



The Core of Inner Ideals of Four Dimensional Real Lie Algebra With Tow Dimensional Derived

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ABSTRACT

Suppose that L is a 4-dimensional Lie algebra with a 2-dimensional derived L' , and E an arbitrary non-trivial inner ideal of L . The core of E is an inner ideal of L . This paper presents a proof. If L is a 4-dimensional Lie algebra with a 2-dimensional derived algebra. Then the core of each inner ideal E of L is zero.

MSC..

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1. Introduction

In 1976, an American scientist named Benkart was the first to identify the concept of the inner ideal (or I -ideal) (see [1]). It is defined as a subspace E of L such that $[E, [E, L]] \subseteq E$, E is called a commutative I -ideal if $[E, E] = 0$. A strong relationship between ad-nilpotent elements in Lie algebras and I -ideals was proven by her [2].

Benkart and Fernandiz Lopes accomplished Benkart's results in [3]. Brox et al (2016) generalized their results to the situation of prime rings that are centrally closed with not 2, 3, or 5 invertible [4].

Inner ideals and one-sided ideals serve analogous functions in both Lie and associative algebras, as evidenced by references [5], [6]. Consequently, by examining the inner ideals of Lie algebras, Artin's theory can be extended to Lie algebras.

[7] An Artinian Lie algebra is a Lie algebra in which every descending chain of I -ideals must eventually stabilize. In [8] demonstrates that every one-sided ideal of the finite dimensional associative algebra A possesses a Levi decomposition and can be created by an idempotent, provided certain minimal requirements are satisfied.

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Baranov and Shlaka demonstrated in [9] that identical results apply to inner ideals, establishing that each inner ideal of the Lie algebra $[A, A]$ permits Levi decomposition and may be created by an idempotent pair, provided certain basic requirements are met. In 2022, Shlaka and Kareem study abelian I -ideals within the Lie subalgebras of finite dimensional associative algebras [10].

Schöbel divided the 4-dimensional (4-dim) Lie algebras based on their derived structure in [11]. A derived subalgebra of L is defined as

$$L' = [L, L] = \text{Span} \{[\ell_1, \ell_2] \mid \ell_1, \ell_2 \in L\}$$

After Schöbel's findings, H. Shlaka and H. Saeed analyzed the inner ideals of four dimensional Lie algebras possessing derived subalgebras of dimensions one and two [12].

In 2021, Baranov and Shlaka defined the core of the inner ideal [9]. Shlaka and Hmood identified the core of the inner ideal of 4-dimensional Lie algebras, characterized by a derived subalgebra of dimension one [13]. They found that the core is zero for every I -ideal of L .

In this paper, we study the core of I -ideals of the 4-dimensional Lie algebras with 2-dimensional derived.

2. Preliminaries

Throughout this paper, unless otherwise stated, L is a Lie algebra over \mathbb{R} with a basis $\{s_1, s_2, s_3, s_4\}$, $C(L)$ is the center of L , Let E be an inner ideal of L . Then $[E, [E, L]] \subseteq E$. $E_1 = [E, [E, L]]$ and $E_2 = [E_1, [E_1, L]]$. It is well know that E_1 is an inner ideal of L (see [2, Lemma 1.1]). Put $E_0 = E$ and consider the following inner ideals of L :

$$E_n = [E_{n-1}, [E_{n-1}, L]] \subseteq E_{n-1} \text{ for } n \geq 1.$$

Then $E = E_0 \supseteq E_1 \supseteq E_2 \supseteq \dots$. As L is finite dimensional, this series erminates, so there is integer n such that $E_n = E_{n+1}$.

Definition 2.1. [9] Assume that E is an inner ideal of L and L as a finite dimensional Lie algebra. Then there is an integer n such that $E_n = E_{n+1}$, so E_n is called the core of E , denoted by $cor_L(E)$.

Schöbel in [11], provides a classification of the real 4-dim Lie algebras depending on the dimension of their derived subalgebras. He classified them in the case when the dimensions of L' are 1, 2, and 3. Therefore, to study the core of inner ideal of 4-dim real Lie algebras with 2-dim derived, one we needs to mention the following theorems in [11].

Case 1: when $L' \not\subseteq C(L)$.

Theorem 2.2. [11] Assume that L is a 4-dim Lie algebra with 2-dim derived L' such that $L' \not\subseteq C(L)$ and let $\{s_1, s_2, s_3, s_4\}$ be a basis of L , depending on the multiplication L may be one of:

$$L_1 : [s_1, s_4] = s_1, [s_2, s_4] = s_2 \text{ and } 0 \text{ otherwise.}$$

$L_2 : [s_1, s_4] = -s_1 + ps_2, [s_2, s_4] = s_1$ and 0 otherwise.

$L_3 : [s_1, s_3] = s_1, [s_2, s_4] = s_2$ and 0 otherwise.

$L_4 : [s_1, s_3] = -s_2, [s_2, s_3] = s_1, [s_1, s_4] = s_1, [s_2, s_4] = s_2$ and 0 otherwise.

$L_5 : [s_1, s_4] = s_1, [s_2, s_3] = s_1, [s_2, s_4] = s_2$ and 0 otherwise.

$L_\varepsilon : [s_1, s_4] = \varepsilon s_2, [s_2, s_4] = s_1$ and 0 otherwise. $\varepsilon = \mp$

Remark 2.3. According to [12] if L is a 4-dim Lie algebra with 2-dim derived L' such that $L' \not\subseteq C(L)$, then the only inner ideals of L are

1- Spans of $\{s_1\}, \{s_2\}, \{s_3\}, \{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_2, s_3\}, \{s_1, s_2, s_4\}$ when $L = L_1, L_2, \text{ or } L_\varepsilon$.

2- Spans of $\{s_1\}, \{s_2\}, \{s_3\}, \{s_1, s_2\}, \{s_1, s_3\}, \{s_1, s_2, s_3\}, \{s_1, s_2, s_4\}$ when $L = L_5$.

3- Spans of $\{s_1\}, \{s_2\}, \{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_4\}, \{s_1, s_2, s_3\}, \{s_1, s_2, s_4\}$ when $L = L_3$.

