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Some Properties of Spectral Theory in Fuzzy Hilbert Spaces

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Abstract
In this paper we give some definitions and properties of spectral theory in fuzzy Hilbert spaces also we introduce definitions Invariant under a linear operator T on fuzzy normed spaces and reduced linear operator on fuzzy Hilbert spaces and we prove theorms related to eigenvalue and eigenvectors ,eigenspace in fuzzy normed , Invariant and reduced in fuzzy Hilbert spaces and show relationship between them.
Keywordes: fuzzy normed spaces, fuzzy Hilbert spaces , eigenvalue and eigenvectors , eigenspace in fuzzy normed , linear operator on fuzzy normed spaces

Mathematics subject classification : 46S40 .

1.Introduction

The theory of a fuzzy sets was introduced by L. A. Zadeh [1] in 1965.Aftar the pioneer work of Zadeh ,many researchers have extended this concept in various branches ,many other mathematicians have studied fuzzy normed space from sereval points of view [2],[7]. Fuzzy Hilbert spaces is an extension to the Hilbert space. The definition of a fuzzy Hilbert space has been introduced by M. Goudarzi and S. M. Vaezpour [9] in 2009 .

2.Preliminaries

Definition (2.1): [3] Let $*$ be a binary operation on the set I, i.e. $*: I \times I \to I$ is a function. Then $*$ is said to be t-norm (triangular-norm) on the set I if the following axioms are satisfied :

(1) $a * 1 = a$, for all $a \in I$.

(2) $*$ is commutative (i.e. $a * b = b * a$, for all $a, b \in I$).

(3) $*$ is monotone (i.e. if $b, c \in I$ such that $b \leq c$, then $a * b \leq a * c$, for all $a \in I$).

(4) $*$ is associative (i.e. $a * (b * c) = (a * b) * c$, for all $a, b, c \in I$).

If, in addition, $*$ is continuous then $*$ is called a continuous t-norm.

Definition (2.2) : [2] Let X be a vector space over F, $*$ be a continuous t-norm on I, a function $N: X \times (0, \infty) \to [0,1]$ is called fuzzy norm if it satisfies the following conditions : for all $x, y \in X$ and $t, s > 0$,

$(N.1)$ $N(x,t) > 0$,
$(N.2)$ $N(x,t) = 1$ if and only if $x = 0$,
$(N.3)$ $N(\alpha x, t) = N\left(x, \frac{t}{|\alpha|}\right)$, for all $\alpha \neq 0$,
$(N.4)$ $N(x,t) * N(y,s) \leq N(x+y, t+s)$,
$(N.5)$ $N(x,.): (0,\infty) \to [0,1]$ is continuous,
$(N.6)$ $\lim_{t \to \infty} N(x,t) = 1$.

$(X, N, *)$ is called fuzzy normed spac**e**

Remark (2.3) : [8]

(1) For any $\alpha_1, \alpha_2 \in (0,1)$ **with** $\alpha_1 > \alpha_2$**, there exists** $\alpha_3 \in (0,1)$ **such that** $\alpha_1 * \alpha_3 \geq \alpha_2$**.**

(2) For any $\alpha_4 \in (0,1)$**, there exists** $\alpha_5 \in (0,1)$ **such that**

$\alpha_5 * \alpha_5 \geq \alpha_4$**.**

Example (2.4) : [11] Let $(X, \|.\|)$ **be a normed space.**

$a * b = a.b$ **for all** $a, b \in X$ **and for all** $x \in X, t > 0$

$$N(x,t) = \begin{cases} \frac{t}{t+\|x\|} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$$

Then $(X, N, *)$ is fuzzy normed space**.**

Definition (2.5) : [4] Let X be a real linear space, $*$ be a continuous t-norm on $I = [0,1]$. A function $H: X \times X \times \mathbb{R} \to [0,1]$ is called a fuzzy pre-Hilbert function if it satisfies the following axioms for every $x, y, z \in X$ and $s, t, r \in \mathbb{R}$:

