Journal of Al-Qadisiyah for Computer Science and Mathematics
3nd. Sinentific Conference 19-20/ APRIL -2011
Vol 3 No.2 Year 2011

Page 37-47

On Uncertain Martingale

Noori .F.A.AL-Mayahi ***Zainab .H.A.AL-Zaubaydi***

Department of Mathematics
College of Computer and Mathematics Science
AL-Qadisiyah University

Abstract

We introduce some a new properties of uncertain conditional expectation, also we give a new kind of martingale and study some theorems related with it.

Key words

uncertain measure, uncertain variable, conditional uncertain measure, uncertain conditional expectation and uncertain martingale.

1-Introduction

Probability theory often profitable to interpret results in terms of a gambling situation . for example ,if $X_1, X_2,....$ is a sequence of random variables, we may think of X_n as our total winnings after n trials in a succession of games. Having survived the first n trial, our expected fortune after trial $n+1$ is $E(X_{n+1} \mid X_1,....X_n)$. If equals X_n, the game is "**fair**" since the expected gain on trial $n+1$ is $E(X_{n+1} - X_n \mid X_1,....X_n) = X_n - X_n = 0$. If $E(X_{n+1} - X_n \mid X_1,....X_n) \geq X_n$.the game is "**favorable**" and $E(X_{n+1} - X_n \mid X_1,....X_n) \leq X_n$, the game is "**unfavorable** "[2].Uncertainty theory was founded by Liu [3] in 2007 and refined by Liu [5] in 2010. Let (Ω, F) be a measurable space, where Ω is a set and F is a $\sigma-$field on Ω. A subset A of Ω is called measurable (measurable with respect to the $\sigma-$field F), if $A \in F$, i.e., any member of F is called a measurable set [2]. A set function μ from F to $[0,1]$ is a real-valued set function μ defined on $\sigma-$field F is called an uncertain measure , if it satisfies the following four axioms:

Axiom 1.(**Normality Axiom**) $\mu(\Omega) = 1$.

Axiom 2.(**Self-duality Axiom**) $\mu(A) + \mu(A^c) = 1$ for any event A.

Journal of Al-Qadisiyah for Computer Science and Mathematics
3nd. Sinentific Conference 19-20/ APRIL -2011
Vol 3 No.2 Year 2011

Axiom 3.(**Countable subadditivity Axiom**) For every countable sequence of events { A_i }, we have $\mu(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} \mu(A_i)$ (1)

Axiom 4.(**Product measure Axiom**) Let Ω_k be a nonempty sets on which μ_k are uncertain measures, $k = 1,2,.....,n$, respectively. Then the product of uncertain measures μ_k is an uncertain measure μ on the product $\sigma -$field $\Omega_1 \times \Omega_2 \times ......\times \Omega_n$ satisfying

$$\mu(\prod_{k=1}^{n} A_k) = \min_{1\le k\le n} \mu_k(A_k) \qquad (2)$$

That is, for each $A \in \Omega$, we have

$$\mu(A) = \begin{cases} \sup\limits_{A_1\times A_2\times......\times A_n \subset A} \left\{\min\limits_{1\le k\le n} \mu_k(A_k)\right\}, & if \sup\limits_{A_1\times A_2\times......\times A_n \subset A} \left\{\min\limits_{1\le k\le n} \mu_k(A_k)\right\} > 0.5 \\ 1 - \sup\limits_{A_1\times A_2\times......\times A_n \subset A^c} \left\{\min\limits_{1\le k\le n} \mu_k(A_k)\right\}, & if \sup\limits_{A_1\times A_2\times.....\times A_n \subset A^c} \left\{\min\limits_{1\le k\le n} \mu_k(A_k)\right\} > 0.5 \\ 0.5, & o.w \end{cases}$$

(3)

Then the triple (Ω, F, μ) is called an uncertainty space .

Remark(1-1)

The probability measure is not uncertain measure as shown by the following example:

Example(1-2)[4]

Let $g: R \to R$ is a nonnegative and integrable function such that $\int_R g(x)dx = 1$. Define $P: \beta(R) \to R$ by $P(A) = \int_A g(x)dx$, is a probability measure but not uncertain measure.