4- Spans of $\{s_1\}, \{s_2\}, \{s_1, s_2\}, \{s_1, s_2, s_3\}, \{s_1, s_2, s_4\}$ when $L = L_4$.

Case 2: when $L' \subseteq C(L)$.

Theorem 2.4. [11] Assume that L is a 4-dim Lie algebra with 2-dim derived L' such that $L' \subseteq C(L)$ and let $\{s_1, s_2, s_3, s_4\}$ be a basis of L , depending on the multiplication L may be one of:

$L_6 : [s_2, s_4] = s_1, [s_3, s_4] = s_2$ and 0 otherwise

$L_7 : [s_2, s_4] = s_2, [s_3, s_4] = s_1$ and 0 otherwise

Remark 2.5. According to [12] if L is a 4-dim Lie algebra with 2-dim derived L' such that $L' \subseteq C(L)$, then the only inner ideals of L are

1-Spans of $\{s_1\}, \{s_2\}, \{s_3\}, \{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_4\}, \{s_1, s_2, s_3\}, \{s_1, s_2, s_4\}, \{s_1, s_3, s_4\}$ when $L = L_6$.

2-Spans of $\{s_1\}, \{s_2\}, \{s_3\}, \{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_2, s_4\}, \{s_1, s_2, s_3\}, \{s_1, s_2, s_4\}, \{s_2, s_3, s_4\}$ when $L = L_7$.

3. The Core of I -Ideals of 4-Dim Real Lie Algebra With 2-Dim Derived.

In this section, we demonstrate several conclusions pertaining to the core of I -ideals of the 4-dimensional Lie algebra L , which has a 2-dimensional derived algebra L' . Schöbel asserts that such a Lie algebra occurs in certain instances (see to theorem 2.2); hence, we shall compute the core of I -ideals corresponding to each Lie algebra accordingly.

Proposition 3.1. Assume that L is a Lie algebra and E is inner ideal of L . If $\exists n \in \mathbb{Z}$ such that $E_n = 0$, then $cor_L(E) = 0$.

Proof. Prove by induction, let $E_n = 0$ for some n .

i) Prove that when $n = 1$, let $E_1 = 0$,

$E_2 = [E_1, [E_1, L]] = [0, [0, L]] = 0$, since $E_1 = E_2$ then by definition 2.1 $cor_L(E) = E_1 = 0$.

ii) When $n = k$, such that $E_k = 0$ suppose that $cor_L(E) = 0$.

iii) Prove that when $n = k + 1$, let $E_{k+1} = 0$

$E_{k+2} = [E_{k+1}, [E_{k+1}, L]] = [0, [0, L]] = 0$

since $E_{k+1} = E_{k+2}$ then by definition 2.1 $cor_L(E) = E_{k+1} = 0$.

Lemma 3.2. Let $L = L_1$,

- 1- If E is a 1-dim I -ideal then $cor_L(E) = 0$.
- 2- If E is a 2-dim I -ideal then $cor_L(E) = 0$.
- 3- If E is a 3-dim I -ideal then $cor_L(E) = 0$.

Proof. By theorem 2.2 if $L = L_1$ and $\{s_1, s_2, s_3, s_4\}$ is a basis of L , then the multiplication of L is as follows

(3.1) $[s_1, s_4] = s_1, [s_2, s_4] = s_2$ and 0 otherwise.

let $\ell \in L$ then $\ell = \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4$ for some $\lambda_i \in \mathbb{R}$

1- By remark 2.3 the only 1-dim inner ideals of L_1 are Spans of $\{s_1\}, \{s_2\}, \{s_3\}$.

Let $E = Span\{s_1\}$ we need to show $cor_L(E) = 0$. Let $x \in E_1$ and $a, b \in E$, then $a = \alpha s_1, b = \beta s_1$ for some $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned} x &= [a, [b, \ell]] = [a, [\beta s_1, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] = [a, \beta \lambda_4 [s_1, s_4]] \\ &= [\alpha s_1, \beta \lambda_4 s_1] = \alpha \beta \lambda_4 [s_1, s_1] = 0, \end{aligned}$$

since $E_1 = 0$, by proposition 3.1 $cor_L(E) = 0$.

In the same way $cor_L(Span\{s_2\}) = 0$ and $cor_L(Span\{s_3\}) = 0$.

2- By remark 2.3 the only 2-dim inner ideals of L_1 are Spans of $\{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}$.

Let $E = Span\{s_1, s_3\}$. we need to show $cor_L(E) = 0$. Let $x \in E_1$ and $a, b \in E$, then $a = \alpha_1 s_1 + \alpha_2 s_3, b = \beta_1 s_1 + \beta_2 s_3$ for some $\alpha_i, \beta_i \in \mathbb{R}$, then

$$\begin{aligned} x &= [a, [b, \ell]] = [a, [\beta_1 s_1 + \beta_2 s_3, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [a, \beta_1 \lambda_4 [s_1, s_4]] = [\alpha_1 s_1 + \alpha_2 s_3, \beta_1 \lambda_4 s_1] = 0, \end{aligned}$$

since $E_1 = 0$, by proposition 3.1 $cor_L(E) = 0$.

In the same way $cor_L(Span\{s_1, s_2\}) = 0$ and $cor_L(Span\{s_2, s_3\}) = 0$.

3- By remark 2.3 the only 3-dim inner ideals of L_1 are Spans of $\{s_1, s_2, s_3\}, \{s_1, s_2, s_4\}$.