Note : $h(t) = \begin{cases} 1 & , t > 0 \\ 0 & , t \le 0 \end{cases}$

(1) $H(x,x,0) = 0$ and $H(x,x,t) > 0$ for each $t > 0$

(2) $H(x,x,t) \neq h(t)$ for some $t \in \mathbb{R}$ if and only if $x \neq 0$

(3) $H(x,y,t) = H(y,x,t)$

(4) For any real number α

$$H(\alpha x, y, t) = \begin{cases} H(x, y, \frac{t}{\alpha}) & , \alpha > 0 \\ h(t) & , \alpha = 0 \\ 1 - H(x, y, \frac{t}{-\alpha}) & , \alpha < 0 \end{cases}$$

(5) $H(x,x,t) * H(y,y,s) \le H(x+y, x+y, t+s)$

(6) $sup_{s+r=t}\big(H(x,z,s) * H(y,z,r)\big) = H(x+y,z,t)$

(7) $H(x,y,.): \mathbb{R} \to [0,1]$ is continuous on $\mathbb{R} \setminus \{0\}$.

(8) $\lim_{t\to+\infty} H(x,y,t) = 1$.

$(X, H, *)$ is a fuzzy pre-Hilbert space.

Example (2.6) : [4] Let $(X, \langle\,,\rangle)$ be an ordinary pre-Hilbert space. We define a function $H : X \times X \times \mathbb{R} \to [0,1]$ as follows :

$$H(\alpha x, y, t) = \begin{cases} \dfrac{t^{\frac{1}{2}}}{t^{\frac{1}{2}} + |\langle \alpha x, y\rangle|^{\frac{1}{2}}} & , \alpha \ge 0, t > 0 \\ 1 - \dfrac{t^{\frac{1}{2}}}{t^{\frac{1}{2}} + |\langle \alpha x, y\rangle|^{\frac{1}{2}}} & , \alpha < 0, t > 0 \\ 0 & , \quad t \le 0 \end{cases}$$

Define $a * b = \min\{a, b\}$ for all $a, b \in X$. This is a fuzzy pre-Hilbert and called the standard fuzzy pre-Hilbert induced by the pre-Hilbert $\langle\,.,.\rangle$.

Definition (2.7) : [4] Let $(X, H, *)$ be a fuzzy pre-Hilbert space. $x, y \in X$ is said to be fuzzy orthogonal if $H(x,y,t) = h(t)(\forall t \in \mathbb{R})$ and it is denoted by $x \perp y$.

Definition (2.8) : [9] Let $(X, H, *)$ be a fuzzy pre-Hilbert space. A subset B of X is called fuzzy orthogonal if $x \perp y$, for each $x, y \in B$.

Lemma (2.9) : [4] If $(X, H, *)$ be a fuzzy pre-Hilbert space , then $(X, H, *)$ is non decreasing with respect to t, for each $x, y \in X$.

Definition (2.10) : [9]If B is a subset of the fuzzy pre-Hilbert space $(X, H, *)$, then $B^{\perp} = \{x \in X : x \perp y, \forall y \in B\}$.

Definition (2.11) : [7] A t-norm $* : [0,1] \times [0,1] \to [0,1]$ is called strong if it has the two following properties :

(1) For all $a, b \in (0,1)$, $a * b > 0$,

(2) For all $a, b, c, d \in [0,1]$ and $a > b$, $c > d$ we have $a * b > c * d$.

Theorem (2.12) : [4] Suppose that $(X, H, *)$ be a fuzzy pre-Hilbert space, where $*$ is a strong t-norm and for each $x, y \in X$,

$$\sup\{t \in \mathbb{R}, H(x,y,t) < 1\} < \infty.$$

Define $\langle\,.,.\rangle : X \times X \to \mathbb{R}$ by $\langle x, y\rangle = \sup\{t \in \mathbb{R}, H(x,y,t) < 1\}$. Then $(X, \langle\,.,.\rangle)$ is a pre-Hilbert space

Corollary (2.13) : [4] Let $(X, H, *)$ be a fuzzy pre-Hilbert space, where $*$ is a strong t-norm and for each $x, y \in X$, $\sup\{t \in \mathbb{R}, H(x,y,t) < 1\} < \infty$. If we define $\|x\| = (\sup\{t \in \mathbb{R}, H(x,x,t) < 1\})^{\frac{1}{2}}$, then $(X, \|.\|)$ is a normed space.