Ans:

Step1: we prove the normality .i.e., $P(R) = 1$.

$$P(R) = \int_R g(x)dx = 1$$

Step2: we prove the self-duality .i.e., $P(A) + P(A^c) = 1$.

Since $P(A \cup A^c) = \int_{A\cup A^c} g(x)dx = \int_R g(x)dx = 1$,and $P(A) + (A^c) = P(A \cup A^c)$

Thus, $P(A) + (A^c) = P(A \cup A^c) = 1$

Step3: Let $\{A_i\}$ be a sequence of sets in $\beta(R)$, then $P(\bigcup_{i=1}^{\infty} A_i) = \int_{\bigcup_{i=1}^{\infty} A_i} g(x)dx = \int_R g(x)dx = 1$, and $\sum_{i=1}^{\infty} P(A_i) = 1$

Then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$, and then P is not uncertain measure.

Example(1-3)[3,4,5,6]

Suppose that $g : R \to R$ is nonnegative satisfying $\sup\{g(x)+g(y) : x \neq y\} = 1$, define $\mu : \beta(R) \to R$, the set function

$$\mu(A) = \begin{cases} \sup\{g(x) : x \in A\}, & \sup\{g(x) : x \in A\} < 0.5 \\ 1-\sup\{g(x) : x \in A^c\}, & \sup\{g(x) : x \in A\} \geq 0.5 \end{cases} \tag{4}$$

for all $A \in \beta(R)$ is an uncertain measure.

Definition(1-4)[2]

Let (Ω, F) and (Ω', F') be two measurable spaces. A function $g : \Omega \to \Omega'$ is said to be a measurable with respect to F and F', if $g^{-1}(A) \in F$ for all $A \in F'$.

Definition(1-5)[5,6]

an uncertain variable is a measurable function from an uncertainty space (Ω, F, μ) to the set of real numbers, i.e., for any Borel set B of real numbers. The set $\{X \in B\} = \{x \in \Omega \setminus X(x) \in B\}$ is an event.

Definition(1-6)[2]

Let $A \subseteq \Omega$. A function $I_A : \Omega \to R$ defined by :

$$I_A(\omega) = \begin{cases} 1 & , \omega \in A \\ 0 & , \omega \notin A \end{cases} \tag{5}$$

Is called indicator function or (characteristic function) of A.

Definition(1-7)

Let μ and λ be two uncertain measures on measurable space (Ω, F). We say that λ is absolute continuous with respect to μ (written $\lambda << \mu$) if $\lambda(A) = 0$

for every $A \in F$ with $\mu(A) = 0$,i.e., for all $A \in F$ with $\mu(A) = 0$,we have $\lambda(A) = 0$.

Definition(1-9)[1]

A filtration $\{F_n, n \in N\}$ is a sequence of sub $\sigma -$fields of F such that for all $n \in N$, $F_n \subset F_{n+1}$.

Definition(1-10)[6]

Let X be an uncertain variable. Then the expected value of X is defined by

$$E(X) = \int_{0}^{+\infty} \mu(X \geq r)dr - \int_{-\infty}^{0} \mu(X \leq r)dr \qquad (6)$$

Provided that at least one of the two integrals is finite.

Lemma(1-11) (Jensen's Inequality)[5,6]

Let X be an uncertain variable and $f : R \to R$ a convex function . If $E(X)$ and $E(f(X))$ are finite, then

$$f(E(X)) \leq E(f(X)) \qquad (7)$$

Especially, when $f(X) = |x|^p$ and $p \geq 1$, we have $|E(X)|^p \leq E(|X|^p)$.

Proof:

Since f is a convex function, for each y, there exists a number K such that $f(x) - f(y) \geq K.(x - y)$. Replacing x with X and y with $E(X)$, we obtain $f(X) - f(E(X)) \geq K.(X - E(X))$.
Taking the expected value on both sides, we have
$E(f(X)) - f(E(X)) \geq K.(E(X) - E(X)) = 0$.
Which proves the inequality.