Let $E = Span\{s_1, s_2, s_4\}$ we need to show $cor_L(E) = 0$, if $x \in E_1$ then exist $a, b \in E$ such that $a = \alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_4, b = \beta_1 s_1 + \beta_2 s_2 + \beta_3 s_4$ for some $\alpha_i, \beta_i \in \mathbb{R}$, then

$$\begin{aligned} x &= [a, [b, \ell]] = [a, [\beta_1 s_1 + \beta_2 s_2 + \beta_3 s_4, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [a, (\beta_1 \lambda_4 - \beta_3 \lambda_1) [s_1, s_4] + (\beta_2 \lambda_4 - \beta_3 \lambda_2) [s_2, s_4]] \\ &= [\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_4, (\beta_1 \lambda_4 - \beta_3 \lambda_1) s_1 + (\beta_2 \lambda_4 - \beta_3 \lambda_2) s_2] \\ &= -\alpha_3 (\beta_1 \lambda_4 - \beta_3 \lambda_1) [s_1, s_4] - \alpha_3 (\beta_2 \lambda_4 - \beta_3 \lambda_2) [s_2, s_4] \\ &= -\alpha_3 (\beta_1 \lambda_4 - \beta_3 \lambda_1) s_1 - \alpha_3 (\beta_2 \lambda_4 - \beta_3 \lambda_2) s_2 \in Span\{s_1, s_2\} \end{aligned}$$

Let $y \in E_2$ then exist $x_1, x_2 \in E_1$ such that $x_1 = \alpha_1 s_1 + \alpha_2 s_2, x_2 = \beta_1 s_1 + \beta_2 s_2$ then

$$\begin{aligned} y &= [x_1, [x_2, \ell]] = [x_1, [\beta_1 s_1 + \beta_2 s_2, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [x_1, \beta_1 \lambda_4 [s_1, s_4] + \beta_2 \lambda_4 [s_2, s_4]] \\ &= [\alpha_1 s_1 + \alpha_2 s_2, \beta_1 \lambda_4 s_1 + \beta_2 \lambda_4 s_2] = 0, \end{aligned}$$

since $E_2 = 0$, by proposition 3.1 $cor_L(E) = 0$. In the same way $cor_L(Span\{s_1, s_2, s_3\}) = 0$.

Lemma 3.3. If $L = L_2$,

- 1- If E is a 1-dim I -ideal then $cor_L(E) = 0$.
- 2- If E is a 2-dim I -ideal then $cor_L(E) = 0$.
- 3- If E is a 3-dim I -ideal then $cor_L(E) = 0$.

Proof. By theorem 2.2 if $L = L_2$ and $\{s_1, s_2, s_3, s_4\}$ is a basis of L , then the multiplication of L is as follows

(3.2) $[s_1, s_4] = -s_1 + ps_2, [s_2, s_4] = s_1$ and 0 otherwise.

let $\ell \in L$ then $\ell = \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4$ for some $\lambda_i \in \mathbb{R}$

1- By remark 2.3 the only 1-dim inner ideals of L_2 are Spans of $\{s_1\}, \{s_2\}, \{s_3\}$.

Let $E = Span\{s_1\}$ we need to show $cor_L(E) = 0$. Let $x \in E_1$ and $a, b \in E$, then $a = \alpha s_1, b = \beta s_1$ for some $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned} x &= [a, [b, \ell]] = [a, [\beta s_1, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] = [a, \beta \lambda_4 [s_1, s_4]] \\ &= [\alpha s_1, \beta \lambda_4 (-s_1 + ps_2)] = 0, \end{aligned}$$

In the same way $cor_L(Span\{s_2\}) = 0$ and $cor_L(Span\{s_3\}) = 0$.

2- By remark 2.3 the only 2-dim inner ideals of L_2 are Spans of $\{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}$.

Let $E = Span\{s_2, s_3\}$. we need to show $cor_L(E) = 0$. Let $x \in E_1$ and $a, b \in E$, then $a = \alpha_1 s_2 + \alpha_2 s_3, b = \beta_1 s_2 + \beta_2 s_3$ for some $\alpha_i, \beta_i \in \mathbb{R}$, then

$$x = [a, [b, \ell]] = [a, [\beta_1 s_2 + \beta_2 s_3, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]]$$

$$= [a, \beta_1 \lambda_4 [s_2, s_4]] = [\alpha_1 s_2 + \alpha_2 s_3, \beta_1 \lambda_4 s_1] = 0,$$

since $E_1 = 0$, by proposition 3.1 $cor_L(E) = 0$.

In the same way $cor_L(Span\{s_1, s_2\}) = 0$ and $cor_L(Span\{s_1, s_3\}) = 0$.

3- By remark 2.3 the only 3- dim inner ideals of L_2 are Spans of $\{s_1, s_2, s_3\}, \{s_1, s_2, s_4\}$.

Let $E = Span\{s_1, s_2, s_4\}$ we need to show $cor_L(E) = 0$, if $x \in E_1$ then exist $a, b \in E$ such that $a = \alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_4, b = \beta_1 s_1 + \beta_2 s_2 + \beta_3 s_4$ for some $\alpha_i, \beta_i \in \mathbb{R}$, then

$$x = [a, [b, \ell]] = [a, [\beta_1 s_1 + \beta_2 s_2 + \beta_3 s_4, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]]$$

$$= [a, (\beta_1 \lambda_4 - \beta_3 \lambda_1) [s_1, s_4] + (\beta_2 \lambda_4 - \beta_3 \lambda_2) [s_2, s_4]]$$

$$= [\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_4, (\beta_1 \lambda_4 - \beta_3 \lambda_1) (-s_1 + ps_2) + (\beta_2 \lambda_4 - \beta_3 \lambda_2) s_1]$$

$$= [\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_4, (-\beta_1 \lambda_4 + \beta_3 \lambda_1 + \beta_2 \lambda_4 - \beta_3 \lambda_2) s_1 + (\beta_1 \lambda_4 - \beta_3 \lambda_1) ps_2]$$

$$= -\alpha_3 (-\beta_1 \lambda_4 + \beta_3 \lambda_1 + \beta_2 \lambda_4 - \beta_3 \lambda_2) (-s_1 + ps_2) - \alpha_3 (\beta_1 \lambda_4 - \beta_3 \lambda_1) ps_1 \in Span\{s_1, s_2\}$$

Let $y \in E_2$ then exist $x_1, x_2 \in E_1$ such that $x_1 = \alpha_1 s_1 + \alpha_2 s_2, x_2 = \beta_1 s_1 + \beta_2 s_2$ then

$$y = [x_1, [x_2, \ell]] = [x_1, [\beta_1 s_1 + \beta_2 s_2, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]]$$

$$= [x_1, \beta_1 \lambda_4 [s_1, s_4] + \beta_2 \lambda_4 [s_2, s_4]]$$

$$= [\alpha_1 s_1 + \alpha_2 s_2, \beta_1 \lambda_4 (-s_1 + ps_2) + \beta_2 \lambda_4 s_1] = 0,$$

since $E_2 = 0$, by proposition 3.1 $cor_L(E) = 0$. In the same way $cor_L(Span\{s_1, s_2, s_3\}) = 0$.