Definition (2.14) : [4] Let $(X, H, *)$ be a fuzzy pre-Hilbert space, where $*$ is a strong t-norm and for each $x, y \in X, \sup\{t \in \mathbb{R}, H(x,y,t) < 1\} < \infty$ and $\|x\| = (\sup\{t \in \mathbb{R}, H(x,x,t) < 1\})^{\frac{1}{2}}$. We say that $(X, H, *)$ is a fuzzy Hilbert space if $(X, \|.\|)$ is complete normed space .

Theorem (2.15) : [6] Let $(X, H,*)$ be a fuzzy pre-Hilbert space . And $A \subset X$;

(1)The relation of Orthogonality symmetric (i.e. if $x \perp y$ then $y \perp x$)

(2)If $x \perp y$ then $\alpha x \perp y \ \forall t \in \mathbb{R}$

(3)Let $A \subset B$ then $B^{\perp} \subset A^{\perp}$

(4) $A \subset A^{\perp\perp}$

(5) $A \subset B^{\perp} \leftrightarrow B \subset A^{\perp}$

(6)If $x \perp x \leftrightarrow x = 0 \quad \forall t \in \mathbb{R}$

(7) $X^{\perp} = \{0\}$ **for all** $t \in \mathbb{R}$

(8) $A \cap A^{\perp} = \{0\}$ **for all** $t \in \mathbb{R}$

(9)For every vector $x \in X$,we have $0 \perp x \ \forall \ x \in X$

3. Eigenvalue and Eigenvector In Fuzzy Normed Spaces.

Definition (3.1) : [5] A function $T : X \to Y$ is called an operator from X into Y if X and Y are linear spaces over the same field F.

Definition (3.2) : [5] A linear operator T is an operator such that $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in X$ and for all $\alpha, \beta \in F$.

Definition (3.3) : Let $(X, N,*)$ be a fuzzy normed spaces over F and $T \in L(X)$ then
(1) A scalar $\lambda \in F$ is called an eigenvalue of T, if there exists non zero $x \in X$ such that $T(x) = \lambda x$

(2) Let x be an eigevector of T corresponding to eigenvalue $\lambda \Longrightarrow T(x) = \lambda x$

Example (3.4) : Let $X = \mathbb{R}^2$ and $T : (X, N,*) \to (X, N,*)$ Define
By $T(x, y) = (-y, x)$ for all $(x, y) \in \mathbb{R}^2$ and $N: \mathbb{R}^2 \times (0, \infty) \to [0,1]$
Define fuzzy norm in example(2.4) and T is linear operator has no eigenvalue

Example (3.5) : Let $X = \mathbb{R}^2$ and $T : (X, N,*) \to (X, N,*)$ Define
By $T(x, y) = (x + 2y, 3x + 2y)$ for all $(x, y) \in \mathbb{R}^2$ and $N: \mathbb{R}^2 \times (0, \infty) \to [0,1]$ Define fuzzy norm in example(2.4) and T is linear operator have eigenvalues $\lambda = -1, \lambda = 4$

Theorem (3.6) : Let $(X, N,*)$ be a fuzzy normed spaces over F and $T \in L(X)$ if x one eigenvector of T corresponding to the eigenvalue λ and α is any non zero scalar then αx is also an eigenvector of T corresponding to the same eigenvalue λ

proof: since x is an eigenvector of T corresponding to the eigenvalue λ then $x \neq 0$ and $T(x) = \lambda x$ since $x \neq 0$ and $\alpha \neq 0 \Longrightarrow \alpha x \neq 0$
$T(\alpha x) = \alpha T(x) = \alpha(\lambda x) = (\alpha\lambda)x = (\lambda\alpha)x = \lambda(\alpha x)$
There fore αx is an eigenvector of T corresponding to the eigenvalue λ