2- Radon-Nikodym theorem on uncertain measure

Let μ and λ be two uncertain measure on the $\sigma -$field F of subsets of Ω. Assume that $\lambda << \mu$, then there is a Borel measurable function $g : \Omega \to R$ such that $\lambda(A)=\int_A g d\mu$, for all $A\in F$. g is called the Radon-Nikodym derivative of λ with respect to μ . It is sometimes denoted by $[\frac{d\lambda}{d\mu}]$,i.e. $g=[\frac{d\lambda}{d\mu}]$.

Proof:

Let $\{g_n\}$ be a bounded increasing sequence of nonnegative measurable function in F .
Since every monotonic and bounded sequence is converge.
Then there exists a unique function g in F such that $\lim_{n\to\infty} g_n = g$
Now, we must to show that $\lambda(A)=\int_A g d\mu$ for all $A\in F$.
Define $\lambda_1(A)=\lambda(A)-\int_A g d\mu$ for all $A\in F$, and $\lambda_1 << \mu$
Since $\mu(A)=0$ for all $A\in F$, then $\lambda_1(A)=0$ .By absolute continuity thus $\lambda(A)=\int_A g d\mu$ for all $A\in F$.

4- The conditional uncertain measure

Definition(3-1)

Let (Ω, F, μ) be an uncertainty space and $A, B\in F$, then the conditional uncertain measure of A given B is defined by

$$\mu_B(A)=\mu(A\,|\,B)=\begin{cases}\dfrac{\mu(A\cap B)}{\mu(B)}, & if\ \dfrac{\mu(A\cap B)}{\mu(B)}<0.5\\ 1-\dfrac{\mu(A^c\cap B)}{\mu(B)}, & if\ \dfrac{\mu(A^c\cap B)}{\mu(B)}<0.5\\ 0.5, & o.w\end{cases} \qquad (8)$$

Provided that $\mu(B)>0$.

Definition (3-2)

Let (Ω, F, μ) be an uncertainty space and let G be a sub $\sigma-$field of F . Then the conditional expectation of X given G is any uncertain variable Z which satisfies the following two properties:

(a) Z is $G-$measurable.
(b) if $A \in G$, then $\int_A X d\mu = \int_A Z d\mu$.
We denote Z by $E(X \mid G)$.

Theorem (3-3)

If X, X_1, X_2, …..be an uncertain variables on (Ω, F, μ), a and b real numbers. Then

1- If $X = a$ a.e., then $E(X \mid G) = a$ a.e.[μ]
2- If X is $G-$measurable, then $E(X \mid G) = X$ a.e.[μ]
3- $E(E(X \mid G)) = E(X)$.[μ]
4- $E(aX_1 + bX_2 \mid G) = aE(X_1 \mid G) + bE(X_2 \mid G)$ a.e.[μ]
5- If $G = \{\phi, \Omega\}$, then $E(X \mid G) = E(X)$ a.e.[μ]
6- If $X \geq 0$ a.e., then $E(X \mid G) \geq 0$ a.e.[μ]
7- If $X_1 \leq X_2$ a.e., $E(X_1 \mid G) \leq E(X_2 \mid G)$ a.e.[μ]
8- $|E(X \mid G)| \leq E(|X| \mid G)$ a.e.[μ]
9- If Y is $G-$measurable and XY is integrable then $E(XY \mid G) = YE(X \mid G)$ a.e.[μ]
10- If X_n and X are integrable, and if either $X_n \uparrow X$ or $X_n \downarrow X$, then $E(X_n \mid G) \to E(X \mid G)$ a.e.[μ]

Proof:

1-If $X = a$ a.e., then it is $G-$ measurable and $\int_A X d\mu = \int_A a d\mu$ for all $A \in G$ and then $E(X \mid G) = a$ a.e. [μ]

2- If X is $G-$measurable and $\int_A X d\mu = \int_A E(X \mid G) d\mu$ for all $A \in G$**,** then $E(X \mid G) = X$ a.e. [μ]

3- Take $A = \Omega$ in condition (b) of definition (3-2), we have that $\int_\Omega X d\mu = \int_\Omega E(X \mid G) d\mu$ and this implies that $E(E(X \mid G)) = E(X)$. a.e. [μ]

4- It's clearly that $E(aX_1 + bX_2 \mid G)$ is $G-$measurable and if $A \in G$, apply condition (b) of definition (3-2) to X_1 and X_2 to see that
$\int_A (aE(X_1 \mid G) + bE(X_2 \mid G)) d\mu = a\int_A X_1 d\mu + b\int_A X_2 d\mu = \int_A (aX_1 + bX_2) d\mu$.