Lemma 3.4. If $L = L_3$,

1- If E is a 1-dim I -ideal then $cor_L(E) = 0$.

2- If E is a 2-dim I -ideal then $cor_L(E) = 0$.

3- If E is a 3-dim I -ideal then $cor_L(E) = 0$.

Proof. By theorem 2.2 if $L = L_3$ and $\{s_1, s_2, s_3, s_4\}$ is a basis of L , then the multiplication of L is as follows

(3.3) $[s_1, s_3] = s_1, [s_2, s_4] = s_2$ and 0 otherwise

let $\ell \in L$ then $\ell = \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4$ for some $\lambda_i \in \mathbb{R}$

1- By remark 2.3 the only 1-dim inner ideals of L_3 are Spans of $\{s_1\}, \{s_2\}$.

Let $E = Span\{s_1\}$ we need to show $cor_L(E) = 0$. Let $x \in E_1$ and $a, b \in E$, then $a = \alpha s_1, b = \beta s_1$ for some $\alpha, \beta \in \mathbb{R}$, then

$$x = [a, [b, \ell]] = [a, [\beta s_1, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]]$$

$$= [a, \beta \lambda_3 [s_1, s_3]] = [\alpha s_1, \beta \lambda_3 s_1] = 0,$$

since $E_1 = 0$, by proposition 3.1 $cor_L(E) = 0$.

In the same way $cor_L(Span\{s_2\}) = 0$.

2- By remark 2.3 the only 2-dim inner ideals of L_3 are Spans of $\{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_4\}$.

Let $E = Span\{s_2, s_4\}$. we need to show $cor_L(E) = 0$. Let $x \in E_1$ and $a, b \in E$, then $a = \alpha_1 s_2 + \alpha_2 s_4, b = \beta_1 s_2 + \beta_2 s_4$ for some $\alpha, \beta \in \mathbb{R}$, then

$$x = [a, [b, \ell]] = [a, [\beta_1 s_2 + \beta_2 s_4, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]]$$

$$= [a, (\beta_1 \lambda_4 - \beta_2 \lambda_2) [s_2, s_4]] = [\alpha_1 s_2 + \alpha_2 s_4, (\beta_1 \lambda_4 - \beta_2 \lambda_2) s_2]$$

$$= -\alpha_2 (\beta_1 \lambda_4 - \beta_2 \lambda_2) s_2 \in Span\{s_2\},$$

Let $y \in E_2$ then exist $x_1, x_2 \in E_1$ such that $x_1 = \alpha_1 s_2, x_2 = \beta_1 s_2$ then

$$y = [x_1, [x_2, \ell]] = [x_1, [\beta_1 s_2, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]]$$

$$= [x_1, \beta_1 \lambda_4 [s_2, s_4]] = [\alpha_1 s_2, \beta_1 \lambda_4 s_2] = 0,$$

since $E_2 = 0$, by proposition 3.1 $cor_L(E) = 0$.

In the same way $cor_L(Span\{s_1, s_2\}) = 0$ and $cor_L(Span\{s_1, s_3\}) = 0$.

3- By remark 2.3 the only 3- dim inner ideals of L_3 are Spans of $\{s_1, s_2, s_3\}, \{s_1, s_2, s_4\}$.

Let $E = Span\{s_1, s_2, s_3\}$ we need to show $cor_L(E) = 0$, if $x \in E_1$ then exist $a, b \in E$ such that $a = \alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3, b = \beta_1 s_1 + \beta_2 s_2 + \beta_3 s_3$ for some $\alpha_i, \beta_i \in \mathbb{R}$, then

$$x = [a, [b, \ell]] = [a, [\beta_1 s_1 + \beta_2 s_2 + \beta_3 s_3, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]]$$

$$= [a, (\beta_1 \lambda_3 - \beta_3 \lambda_1) [s_1, s_3] + \beta_2 \lambda_4 [s_2, s_4]]$$

$$= [\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3, (\beta_1 \lambda_3 - \beta_3 \lambda_1) s_1 + \beta_2 \lambda_4 s_2]$$

$$= -\alpha_3 (\beta_1 \lambda_3 - \beta_3 \lambda_1) s_1 \in Span\{s_1\}$$

Let $y \in E_2$ then exist $x_1, x_2 \in E_1$ such that $x_1 = \alpha_1 s_1, x_2 = \beta_1 s_1$ then

$$y = [x_1, [x_2, \ell]] = [x_1, [\beta_1 s_1, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]]$$

$$= [x_1, \beta_1 \lambda_3 [s_1, s_3]] = [\alpha_1 s_1, \beta_1 \lambda_3 s_1] = 0$$

since $E_2 = 0$, by proposition 3.1 $cor_L(E) = 0$. In the same way $cor_L(Span\{s_1, s_2, s_4\}) = 0$.

Lemma 3.5. Let $L = L_4$,

- 1- If E is a 1-dim I -ideal then $cor_L(E) = 0$.
- 2- If E is a 2-dim I -ideal then $cor_L(E) = 0$.
- 3- If E is a 3-dim I -ideal then $cor_L(E) = 0$.

Proof. By theorem 2.2 if $L = L_4$ and $\{s_1, s_2, s_3, s_4\}$ is a basis of L , then the multiplication of L is as follows

$$(3.4) [s_1, s_3] = -s_2, [s_2, s_3] = s_1, [s_1, s_4] = s_1, [s_2, s_4] = s_2 \text{ and } 0 \text{ otherwise.}$$

let $\ell \in L$ then $\ell = \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4$ for some $\lambda_i \in \mathbb{R}$

1- By remark 2.3 the only 1-dim inner ideals of L_4 are Spans of $\{s_1\}, \{s_2\}$.

Let $E = Span\{s_1\}$ we need to show $cor_L(E) = 0$. Let $x \in E_1$ and $a, b \in E$, then $a = \alpha s_1, b = \beta s_1$ for some $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned} x &= [a, [b, \ell]] = [a, [\beta s_1, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [a, \beta \lambda_3 [s_1, s_3] + \beta \lambda_4 [s_1, s_4]] = [\alpha s_1, -\beta \lambda_3 s_2 + \beta \lambda_4 s_1] = 0, \end{aligned}$$

since $E_1 = 0$, by proposition 3.1 $cor_L(E) = 0$.