Remark(3.7): Corresponding to an eigenvalue λ there may correspond more Than one eigenvectors

Theorem (3.8) : Let $(X, N,*)$ be a fuzzy normed spaces over F and $T \in L(X)$ if x an eigenvector of T,then x cannot correspond to more than one eigenvalues of T

proof: Let be an eigenvector of T corresponding to two distinct eigenvalues
λ_1 and λ_2 of T $T(x) = \lambda_1 x$ and also $T(x) = \lambda_2 x$ .therefore we have
$\lambda_1 x = \lambda_2 x \Longrightarrow \lambda_1 x - \lambda_2 x = 0 \Longrightarrow (\lambda_1 - \lambda_2) x = 0$
since $x \neq 0 \Longrightarrow \lambda_1 - \lambda_2 = 0 \Longrightarrow \lambda_1 = \lambda_2$
and α is any non-zero scalar ,then αx is also an eigenvector of T corresponding to the same eigenvector λ

Definition (3.9) : [10] Let $(X, N_1,*)$ and $(Y, N_2,*)$ be a fuzzy normed spaces .A linear operator $T : (X, N_1,*) \to (Y, N_2,*)$ is said to be fuzzy bounded if and only if there exists $r > 0$, such that for each $t > 0$

$$N_2(T(x), t) \geq N_1\left(x, \frac{t}{r}\right), \quad \forall x \in X$$

Remark (3.10) : [13] Let $(X, N_1,*)$ and $(Y, N_2,*)$ be a fuzzy normed spaces over F, $FB(X, Y)$ is the space of all fuzzy bounded linear operator from X in to Y.

Definition (3.11) : [13] Let $(X, H,*)$ and $(Y, H,*)$ be a fuzzy Hilbert spaces over F, and let $T \in FB(X, Y)$. A fuzzy Hilbert-adjoint operator T^* of T is the operator $T^*: (Y, H,*) \to (X, H,*)$ such that :

$\sup\{t \in \mathbb{R}, H(T(x), y, t) < 1\} =$

$\sup\{t \in \mathbb{R}, H(x, T^*(y), t) < 1\}$ for all $x \in X$ and $y \in Y$.

Remark (3.12) : [13] We denoted $FB(X, X)$ by $FB(X)$.

Theorem (3.13) : [13] (Some Properties of fuzzy Hilbert-adjoint operator**)**

Let $(X, H,*)$ and $(Y, H,*)$ be a fuzzy Hilbert spaces over F, and let $S, T \in FB(X, Y)$. Then we have :

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(a)$\sup\{t \in \mathbb{R}, H(T^*(y), x, t) < 1\} = \sup\{t \in \mathbb{R}, H(y, T(x), t) < 1\}$ for all $x \in X$ and $y \in Y$

(b)$(T+S)^* = T^* + S^*$

(c)$(T^*)^* = T$

Definition (3.14) : [13] Let $(X, H, *)$ be a fuzzy Hilbert space over F and let $T \in B(X)$. T is said to be Normal if $T \circ T^* = T^* \circ T$.

Theorem (3.15) : [13] Let $(X, H, *)$ be a fuzzy Hilbert space over F, and $T \in FB(X)$. Then $T = 0$ if and only if $\sup\{t \in \mathbb{R}, H(T(x), T(x), t) < 1\} = 0$ for all $x \in X$.

Theorem (3.16) : Let T be a normal operator on a Fuzzy finite dimensional Hilbert space X over F then

(1) $T - \lambda\,\mathrm{I}$ is normal

(2) Every eigenvector of T is also eigenvector for T^*

Proof:

(1) since T is normal $\Longrightarrow T \circ T^* = T^* \circ T$

$(T - \lambda\mathrm{I})^* = T^* - \bar{\lambda}\,\mathrm{I}$

$(T - \lambda\mathrm{I}) \circ (T - \lambda\mathrm{I})^* = (T - \lambda\mathrm{I}) \circ (T^* - \bar{\lambda}\,\mathrm{I})$

