\mathfrak{I}_2)$ and $(\mathcal{L}_1+\mathcal{L}_2)/(\mathfrak{I}_1+\mathfrak{I}_2)\sqsubseteq^{s.a.l.s}\mathfrak{H}/(\mathfrak{I}_1+\mathfrak{I}_2)$. Then by proposition 3.4 we have $(\mathfrak{I}_1+\mathcal{L}_2)/(\mathfrak{I}_1+\mathfrak{I}_2)\sqsubseteq^{s.a.l.s}\mathfrak{H}/(\mathfrak{I}_1+\mathfrak{I}_2)$. Thus by proposition 3.2 $(\mathfrak{I}_1+\mathcal{L}_2)/\mathfrak{I}_2={\eta_2}^{-1}(\mathfrak{I}_1+\mathcal{L}_2/\mathfrak{I}_1+\mathfrak{I}_2)\sqsubseteq^{s.a.l.s}\mathfrak{H}/\mathfrak{I}_2$.

3. Its comes directly from (1) and (2) and proposition 3.7.

Proposition 3.9.

Take $\mathfrak{H}=\sum_{i=1}^{k}\mathfrak{R}c_i$ be an $\mathfrak{R}$-module and $w\in\mathfrak{H}$, if

1. $\mathfrak{R}w$ is SALS of $\mathfrak{H}$.

2. $\cap\, i\in I\Lambda_{\mathfrak{R}}(c_i-d_iw)\sqsubseteq^{l.s}\mathfrak{R}$ for $d_i\in\mathfrak{R}$.

3. There exists $i\in I$ such that $dc_i\notin\mathfrak{R}dw$, for all $0\neq d\notin GJ(\mathfrak{R})$.

Then **(1)⟺(2)⟹(3)** and if $GJ(\mathfrak{R})=0$ then **(3)⟹(2).**

Proof.

(1)⟹(2) for $i\in I, c_i=c_i-d_iw+d_iw$ and hence $\mathfrak{H}=\sum\mathfrak{R}(c_i-d_iw)+\mathfrak{R}w$. By (1) we have $\mathfrak{R}w\sqsubseteq^{s.a.l.s}\mathfrak{H}$ thus $\Lambda_{\mathfrak{R}}(\sum\mathfrak{R}(c_i-d_iw))=\cap\, i\Lambda_{\mathfrak{R}}(c_i-d_iw)\sqsubseteq^{l.s}\mathfrak{R}$.

(2)⟹(1) Let $\aleph\sqsubseteq\mathfrak{H}$, with $\aleph+\mathfrak{R}w=\mathfrak{H}$. Then for each $i\in I, c_i=h_i+d_iw$ and $h_i\in\aleph$. Assume $k\in\Lambda_{\mathfrak{R}}(\aleph)$, then $kc_i=kd_iw+kh_i$ since $kh_i=0$, thus $k(c_i-d_iw)=0$, for each $i\in I$, then $k\in\Lambda_{\mathfrak{R}}(c_i-d_iw)$, this imply $\Lambda_{\mathfrak{R}}(\aleph)\sqsubseteq\Lambda_{\mathfrak{R}}(c_i-d_iw)$, hence $\Lambda_{\mathfrak{R}}(\aleph)\sqsubseteq\Lambda_{\mathfrak{R}}(c_i-d_iw)=\cap\, i\in I\Lambda_{\mathfrak{R}}(c_i-d_iw)\sqsubseteq^{l.s}\mathfrak{R}$. Therefore, $\Lambda_{\mathfrak{R}}(\aleph)\sqsubseteq^{l.s}\mathfrak{R}$ by [8, proposition 2.5(1(a))]. Implies $\mathfrak{R}w\sqsubseteq^{s.a.l.s}\mathfrak{H}$.

(2)⟹(3) Assume that $d\notin GJ(\mathfrak{R})$ and let $dc_i\in\mathfrak{R}dw$ for all $i\in I$, thus $dc_i=d_idw=dd_iw$ for all $i\in I$, thus $dc_i-d_idw=0$, hence $d(c_i-d_iw)=0$ which means $d\in\Lambda_{\mathfrak{R}}(c_i-d_iw)$. By (2) $d\in\cap\, i\in I\Lambda_{\mathfrak{R}}(c_i-d_iw)\sqsubseteq^{l.s}\mathfrak{R}$. Thus $d\in GJ(\mathfrak{R})$ which is a contradiction.