5- Since $\int_A X d\mu = \int_A E(X) d\mu$ for all $A=\phi$ or $A=\Omega$, we have $E(X \mid G)=E(X)$ a.e.[μ]

6- Take $A=\{E(X \mid G)<0\} \in G$. then by condition (b) of definition (3-2), we have, $0 \geq \int_A E(X \mid G) d\mu = \int_A X d\mu \geq 0 \Rightarrow \mu(A)=0$.

7- Since $X_1 \leq X_2$ a.e., then $Z=X_2-X_1 \geq 0$ a.e., by (6) we have $E(X_2 \mid G)-E(X_1 \mid G) \geq 0$ a.e., thus $E(X_1 \mid G) \leq E(X_2 \mid G)$ a.e.[μ]

8- Since $-|X| \leq X \leq |X|$, it follows from (7), that $|E(X \mid G)| \leq E(|X| \mid G)$ a.e. [μ]

9- Let $Y=I_B$ (I_B is indicator function)for some $B \in G$. Then $\int_A Y E(X \mid G) d\mu = \int_A I_B (X \mid G) d\mu = \int_{A\cap B} E(X \mid G) d\mu = \int_{A\cap B} X d\mu = \int_A Y X d\mu$

Thus the condition (b) of definition (3-2), holds in this case.

10- It follows from definition (3-2), by letting $Z=E(X \mid G)$ and for all $A \in G$. We have $\int_A Z d\mu = \lim_{n\to\infty} \int_A Z_n d\mu = \int_A Z d\mu$ .a.e[μ]

Thus Z satisfies both (a) and (b) of definition (3-2), and therefore equals $E(X \mid G)$

4-Uncertain martingale

Definition(4-1)

An uncertain stochastic process is a family of uncertain variables defined on the same uncertainty space.

Definition(4-2)

An uncertain stochastic process $X=\{X_n, n \in N\}$ is an uncertain adapted to the filtration $\{F_n, n \in N\}$ if for all n, X is F_n −measurable.

Definition (4-3)

An uncertain stochastic process $X=\{X_n, F_n, n \in N\}$ is said to be an uncertain martingale if it is satisfying the following conditions:

(1) X is an uncertain adapted to filtration $\{F_n, n \in N\}$.

(2) X_n is integrable for all $n \in N$.

(3) $E\{X_{n+1} \mid F_n\} = X_n$ a.e.[μ], for all $n \in N$ (9)

Definition(4-4)

An uncertain stochastic process $X = \{X_n, F_n, n \in N\}$ is said to be an uncertain sub-martingale (resp, uncertain super-martingale) with respect to the filtration $\{F_n, n \in N\}$ if it is satisfying the following conditions:

(1) X is an uncertain adapted to filtration $\{F_n, n \in N\}$.

(2) X_n is integrable for all $n \in N$.

(3) $E\{X_{n+1} \mid F_n\} \geq X_n$ a.e.[μ], (resp, $E\{X_{n+1} \mid F_n\} \leq X_n$ a.e. [μ]) for all $n \in N$ (10)

Example(4-5)

An uncertain stochastic process $X = \{X_n, F_n, n \in N\}$ is an uncertain sub-martingale iff $-X = \{-X_n, F_n, n \in N\}$ is an uncertain super-martingale.

Ans:

Since $E\{X_{n+1} \mid F_n\} \geq X_n$ a.e., iff $E\{-X_{n+1} \mid F_n\} \leq -X_n$ a.e. for all $n \in N$.