$= T \circ T^* - \bar{\lambda}T - \lambda T^* + \lambda\bar{\lambda}$

$(T - \lambda\mathrm{I})^* \circ (T - \lambda\mathrm{I}) = (T^* - \bar{\lambda}\,\mathrm{I}) \circ (T - \lambda\mathrm{I})$

$= T^* \circ T - \bar{\lambda}T - \lambda T^* + \bar{\lambda}\lambda$

$(T - \lambda\mathrm{I})^* \circ (T - \lambda\mathrm{I}) = T \circ T^* - \bar{\lambda}T - \lambda T^* + \lambda\bar{\lambda}$

$\Longrightarrow (T - \lambda\mathrm{I})^* \circ (T - \lambda\mathrm{I}) = (T - \lambda\mathrm{I}) \circ (T - \lambda\mathrm{I})^*$

Therefore $T - \lambda\mathrm{I}$ is normal

$\sup\{t \in \mathbb{R}, H(T(x), T(x), t) < 1\}$
$= \sup\{t \in \mathbb{R}, H(x, T^*(T(x)), t) < 1\}$
$= \sup\{t \in \mathbb{R}, H(x, T^* \circ T(x), t) < 1\}$
$= \sup\{t \in \mathbb{R}, H(x, T \circ T^*(x), t) < 1\}$
$= \sup\{t \in \mathbb{R}, H(x, T(T^*(x)), t) < 1\}$

$= \sup\{t \in \mathbb{R}, H(T^*(x), T^*(x), t) < 1\}$

Since $T - \lambda\mathrm{I}$ is normal, therefore $x \in X$ we have
$\sup\{t \in \mathbb{R}, H((T - \lambda\mathrm{I})(x), (T - \lambda\mathrm{I})(x), t) < 1\}$
$= \sup\{t \in \mathbb{R}, H((T - \lambda\mathrm{I})^*(x), (T - \lambda\mathrm{I})^*(x), t) < 1\}$

Since $T(x) = \lambda x \Longrightarrow T(x) - \lambda\mathrm{I}(x) = 0 \Longrightarrow (T - \lambda\mathrm{I})(x) = 0 \Longrightarrow (T - \lambda\mathrm{I}) = 0$ then by theorem (3.15)

$\sup\{t \in \mathbb{R}, H((T - \lambda\mathrm{I})(x), (T - \lambda\mathrm{I})(x), t) < 1\} = 0$
$\Longrightarrow \sup\{t \in \mathbb{R}, H((T - \lambda\mathrm{I})^*(x), (T - \lambda\mathrm{I})^*(x), t) < 1\} = 0 \Longrightarrow$
$(T - \lambda\mathrm{I})^* = 0$ by theorem (3.15), for each $x \in X \Longrightarrow$
$(T - \lambda\mathrm{I})^*(x) = 0$
$\Longrightarrow T^*(x) - \bar{\lambda}\,\mathrm{I}(x) = 0 \Longrightarrow T^*(x) = \bar{\lambda}\,\mathrm{I}(x) \Longrightarrow T^*(x) = \bar{\lambda}x$

Therefore x is eigenvector of T^* and corresponding eigenvalue is $\bar{\lambda}$

Theorem (3.17) : [6] Let B be a non-empty subset of a fuzzy pre-Hilbert space $(X, H, *)$, then $B^{\perp}$ is closed fuzzy subspace

Proof: Since $H(0, y, t) = h(t)$, $\forall y \in B \Longrightarrow 0 \in B^{\perp}$ then $B^{\perp} \neq \emptyset$

Let $x, y \in B^{\perp}$ and $\alpha, \beta, r \in \mathbb{R}$

$H(x, z, t) = h(r) \quad \forall z \in B$
$H(y, z, t) = h(r) \quad \forall z \in B$

For every $\forall z \in B$ we have:

If $\alpha>0, \beta>0$

$$H(\alpha x + \beta y, z, t) = sup_{s+t=r}\left(H\left(x, z, \frac{t}{\alpha}\right) * H\left(y, z, \frac{s}{\beta}\right)\right)$$

$$= h\left(\frac{t}{\alpha}\right) * h\left(\frac{s}{\beta}\right) == h(t) * h(s) = h(r) \quad \forall r \in \mathbb{R}$$