(3)⟹(2) Assume that $d\in\cap\, i\in I\Lambda_{\mathfrak{R}}(c_i-d_iw)$ and hence $d\in\Lambda_{\mathfrak{R}}(c_i-d_iw)$ for all $i\in I$. Thus $dc_i=d_idw=dd_iw$ for all $i\in I$, so $dc_i\in\mathfrak{R}dw$. By (3) $d\in GJ(\mathfrak{R})$ then $\cap\, i\in I\Lambda_{\mathfrak{R}}(c_i-d_iw)\sqsubseteq GJ(\mathfrak{R})=0$ thus $\cap\, i\in I\Lambda_{\mathfrak{R}}(c_i-d_iw)\sqsubseteq^{l.s}\mathfrak{R}$.

Proposition 3.10.

Let $\mathfrak{H}=\sum_{i\in I}\mathfrak{R}n_i$ be an $\mathfrak{R}$-module where R is commutative ring, and $\aleph\sqsubseteq\mathfrak{H}$. Then the following are equivalent

1. $\aleph$ is SALS of $\mathfrak{H}$.

2. $\cap\, i \in I \Lambda_{\mathfrak{R}}(n_i - h_i) \sqsubseteq^{l.s} \mathfrak{R}$ for each $h_i \in \aleph$.

Proof.

(1)⟹(2) Let $h_i \in \aleph$, for all $i \in I$. Thus $n_i = n_i - h_i + h_i$ for each $i \in I$. Thus $\mathfrak{H} = \sum \mathfrak{R}(n_i - h_i) + \aleph$, by (1) $\aleph$ is SALS of $\mathfrak{H}$. Thus $\Lambda_{\mathfrak{R}}(\sum \mathfrak{R}(n_i - r_i)) = \cap\, i \in I \Lambda_{\mathfrak{R}}(\mathfrak{R}(n_i - r_i)) \sqsubseteq^{l.s} \mathfrak{R}$.

(2) ⟹ (1) Assume $\mathfrak{H} = \mathcal{L} + \aleph$. Then $n_i = b_i + h_i$ for all $i \in I, n_i \in \mathfrak{H}, b_i \in \mathcal{L}, h_i \in \aleph$. Hence $b_i = n_i - h_i$. So, $\mathfrak{H} = \sum_{i \in I}(n_i - h_i) + \aleph$. Assume that $s \in \Lambda_{\mathfrak{R}}(\mathcal{L})$ thus $0 = sb_i = s(n_i - h_i)$ for $i \in I$. Thus $s \in \Lambda_{\mathfrak{R}}(\mathfrak{R}(n_i - h_i))$. By (2), $s \in \Lambda_{\mathfrak{R}}(\mathfrak{R}(n_i - h_i)) \sqsubseteq^{l.s} \mathfrak{R}$. This imply $\Lambda_{\mathfrak{R}}(\mathcal{L}) \sqsubseteq \Lambda_{\mathfrak{R}}(\mathfrak{R}(n_i - h_i)) \sqsubseteq^{l.s} \mathfrak{R}$, thus by [8, proposition 2.5(1(a))] $\Lambda_{\mathfrak{R}}(\mathcal{L}) \sqsubseteq^{l.s} \mathfrak{R}$. Therefore, $\aleph \sqsubseteq^{s.a.l.s} \mathfrak{H}$.

Recall [9], an $\mathfrak{R}$-module T is referred to be faithfully flat if $\mathfrak{I} \to \mathcal{L} \to C$ is an exact sequence of $\mathfrak{R}$-modules if and only if $T \otimes \mathfrak{I} \to T \otimes \mathcal{L} \to T \otimes C$ is an exact sequence.

Lemma 3.11.

Let $\mathfrak{H}$ is an $\mathfrak{R}$-module and $\mathfrak{I} \sqsubseteq \mathfrak{H}$, and K is any faithfully flat $\mathfrak{R}$-module then $\Lambda_{\mathfrak{R}}(\mathfrak{I}) = \Lambda_{\mathfrak{R}}(K \otimes \mathfrak{I})$.

Proof.