In the same way $cor_L(Span\{s_2\}) = 0$.

2- By remark 2.3 the only 2-dim inner ideals of L_4 are Spans of $\{s_1, s_2\}$. Let $E = Span\{s_1, s_2\}$. we need to show $cor_L(E) = 0$. Let $x \in E_1$ and $a, b \in E$, then $a = \alpha_1 s_1 + \alpha_2 s_2, b = \beta_1 s_1 + \beta_2 s_2$ for some $\alpha_i, \beta_i \in \mathbb{R}$, then

$$\begin{aligned} x &= [a, [b, \ell]] = [a, [\beta_1 s_1 + \beta_2 s_2, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [a, \beta_1 \lambda_3 [s_1, s_3] + \beta_1 \lambda_4 [s_1, s_4] + \beta_2 \lambda_3 [s_2, s_3] + \beta_2 \lambda_4 [s_2, s_4]] \\ &= [\alpha_1 s_1 + \alpha_2 s_2, (-\beta_1 \lambda_3 + \beta_2 \lambda_4) s_2 + (\beta_1 \lambda_4 + \beta_2 \lambda_3) s_1] = 0, \end{aligned}$$

since $E_1 = 0$, by proposition 3.1 $cor_L(E) = 0$.

3- By remark 2.3 the only 3-dim inner ideals of L_4 are Spans of $\{s_1, s_2, s_3\}, \{s_1, s_2, s_4\}$.

Let $E = Span\{s_1, s_2, s_3\}$ we need to show $cor_L(E) = 0$, if $x \in E_1$ then exist $a, b \in E$ such that $a = \alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3, b = \beta_1 s_1 + \beta_2 s_2 + \beta_3 s_3$ for some $\alpha_i, \beta_i \in \mathbb{R}$, then

$$\begin{aligned} x &= [a, [b, \ell]] = [a, [\beta_1 s_1 + \beta_2 s_2 + \beta_3 s_3, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [a, (\beta_1 \lambda_3 - \beta_3 \lambda_1) [s_1, s_3] + \beta_1 \lambda_4 [s_1, s_4] + (\beta_2 \lambda_3 - \beta_3 \lambda_2) [s_2, s_3] + \beta_2 \lambda_4 [s_2, s_4]] \\ &= [\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3, -(\beta_1 \lambda_3 - \beta_3 \lambda_1 + \beta_2 \lambda_4) s_2 + (\beta_1 \lambda_4 + \beta_2 \lambda_3 - \beta_3 \lambda_2) s_1] \\ &= -\alpha_3 (-(\beta_1 \lambda_3 - \beta_3 \lambda_1 + \beta_2 \lambda_4) s_1 - (\beta_1 \lambda_4 + \beta_2 \lambda_3 - \beta_3 \lambda_2) s_2) \in Span\{s_1, s_2\} \end{aligned}$$

Let $y \in E_2$ then exist $x_1, x_2 \in E_1$ such that $x_1 = \alpha_1 s_1 + \alpha_2 s_2, x_2 = \beta_1 s_1 + \beta_2 s_2$ then

$$\begin{aligned} y &= [x_1, [x_2, \ell]] = [x_1, [\beta_1 s_1 + \beta_2 s_2, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [x_1, \beta_1 \lambda_3 [s_1, s_3] + \beta_2 \lambda_3 [s_2, s_3] + \beta_1 \lambda_4 [s_1, s_4] + \beta_2 \lambda_4 [s_2, s_4]] \\ &= [\alpha_1 s_1 + \alpha_2 s_2, (-\beta_1 \lambda_3 + \beta_2 \lambda_4) s_2 + (\beta_2 \lambda_3 + \beta_1 \lambda_4) s_1] = 0, \end{aligned}$$

since $E_2 = 0$, by proposition 3.1 $cor_L(E) = 0$. In the same way $cor_L(Span\{s_1, s_2, s_4\}) = 0$.

Lemma 3.6. Let $L = L_5$,

- 1- If E is a 1-dim I -ideal then $cor_L(E) = 0$.
- 2- If E is a 2-dim I -ideal then $cor_L(E) = 0$.
- 3- If E is a 3-dim I -ideal then $cor_L(E) = 0$.

Proof. By theorem 2.2 if $L = L_5$ and $\{s_1, s_2, s_3, s_4\}$ is a basis of L , then the multiplication of L is as follows

$$(3.5) [s_1, s_4] = s_1, [s_2, s_3] = s_1, [s_2, s_4] = s_2 \text{ and } 0 \text{ otherwise.}$$

let $\ell \in L$ then $\ell = \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4$ for some $\lambda_i \in \mathbb{R}$

1- By remark 2.3 the only 1-dim inner ideals of L_5 are Spans of $\{s_1\}, \{s_2\}, \{s_3\}$.

Let $E = Span\{s_1\}$ we need to show $cor_L(E) = 0$. Let $x \in E_1$ and $a, b \in E$, then $a = \alpha s_1, b = \beta s_1$ for some $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned} x &= [a, [b, \ell]] = [a, [\beta s_1, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [a, \beta \lambda_4 [s_1, s_4]] = [\alpha s_1, \beta \lambda_4 s_1] = 0, \end{aligned}$$

since $E_1 = 0$, by proposition 3.1 $cor_L(E) = 0$.

In the same way $cor_L(Span\{s_2\}) = 0$ and $cor_L(Span\{s_3\}) = 0$.

2- By remark 2.3 the only 2-dim inner ideals of L_5 are Spans of $\{s_1, s_2\}, \{s_1, s_3\}$.

Let $E = Span\{s_1, s_3\}$. we need to show $cor_L(E) = 0$. Let $x \in E_1$ and $a, b \in E$, then $a = \alpha_1 s_1 + \alpha_2 s_3, b = \beta_1 s_1 + \beta_2 s_3$ for some $\alpha_i, \beta_i \in \mathbb{R}$, then

$$\begin{aligned} x &= [a, [b, \ell]] = [a, [\beta_1 s_1 + \beta_2 s_3, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [a, \beta_1 \lambda_4 [s_1, s_4] - \beta_2 \lambda_2 [s_2, s_3]] \end{aligned}$$

$$= [\alpha_1 s_1 + \alpha_2 s_3, (\beta_1 \lambda_4 - \beta_2 \lambda_2) s_1] = 0,$$

since $E_1 = 0$, by proposition 3.1 $cor_L(E) = 0$.