If $\alpha<0, \beta<0$

$$H(\alpha x + \beta y, z, t) = sup_{s+t=r}\left(1 - H\left(x, z, \frac{t}{-\alpha}\right) * 1 - H\left(y, z, \frac{s}{-\beta}\right)\right) = h(r) \quad \forall r \in \mathbb{R}$$

If $\alpha<0, \beta = 0$ or $\alpha = 0, \beta<0$, or $\alpha>0, \beta = 0, \alpha = 0, \beta>0$
$H(\alpha x + \beta y, z, t) = h(r) \quad \forall\ r \in \mathbb{R}$
$\Longrightarrow \alpha x + \beta y \in B^{\perp}$

Therefore $B^{\perp}$ is a fuzzy subspace

Let $x \in \overline{B^{\perp}}$ $\exists$ $\{x_n\}$ in $B^{\perp}$ such that $x_n \longrightarrow x$
Let $y \in B \Longrightarrow H(x_n, y, t) = h(t)$ $\forall n \in \mathbb{Z}^+$
And $t \in \mathbb{R}$ $(x_n \in B^{\perp}$ $\forall n \in \mathbb{Z}^+)$
Since $x_n \longrightarrow x \Longrightarrow H(x_n, y, t) \longrightarrow H(x, y, t)$
$\Longrightarrow H(x, y, t) = h(t)$ for all $y \in B$
$\Longrightarrow x \in B^{\perp} \Longrightarrow \overline{B^{\perp}} = B^{\perp}$
$\Longrightarrow B^{\perp}$ is closed fuzzy subspace

Definition (3.18): Let M be a closed of a fuzzy Hilbert space X and $x \notin \mathrm{M}$ said that projection of $x \in X$ onto M if there is $z \in \mathrm{M}$

$$N(x - z, t) = sup\left\{\frac{t}{t + \|x - y\|} : y \in \mathrm{M}, t>0\right\}$$,we write

$y = P_{\mathrm{M}}(x)$

Theorem (3.19): If M is subspace of a fuzzy Hilbert space X,for $x \in X$ there exist a unique $y \in \mathrm{M}$ such that $x-y \perp \mathrm{M}$ and $y = P_{\mathrm{M}}(x)$

Proof: Define $\langle .,. \rangle : X \times X \to \mathbb{R}$ by $\langle x,y \rangle = \sup\{\, t \in \mathbb{R}, H(x,y,t) < 1\,\}$. from theorem (2.12) ,we have $(X, \langle .,. \rangle)$ is a pre-Hilbert space. Also $\|x\| = (\sup\{\, t \in \mathbb{R}\,, H(x,x,t) < 1\})^{\frac{1}{2}}$ from corollary (2.13) we have $(X, \|.\|)$ is a normed space since X is a fuzzy Hilbert space then $(X, \|.\|)$ is complete normed space then X is Hilbert space then by using [12] for $x \in X$ there exist a unique $y \in \mathrm{M}$ such that $x - y \perp \mathrm{M}$ and $y = P_{\mathrm{M}}(x)$

Then $\langle x-y, z \rangle = 0$ $\forall z \in \mathrm{M}$ then $\sup\{\, t \in \mathbb{R}, H(x,y,t) < 1\,\} = 0$ there fore $x - y \perp \mathrm{M}$ $\forall z \in \mathrm{M}$ in X fuzzy Hilbert space , since $y = P_{\mathrm{M}}(x)$ then by [12] there is $b \in \mathrm{M}$ such that

$\|x-b\| = inf\{\|x-y\| : y \in \mathrm{M}\}$ then

$\|x-b\| \leq \|x-y\|, y \in \mathrm{M}$ $\Longrightarrow$ $t + \|x-b\| \leq t + \|x-y\|$, t>0 $\Longrightarrow$ $\frac{t}{t+\|x-y\|} \leq \frac{t}{t+\|x-b\|}$, $y \in \mathrm{M}$, t>0

there fore $N(x-b,t) = sup\left\{\frac{t}{t+\|x-y\|} : y \in \mathrm{M}, t>0\right\}$