$\Rightarrow$ $X = \{X_n, F_n, n \in N\}$ is an uncertain sub-martingale iff $-X = \{-X_n, F_n, n \in N\}$ is an uncertain super-martingale.

Example(4-6)

If $X = \{X_n, F_n, n \in N\}$ is an uncertain sub-martingale and K a constant, then $\max\{X, k\} = \{\max\{X_n, K\}, F_n, n \in N\}$ is an uncertain sub-martingale.

Ans:

$\max\{X_{n+1}, K\} \geq X_{n+1} \Rightarrow E\{\max\{X_{n+1}, K\} \mid F_n\} \geq E\{X_{n+1} Z F_n\}$

Since $E\{X_{n+1} \mid F_n\} \geq X_n \Rightarrow E\{\max\{X_{n+1}, K\} \mid F_n\} \geq X_n$

And similarly

$E\{\max\{X_{n+1}, K\} \mid F_n\} \geq K \Rightarrow E\{\max\{X_{n+1}, K\} \mid F_n\} \geq \max\{X_n, K\}$

$$\Rightarrow \max\{X, K\} = \{\max\{X_n, K\}, F_n, n \in N\}$$

is an uncertain sub-martingale.

Example(4-7)

If $X = \{X_n, F_n, n \in N\}$ is an uncertain super-martingale and K a constant, then $\min\{X, k\} = \{\min\{X_n, K\}, F_n, n \in N\}$ is an uncertain super-martingale.

Ans:
By the same way we get the answer.

Theorem (4-8)

Let $X=\{X_n,F_n,n\in N\}$ is an uncertain martingale and $f:R\to R$ a convex function, such that $f(X_n)$ is integrable for all n, then $f(X)=\{f(X_n),F_n,n\in N\}$ is an uncertain sub-martingale.

Proof:
By Jensen's inequality for uncertain conditional expectations we have,
$E\{f(X_{n+1}\,|\,F_n\}\geq f\{E(X_{n+1}\setminus F_n)\}$. Since $X=\{X_n,F_n,n\in N\}$ is an uncertain martingale, it follows that $E\{X_{n+1}\,|\,F_n\}=X_n$
then $E\{f(X_{n+1}\,|\,F_n)\}\geq f\{E(X_{n+1}\,|\,F_n)\}=f(X_n)$
thus $f(X)=\{f(X_n),F_n,n\in N\}$ is an uncertain sub-martingale.

Theorem (4-9)

Let $X=\{X_n,F_n,n\in N\}$ is an uncertain sub- martingale and $f:R\to R$ a convex increasing function, such that $f\left(X_n\right)$ is integrable for all n, then $f(X)=\{f(X_n),F_n,n\in N\}$is an uncertain sub-martingale.

Proof:
By Jensen's inequality for uncertain conditional expectations we have,

$E\{f(X_{n+1}\,|\,F_n)\}\geq f\{E(X_{n+1}\,|\,F_n)\}$
Since $X=\{X_n,F_n,n\in N\}$ is an uncertain
sub- martingale, it follows that $E\{X_{n+1}\,|\,F_n\}\geq X_n$ a.e.,
since is increasing function, then $f\{E(X_{n+1}\,|\,F_n)\}\geq f(X_n)$ and then
$E\{f(X_{n+1}\,|\,F_n)\}\geq f(X_n)$
Thus $f(X)=\{f(X_n),F_n,n\in N\}$ is an uncertain sub-martingale.

Theorem (4-10) ***(*** **Doop Decomposition** ***)***

Let $X = \{X_n, F_n, n \in N\}$ is an uncertain sub- martingale with respect to the filtration $\{F_n, n \in N\}$. Then there exists an uncertain martingale $M = \{M_n, F_n, n \in N\}$ and an uncertain process $A = \{A_n, n \in N\}$ such that

1- M is an uncertain martingale relative to $\{F_n, n \in N\}$.

2- A is an increasing uncertain process: $A_n \le A_{n+1}$ a.e.

3- A_n is F_{n-1}-measurable for all $n \in N$.

4- $X_n = M_n + A_n$.