In the same way $cor_L(Span\{s_1, s_2\}) = 0$.

3- By remark 2.3 the only 3- dim inner ideals of L_5 are Spans of $\{s_1, s_2, s_3\}, \{s_1, s_2, s_4\}$.

Let $E = Span\{s_1, s_2, s_4\}$ we need to show $cor_L(E) = 0$, if $x \in E_1$ then exist $a, b \in E$ such that $a = \alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_4, b = \beta_1 s_1 + \beta_2 s_2 + \beta_3 s_4$ for some $\alpha_i, \beta_i \in \mathbb{R}$, then

$$\begin{aligned} x &= [a, [b, \ell]] = [a, [\beta_1 s_1 + \beta_2 s_2 + \beta_3 s_4, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [a, (\beta_1 \lambda_4 - \beta_3 \lambda_1)[s_1, s_4] + \beta_2 \lambda_3 [s_2, s_3] + (\beta_2 \lambda_4 - \beta_3 \lambda_2)[s_2, s_4]] \\ &= [\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_4, (\beta_1 \lambda_4 - \beta_3 \lambda_1 + \beta_2 \lambda_3) s_1 + (\beta_2 \lambda_4 - \beta_3 \lambda_2) s_2] \\ &= -\alpha_3 (\beta_1 \lambda_4 - \beta_3 \lambda_1 + \beta_2 \lambda_3) s_1 - \alpha_2 (\beta_2 \lambda_4 - \beta_3 \lambda_2) s_2 \in Span\{s_1, s_2\} \end{aligned}$$

Let $y \in E_2$ then exist $x_1, x_2 \in E_1$ such that $x_1 = \alpha_1 s_1 + \alpha_2 s_2, x_2 = \beta_1 s_1 + \beta_2 s_2$ then

$$\begin{aligned} y &= [x_1, [x_2, \ell]] = [x_1, [\beta_1 s_1 + \beta_2 s_2, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [x_1, \beta_1 \lambda_4 [s_1, s_4] + \beta_2 \lambda_3 [s_2, s_3] + \beta_2 \lambda_4 [s_2, s_4]] \\ &= [\alpha_1 s_1 + \alpha_2 s_2, (\beta_1 \lambda_4 + \beta_2 \lambda_3) s_1 + \beta_2 \lambda_4 s_2] = 0, \end{aligned}$$

since $E_2 = 0$, by proposition 3.1 $cor_L(E) = 0$. In the same way $cor_L(Span\{s_1, s_2, s_3\}) = 0$.

Lemma 3.7. Let $L = L_6$,

1- If E is a 1-dim I -ideal then $cor_L(E) = 0$.

2- If E is a 2-dim I -ideal then $cor_L(E) = 0$.

3- If E is a 3-dim I -ideal then $cor_L(E) = 0$.

Proof. By theorem 2.4 if $L = L_6$ and $\{s_1, s_2, s_3, s_4\}$ is a basis of L , then the multiplication of L is as follows

(3.6) $[s_2, s_4] = s_1, [s_3, s_4] = s_2$ and 0 otherwise.

let $\ell \in L$ then $\ell = \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4$ for some $\lambda_i \in \mathbb{R}$

1- By remark 2.5 the only 1-dim inner ideals of L_6 are Spans of $\{s_1\}, \{s_2\}, \{s_3\}$.

Let $E = Span\{s_1\}$ we need to show $cor_L(E) = 0$. Let $x \in E_1$ and $a, b \in E$, then $a = \alpha s_1, b = \beta s_1$ for some $\alpha, \beta \in \mathbb{R}$, then

$$x = [a, [b, \ell]] = [a, [\beta s_1, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] = 0$$

since $E_1 = 0$, by proposition 3.1 $cor_L(E) = 0$.

In the same way $cor_L(Span\{s_2\}) = 0$ and $cor_L(Span\{s_3\}) = 0$.

2- By remark 2.5 the only 2-dim inner ideals of L_6 are Spans of $\{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_4\}$.

Let $E = Span\{s_1, s_4\}$. we need to show $cor_L(E) = 0$. Let $x \in E_1$ and $a, b \in E$, then $a = \alpha_1 s_1 + \alpha_2 s_4, b = \beta_1 s_1 + \beta_2 s_4$ for some $\alpha_i, \beta_i \in \mathbb{R}$, then

$$\begin{aligned} x &= [a, [b, \ell]] = [a, [\beta_1 s_1 + \beta_2 s_4, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [a, -\beta_2 \lambda_2 [s_2, s_4] - \beta_2 \lambda_3 [s_3, s_4]] = [\alpha_1 s_1 + \alpha_2 s_4, -\beta_2 \lambda_2 s_1 - \beta_2 \lambda_3 s_2] \\ &= \alpha_2 \beta_2 \lambda_3 [s_2, s_4] = \alpha_2 \beta_2 \lambda_3 s_1 \in Span\{s_1\}, \end{aligned}$$

Let $y \in E_2$ then exist $x_1, x_2 \in E_1$ such that $x_1 = \alpha_1 s_1, x_2 = \beta_1 s_1$ then

$$\begin{aligned} y &= [x_1, [x_2, \ell]] = [x_1, [\beta_1 s_1, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [x_1, 0] = 0, \end{aligned}$$

since $E_1 = 0$, by proposition 3.1 $cor_L(E) = 0$.

In the same way $cor_L(Span\{s_1, s_2\}) = 0, cor_L(Span\{s_1, s_3\}) = 0$ and $cor_L(Span\{s_2, s_3\}) = 0$.

3- By remark 2.5 the only 3- dim inner ideals of L_6 are Spans of $\{s_1, s_2, s_3\}, \{s_1, s_2, s_4\}, \{s_1, s_3, s_4\}$.