Let $r \in \Lambda_{\mathfrak{R}}(\mathfrak{I})$, thus $r.\mathfrak{I} = 0$, $r(K \otimes \mathfrak{I}) = K \otimes r\mathfrak{I} = K \otimes 0 = 0$, thus $r \in \Lambda_{\mathfrak{R}}(K \otimes \mathfrak{I})$, imply $\Lambda_{\mathfrak{R}}(\mathfrak{I}) \subseteq \Lambda_{\mathfrak{R}}(K \otimes \mathfrak{I})$. Now, let $r \in \Lambda_{\mathfrak{R}}(K \otimes \mathfrak{I})$, thus $r.(K \otimes \mathfrak{I}) = K \otimes r\mathfrak{I} = 0$. Thus $0 \longrightarrow K \otimes r\mathfrak{I} \longrightarrow 0$ is exact sequence. Since K is faithfully flat thus $0 \longrightarrow r\mathfrak{I} \longrightarrow 0$ is exact sequence. Thus $r\mathfrak{I} = 0$, which means $r \in \Lambda_{\mathfrak{R}}(\mathfrak{I})$. imply $\Lambda_{\mathfrak{R}}(K \otimes \mathfrak{I}) \subseteq \Lambda_{\mathfrak{R}}(\mathfrak{I})$. Therefore, $\Lambda_{\mathfrak{R}}(\mathfrak{I}) = \Lambda_{\mathfrak{R}}(K \otimes \mathfrak{I})$.

Proposition 3.12.

Let $\mathfrak{H}$ is an $\mathfrak{R}$-module and $\mathfrak{I} \sqsubseteq \mathfrak{H}$. If T is a faithfully flat $\mathfrak{R}$-module. Then $\mathfrak{I} \sqsubseteq^{s.a.l.s} \mathfrak{H}$ if and only if $T \otimes \mathfrak{I} \sqsubseteq^{s.a.l.s} T \otimes \mathfrak{H}$.

Proof.

⟹) Let $\mathfrak{I} \sqsubseteq^{s.a.l.s} \mathfrak{H}$ and $T \otimes \mathfrak{I} + T \otimes \mathcal{L} = T \otimes \mathfrak{H}$, for $T \otimes \mathcal{L} \sqsubseteq T \otimes \mathfrak{H}$. So $T \otimes (\mathfrak{I} + \mathcal{L}) = T \otimes \mathfrak{I} + T \otimes \mathcal{L} = T \otimes \mathfrak{H}$. Hence, $0 \longrightarrow T \otimes (\mathfrak{I} + \mathcal{L}) \longrightarrow T \otimes \mathfrak{H} \longrightarrow 0$ is exact sequence. Thus $0 \longrightarrow \mathfrak{I} + \mathcal{L} \longrightarrow \mathfrak{H} \longrightarrow 0$ is exact sequence also since T is a faithfully flat. This imply $\mathfrak{I} + \mathcal{L} = \mathfrak{H}$. Hence $\Lambda_{\mathfrak{R}}(\mathcal{L}) \sqsubseteq^{l.s} \mathfrak{R}$, since $\mathfrak{I} \sqsubseteq^{s.a.l.s} \mathfrak{H}$. By lemma 3.11, $\Lambda_{\mathfrak{R}}(\mathcal{L}) = \Lambda_{\mathfrak{R}}(T \otimes \mathcal{L})$. Thus $\Lambda_{\mathfrak{R}}(T \otimes \mathcal{L}) \sqsubseteq^{l.s} \mathfrak{R}$, then $T \otimes \mathfrak{I} \sqsubseteq^{s.a.l.s} T \otimes \mathfrak{H}$.

⟸) Let $T \otimes \mathfrak{I} \sqsubseteq^{s.a.l.s} T \otimes \mathfrak{H}$, and let $\mathfrak{I} + \mathcal{L} = \mathfrak{H}$, for $\mathcal{L} \sqsubseteq \mathfrak{H}$. Then $T \otimes \mathfrak{H} = T \otimes (\mathfrak{I} + \mathcal{L}) = (T \otimes \mathfrak{I}) + (T \otimes \mathcal{L})$, so $\Lambda_{\mathfrak{R}}(T \otimes \mathcal{L}) = \Lambda_{\mathfrak{R}}(\mathcal{L}) \sqsubseteq^{l.s} \mathfrak{R}$, since $T \otimes \mathfrak{I} \sqsubseteq^{s.a.l.s} T \otimes \mathfrak{H}$. Therefore, $\mathfrak{I} \sqsubseteq^{s.a.l.s} \mathfrak{H}$.

4. Conclusions

Throughout this paper we presented the concept of semi-annihilator large small submodules as a generalization of the concept of semi-annihilator small submodules. We also discussed the behavior of these submodules under the influence of this concept. Also presented a range of characteristics and we obtained many interesting results.

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