Proof*:*

1- Set $A_0 = 0$, and $A_n = A_{n-1} - (X_{n-1} - E(X_n \mid F_{n-1}))$ for all $n \ge 0$.

$\Rightarrow A_n = \sum_{k=0}^{n-1} (E(X_{k+1} \mid F_n) - X_k)$ for all $n \ge 0$ and $A_{n+1} - A_n = E(X_{n+1} \mid F_n) - X_n$ for all $n \ge 1$.

Since $X = \{X_n, F_n, n \in N\}$ is an uncertain sub- martingale

$A_{n+1} - A_n = E(X_{n+1} \mid F_n) - X_n \Rightarrow E\{X_{n+1} \mid F_n\} \ge X_n$

$\Rightarrow E\{X_{n+1} \mid F_n\} - X_n \ge 0$

$\Rightarrow A_{n+1} - A_n \ge 0$

$\Rightarrow \{A_n\}$ is increasing sequence of uncertain variables.

Take, $M_n = X_n - A_n$ for all n

$$\Rightarrow M_{n+1} - M_n = (X_{n+1} - A_{n+1}) - (X_n - A_n)$$
$$= X_{n+1} - X_n - (A_{n+1} - A_n)$$
$$= X_{n+1} - E(X_{n+1} \mid F_n)$$

$\Rightarrow E(M_{n+1} - M_n \mid F_n) = E(X_{n+1} \mid F_n) - E(E(X_{n+1} \mid F_n) \mid F_n)$

$\Rightarrow E(M_{n+1} \mid F_n) - E(M_n \mid F_n) = E(X_{n+1} \mid F_n) - E(X_{n+1} \mid F_n)$

$\Rightarrow E(M_{n+1} \mid F_n) - M_n = 0$

$\Rightarrow E(M_{n+1} \mid F_n) = \mu_n$

$\Rightarrow M = \{M_n, F_n, n \in N\}$ is an uncertain martingale.

2- Set $A_0 = 0$, and $A_n = A_{n-1} - (X_{n-1} - E(X_n \mid F_{n-1})$ for all $n \ge 0$.

$\Rightarrow A_n = \sum_{k=0}^{n-1} (E(X_{k+1} \mid F_n) - X_k)$ for all $n \ge 0$ and $A_{n+1} - A_n = E(X_{n+1} \mid F_n) - X_n$ for all $n \ge 1$.

Since $X = \{X_n, F_n, n \in N\}$ is an uncertain sub- martingale

$A_{n+1} - A_n = E(X_{n+1} \mid F_n) - X_n \Rightarrow E\{X_{n+1} \setminus F_n\} \ge X_n$

$\Rightarrow E\{X_{n+1} \mid F_n\} - X_n \geq 0$
$\Rightarrow A_{n+1} - A_n \geq 0$
$\Rightarrow \{A_n\}$ is increasing sequence of uncertain variables.

3- since A_n is F_n −measurable and $F_n \subseteq F_{n-1}$, then A_n is F_{n-1}-measurable for all $n \in N$.

4- Let $X = \{X_n, F_n, n \in N\}$ is an uncertain sub- martingale
Set $A_0 = 0$, and $A_n = A_{n-1} - (X_{n-1} - E(X_n \mid F_{n-1})$ for all $n \geq 0$.
$\Rightarrow \{A_n\}$ is increasing sequence of uncertain variables.
Take, $M_n = X_n - A_n$ for all n
$\Rightarrow M = \{M_n, F_n, n \in N\}$ is an uncertain martingale.

Reference

[1] Kannan .D. "An introduction to stochastic process", Elsevier North Holland, Inc,1979.
[2] R.B. Ash," Real analysis and probability ", Academic process, New York, 1992.
[3] Liu .B. "Uncertainty theory ", 2 nd ed ., Springer-Verlag, Berlin, 2007.
[4] Liu .B. "Theory and practice of uncertain programming", 3ed., UTLAB, 2009.
[5] Liu .B. " Uncertainty theory: A Branch of Mathematics for Modeling Human Uncertainty, Springer-Verlag, Berlin, 2010.
[6] Liu .B. "Uncertainty theory", 4th ed ., UTLAB, 2011.