Let $E = Span\{s_1, s_3, s_4\}$ we need to show $cor_L(E) = 0$, if $x \in E_1$ then exist $a, b \in E$ such that $a = \alpha_1 s_1 + \alpha_2 s_3 + \alpha_3 s_4, b = \beta_1 s_1 + \beta_2 s_3 + \beta_3 s_4$ for some $\alpha_i, \beta_i \in \mathbb{R}$, then

$$\begin{aligned} x &= [a, [b, \ell]] = [a, [\beta_1 s_1 + \beta_2 s_3 + \beta_3 s_4, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [a, (\beta_2 \lambda_4 - \beta_3 \lambda_3)[s_3, s_4] - \beta_3 \lambda_2 [s_2, s_4]] \\ &= [\alpha_1 s_1 + \alpha_2 s_3 + \alpha_3 s_4, -\beta_3 \lambda_2 s_1 + (\beta_2 \lambda_4 - \beta_3 \lambda_3) s_2] \\ &= -\alpha_3 (\beta_2 \lambda_4 - \beta_3 \lambda_3) s_1 \in Span\{s_1\} \end{aligned}$$

Let $y \in E_2$ then exist $x_1, x_2 \in E_1$ such that $x_1 = \alpha_1 s_1, x_2 = \beta_1 s_1$ then

$$\begin{aligned} y &= [x_1, [x_2, \ell]] = [x_1, [\beta_1 s_1, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [x_1, 0] = 0, \end{aligned}$$

since $E_2 = 0$, by proposition 3.1 $cor_L(E) = 0$. In the same way $cor_L(Span\{s_1, s_2, s_3\}) = 0$ and $cor_L(Span\{s_1, s_2, s_4\}) = 0$.

Lemma 3.8. Let $L = L_7$,

- 1- If E is a 1-dim I -ideal then $cor_L(E) = 0$.
- 2- If E is a 2-dim I -ideal then $cor_L(E) = 0$.
- 3- If E is a 3-dim I -ideal then $cor_L(E) = 0$.

Proof. By theorem 2.4 if $L = L_7$ and $\{s_1, s_2, s_3, s_4\}$ is a basis of L , then the multiplication of L is as follows

$$(3.7) [s_2, s_4] = s_2, [s_3, s_4] = s_1 \text{ and } 0 \text{ otherwise}$$

let $\ell \in L$ then $\ell = \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4$ for some $\lambda_i \in \mathbb{R}$

1- By remark 2.5 the only 1-dim inner ideals of L_7 are Spans of $\{s_1\}, \{s_2\}, \{s_3\}$.

Let $E = Span\{s_1\}$ we need to show $cor_L(E) = 0$. Let $x \in E_1$ and $a, b \in E$, then $a = \alpha s_1, b = \beta s_1$ for some $\alpha, \beta \in \mathbb{R}$, then

$$x = [a, [b, \ell]] = [a, [\beta s_1, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] = 0$$

since $E_1 = 0$, by proposition 3.1 $cor_L(E) = 0$.

In the same way $cor_L(Span\{s_2\}) = 0$ and $cor_L(Span\{s_3\}) = 0$.

2- By remark 2.5 the only 2-dim inner ideals of L_7 are Spans of $\{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_2, s_4\}$.

Let $E = Span\{s_2, s_4\}$. we need to show $cor_L(E) = 0$. Let $x \in E_1$ and $a, b \in E$, then $a = \alpha_1 s_2 + \alpha_2 s_4, b = \beta_1 s_2 + \beta_2 s_4$ for some $\alpha_i, \beta_i \in \mathbb{R}$, then

$$\begin{aligned} x &= [a, [b, \ell]] = [a, [\beta_1 s_2 + \beta_2 s_4, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [a, (\beta_1 \lambda_4 - \beta_2 \lambda_2)[s_2, s_4] - \beta_2 \lambda_3 [s_3, s_4]] \\ &= [\alpha_1 s_2 + \alpha_2 s_4, (\beta_1 \lambda_4 - \beta_2 \lambda_2)s_2 - \beta_2 \lambda_3 s_1] \\ &= -\alpha_2 (\beta_1 \lambda_4 - \beta_2 \lambda_2) s_2 \in Span\{s_2\}, \end{aligned}$$

Let $y \in E_2$ then exist $x_1, x_2 \in E_1$ such that $x_1 = \alpha_1 s_2, x_2 = \beta_1 s_2$ then

$$\begin{aligned} y &= [x_1, [x_2, \ell]] = [x_1, [\beta_1 s_2, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [x_1, \beta_1 \lambda_4 [s_2, s_4]] = [\alpha_1 s_2, \beta_1 \lambda_4 s_2] = 0, \end{aligned}$$

since $E_2 = 0$, by proposition 3.1 $cor_L(E) = 0$.

In the same way $cor_L(Span\{s_1, s_2\}) = 0, cor_L(Span\{s_1, s_3\}) = 0$ and $cor_L(Span\{s_2, s_3\}) = 0$.

3- By remark 2.5 the only 3- dim inner ideals of L_7 are Spans of $\{s_1, s_2, s_3\}, \{s_1, s_2, s_4\}, \{s_2, s_3, s_4\}$.

Let $E = Span\{s_2, s_3, s_4\}$ we need to show $cor_L(E) = 0$, if $x \in E_1$ then exist $a, b \in E$ such that $a = \alpha_1 s_2 + \alpha_2 s_3 + \alpha_3 s_4, b = \beta_1 s_2 + \beta_2 s_3 + \beta_3 s_4$ for some $\alpha_i, \beta_i \in \mathbb{R}$, then

$$\begin{aligned} x &= [a, [b, \ell]] = [a, [\beta_1 s_2 + \beta_2 s_3 + \beta_3 s_4, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [a, (\beta_2 \lambda_4 - \beta_3 \lambda_3)[s_3, s_4] + (\beta_1 \lambda_4 - \beta_3 \lambda_2)[s_2, s_4]] \\ &= [\alpha_1 s_1 + \alpha_2 s_3 + \alpha_3 s_4, (\beta_2 \lambda_4 - \beta_3 \lambda_3)s_1 + (\beta_1 \lambda_4 - \beta_3 \lambda_2)s_2] \\ &= -\alpha_3 (\beta_1 \lambda_4 - \beta_3 \lambda_2) s_2 \in Span\{s_2\} \end{aligned}$$

Let $y \in E_2$ then exist $x_1, x_2 \in E_1$ such that $x_1 = \alpha_1 s_2, x_2 = \beta_1 s_2$ then

$$\begin{aligned} y &= [x_1, [x_2, \ell]] = [x_1, [\beta_1 s_2, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [x_1, \beta_1 \lambda_4 [s_2, s_4]] = [\alpha_1 s_2, \beta_1 \lambda_4 s_2] = 0 \end{aligned}$$

since $E_2 = 0$, by proposition 3.1 $cor_L(E) = 0$. In the same way $cor_L(Span\{s_1, s_2, s_3\}) = 0$ and $cor_L(Span\{s_1, s_2, s_4\}) = 0$.