Theorem (3.20): If M is subspace of a fuzzy Hilbert space X,then $X = \mathrm{M} \oplus \mathrm{M}^{\perp}$, that is each $x \in X$ can be uniqully decomposed from $x = y + z$ with $y \in \mathrm{M}$, $z \in \mathrm{M}^{\perp}$

Proof: For all $x \in X$ and M is subspace there exist y so that $x = x - y + y$ with

$x - y \in \mathrm{M}^{\perp}$ and $y \in \mathrm{M}$ such that $y = P_{\mathrm{M}}(x)$ and

$z = x - y \Longrightarrow x = y + z \Longrightarrow$

$X = \mathrm{M} + \mathrm{M}^{\perp}$ also since $\mathrm{M} \cap \mathrm{M}^{\perp} = \{0\}$ by theorem (2.15) ,there fore $X = \mathrm{M} \oplus \mathrm{M}^{\perp}$

Theorem (3.21): If M is subspace of a fuzzy Hilbert space X, then M is fuzzy closed iff $\mathrm{M} = \mathrm{M}^{\perp\perp}$

Proof: Since $\mathrm{M} \subset \mathrm{M}^{\perp\perp}$ by theorem (2.15) ,we show that $\mathrm{M}^{\perp\perp} \subset \mathrm{M}$

Let $x \in \mathrm{M}^{\perp\perp}$ then by theorem(3.20) $x = y + z$,where $y \in \mathrm{M}$,$z \in \mathrm{M}^{\perp}$ since

$\mathrm{M} \subset \mathrm{M}^{\perp\perp}$ and $\mathrm{M}^{\perp\perp}$ is subspace $z = x - y$ $\mathrm{M}^{\perp\perp}$ but $z \in \mathrm{M}^{\perp} \Longrightarrow z \in \mathrm{M}^{\perp\perp} \cap \mathrm{M}^{\perp}$

Since $\mathrm{M}^{\perp\perp} \cap \mathrm{M}^{\perp} = \{0\}$ then $z = 0$,thus $x = y \in \mathrm{M}$ there fore $\mathrm{M}^{\perp\perp} \subset \mathrm{M}$ thus $\mathrm{M} = \mathrm{M}^{\perp\perp}$

Conversely suppose $\mathrm{M} = \mathrm{M}^{\perp\perp}$ since $(\mathrm{M}^{\perp})^{\perp} = \mathrm{M}^{\perp\perp}$ is close set then M is close set .

Theorem (3.22): Let M be a closed subspace of a fuzzy Hilbert space X over F, and let $T \in FB(X)$.Then M is invariant under T iff $\mathrm{M}^{\perp}$ is invariant under T^*

Proof: Suppose M is invariant under T

Let $y \in \mathrm{M}^{\perp}$ .To prove that $T^*(y) \in \mathrm{M}^{\perp}$(i.e. $T^*(y) \perp \mathrm{M}$)

Let $x \in \mathrm{M}$,since M is invariant under $T \Longrightarrow T(x) \in \mathrm{M}$

Since $y \in \mathrm{M}^{\perp} \Longrightarrow \sup\{\, t \in \mathbb{R}\,, H(T(x),y,t) < 1\} = 0 \Longrightarrow$

$\sup\{\, t \in \mathbb{R}\,, H(x, T^*(y), t) < 1\} = 0$ .Thus $T^*(y) \perp \mathrm{M}$

Conversely suppose that $\mathrm{M}^{\perp}$ is invariant under T^*.