Theorem 3.9. Assume that L is a 4-dim Lie algebra with 2-dim derived L' . Then $cor_L(E) = 0$ for every inner ideal E of L .

Proof. Recall that if L be a 4-dim Lie algebra with 2-dim derived L' . Then, either $L' \subseteq C(L)$ or $L' \not\subseteq C(L)$.

i) If $L' \not\subseteq C(L)$, then by theorem 2.2, $L = L_i (1 \leq i \leq 5)$, or $L = L_\varepsilon (\varepsilon = \pm)$. When $L = L_i$ we prove that $cor_L(E) = 0$ whether dimensional of E is 1,2 or 3, by Lemma 3.2, Lemma 3.3, Lemma 3.4, Lemma 3.5, Lemma 3.6, respectively.

If $L = L_\varepsilon$ and $\{s_1, s_2, s_3, s_4\}$ is a basis of L , then by theorem 2.2 the multiplication of L is as follows

$$(3.8) [s_1, s_4] = \varepsilon s_2, [s_2, s_4] = s_1 \text{ and } 0 \text{ otherwise. } \varepsilon = \mp.$$

And by remark 2.3 the only inner ideals of $L = L_\varepsilon$ are spans of $\{s_1\}, \{s_2\}, \{s_3\}, \{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_2, s_3\}, \{s_1, s_2, s_4\}$.

1- Let $E = Span\{s_1\}$ we need to show $cor_L(E) = 0$. Let $x \in E_1$ and $a, b \in E$, then $a = \alpha s_1, b = \beta s_1$ for some $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned} x &= [a, [b, \ell]] = [a, [\beta s_1, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [a, \beta \lambda_4 [s_1, s_4]] = [\alpha s_1, \beta \lambda_4 \varepsilon s_2] = \alpha \beta \lambda_4 \varepsilon [s_1, s_2] = 0, \end{aligned}$$

since $E_1 = 0$, by proposition 3.1 $cor_L(E) = 0$.

In the same way $cor_L(Span\{s_2\}) = 0$ and $cor_L(Span\{s_3\}) = 0$.

2- Let $E = \text{Span} \{s_2, s_3\}$. we need to show $\text{cor}_L(E) = 0$. Let $x \in E_1$ and $a, b \in E$, then $a = \alpha_1 s_2 + \alpha_2 s_3, b = \beta_1 s_2 + \beta_2 s_3$ for some $\alpha_i, \beta_i \in \mathbb{R}$, then

$$\begin{aligned} x &= [a, [b, \ell]] = [a, [\beta_1 s_2 + \beta_2 s_3, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [a, \beta_1 \lambda_4 [s_2, s_4]] = [\alpha_1 s_2 + \alpha_2 s_3, \beta_1 \lambda_4 s_1] = 0, \end{aligned}$$

since $E_1 = 0$, by proposition 3.1 $\text{cor}_L(E) = 0$.

In the same way $\text{cor}_L(\text{Span} \{s_1, s_2\}) = 0$ and $\text{cor}_L(\text{Span} \{s_1, s_3\}) = 0$.

3- Let $E = \text{Span} \{s_1, s_2, s_4\}$ we need to show $\text{cor}_L(E) = 0$, if $x \in E_1$ then exist $a, b \in E$ such that $a = \alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_4, b = \beta_1 s_1 + \beta_2 s_2 + \beta_3 s_4$ for some $\alpha_i, \beta_i \in \mathbb{R}$, then

$$\begin{aligned} x &= [a, [b, \ell]] = [a, [\beta_1 s_1 + \beta_2 s_2 + \beta_3 s_4, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [a, (\beta_1 \lambda_4 - \beta_3 \lambda_1) [s_1, s_4] + (\beta_2 \lambda_4 - \beta_3 \lambda_2) [s_2, s_4]] \\ &= [\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_4, (\beta_1 \lambda_4 - \beta_3 \lambda_1) \varepsilon s_2 + (\beta_2 \lambda_4 - \beta_3 \lambda_2) s_1] \\ &= -\alpha_3 ((\beta_1 \lambda_4 - \beta_3 \lambda_1) \varepsilon s_1 + (\beta_2 \lambda_4 - \beta_3 \lambda_2) \varepsilon s_2) \in \text{Span} \{s_1, s_2\} \end{aligned}$$

Let $y \in E_2$ then exist $x_1, x_2 \in E_1$ such that $x_1 = \alpha_1 s_1 + \alpha_2 s_2, x_2 = \beta_1 s_1 + \beta_2 s_2$ then

$$\begin{aligned} y &= [x_1, [x_2, \ell]] = [x_1, [\beta_1 s_1 + \beta_2 s_2, \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \lambda_4 s_4]] \\ &= [x_1, \beta_1 \lambda_4 [s_1, s_4] + \beta_2 \lambda_4 [s_2, s_4]] \\ &= [\alpha_1 s_1 + \alpha_2 s_2, \beta_1 \lambda_4 \varepsilon s_2 + \beta_2 \lambda_4 s_1] = 0, \end{aligned}$$

since $E_2 = 0$, by proposition 3.1 $\text{cor}_L(E) = 0$. In the same way $\text{cor}_L(\text{Span} \{s_1, s_2, s_3\}) = 0$.

ii) If $L \subseteq C(L)$, then by theorem 2.4, $L = L_6$ or $L = L_7$. And $\text{cor}L(E) = 0$ whether dimensional of E is 1,2 or 3, by Lemma3.7, Lemma3.8 respectively.

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