Since $\mathrm{M}^{\perp}$ is closed subspace of a fuzzy Hilbert space X by theorem (3.17) and since $\mathrm{M}^{\perp}$ is invariant under T^*, therefore by first case $(\mathrm{M}^{\perp})^{\perp}$ is invariant under $(T^*)^*$but $(\mathrm{M}^{\perp})^{\perp} = \mathrm{M}^{\perp\perp} = \mathrm{M}$ and $(T^*)^* = T^{**} = T$

Therefore M is invariant under T

Definition (3.23) : Let M be a closed subspace of a fuzzy Hilbert X over F And let $T \in FB(X)$ .We say that T is reduced by M if both M and $\mathrm{M}^{\perp}$ are Invariant under T .If T is reduced by M , then some times we also say that M reduces T

Theorem (3.24):A closed subspace M of a fuzzy Hilbert X over F reduces an operator T iff M is invariant under both T and T^*

Proof: Suppose M reduces .Then by the definition of reducibility both M and $\mathrm{M}^{\perp}$ are invariant under T by theorem (3.22) ,if $\mathrm{M}^{\perp}$ is invariant under T

Then $(\mathrm{M}^{\perp})^{\perp}$, i.e. M is invariant T^* .then M is invariant under both $T$and T^*

Conversely suppose that M is invariant under both T and T^*

Since M is invariant under T^* ,therefore by theorem (3.22), $\mathrm{M}^{\perp}$ is invariant under $(T^*)^*$,i.e. T .thus both M and $\mathrm{M}^{\perp}$ are invariant under T .therefore M reduces T

Definition (3.25) : Let X be a fuzzy normed space over F ,$T \in FB(X)$ and let λ be eigenvalue of T then set consting of all eigenvectors of T which correspond to eigenvalue λ together with the vector 0 is called eigenspace of T corresponding to the eigenvalue λ and is denoted by M_{λ}

(1)Since by definition an eigenvector is non zero vector,there fore the set M_{λ} necessary contains some non zero vector

(2)Since by definition of M_{λ} a non zero vector x is in M_{λ} iff $T(x) = \lambda x$

Also it is given that the vector 0 is in M_{λ} the vector 0 defintly satisfies

The equation $T(x) = \lambda x$ there for

$\mathrm{M}_{\lambda} = \{x \in X : T(x) = \lambda x\,\}$
$\quad = \{x \in X : (T - \lambda\mathrm{I})(x) = 0\,\}$

Thus M_{λ} is null space (or kernel of linear operator $T - \lambda\mathrm{I}$ on X). Hence M_{λ} is a subspace of X

(3)Let $x \in X$ since M_λ is a subspace of X and $\lambda \in F$ $\Longrightarrow \lambda x \in M_\lambda$ since $x \in M_\lambda \Longrightarrow T(x) = \lambda x \Longrightarrow T(x) \in M_\lambda \Longrightarrow M_\lambda$ is an invariant under T

from (1),(2) and (3) we have M_λ is non zero subspace of X invariant under of T

(4)If $T \in FB(X)$ then M_λ is closed subspace of X , M_λ is called eigenspace of T ,corresponding to the eigenvalue λ

Theorem (3.26): If T be a normal operator on n dimensional fuzzy Hilbert Space X over F ,then each eigenspace reduces T

Proof: Let x_i belong to M_i the eigenspace of T and corresponding eigenvalue be λ_i, so that $T(x_i) = \lambda_i x_i$ since T is normal then by theorem(3.16) eigenvalue for T^*(i.e. $T^*(x_i) = \overline{\lambda_i} x_i$)since M_i is a subspace $\Longrightarrow$ $\overline{\lambda_i} x_i \in M_i \Longrightarrow T^*(x_i) \in M_i \Longrightarrow M_i$ is invariant under T^*,but M_i is invariant under T then by theorem(3.24) M_i is reduces T

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بعض الخصائص لنظرية الاطياف في فضاء هلبرت الضبابي

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المستخلص :

في هذا البحث قدمنا بعض التعاريف المتعلقة بنظرية الطيفية للمؤثر الخطي T المعرف على الفضاء المعياري الضبابي وكما سنبرهن بعض الحقائق المتعلقة بلقيم الذاتية والمتجه الذاتي في فضاء هلبرت الضبابي وكما نبرهن كل فضاء ذاتي في فضاء هلبرت الضبابي ذو البعد n يختزل المؤثر الخطي T اذا كان T مؤثر خطي سوي .