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On Differential Sandwich Results for Analytic p-valent Meromorphic Functions Defined by Generalized Rafid Operator

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ABSTRACT

In the paper, we obtain some differential sandwich results for analytic meromorphic p-valent functions using results of differential subordinations and superordinations defined by generalized Rafid operator in the punctured open unit disk U^*.

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1-Introduction:

Let $\sum_p$ denote the class of function of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k z^k, (p, k \in \mathbb{N} \; and \; p < k), \tag{1.1}$$

which are meromorphic multivalent in the punctured open unit disk $U^* = \{z: z \in \mathbb{C}, 0 < |z| < 1\}$.

Several authors studied meromorphic functions for another classes and conditions, see [8,9,17,19,21].

Let H is the linear space of all analytic functions in U. For a positive integer number n and $a \in \mathbb{C}$, we let

$$H[a, n] = \{f \in H : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

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For f and G analytic functions in H, we say that f is subordinate to G in U and write $f(z) \prec G(z)$, if there exists a Schwarz function w, which is analytic in U with $w(0) = 0$, and $|w(z)| < 1, (z \in U)$, such that $f(z) = G(w(z))$, $(z \in U)$.

Furthermore, if the function G is univalent in U, we have the following equivalence relationship (cf. , e.g.[10,11,15,16]):

$$f(z) \prec G(z) \leftrightarrow f(0) = G(0) \; and \; f(U) \subset G(U), (z \in U).$$

Miller and Mocanu [15] and other authors [1,3,5,8,7,10,14,16,18] and also [2,4,6,9,12,20,21,23] discovered sufficient conditions for the functions h, p and $\emptyset$ for which the following result:

$$h(z) \prec \Psi(p(z), zp'(z), z^2p''(z); z) \Longrightarrow q(z) \prec p(z)(z \in U). \tag{1.2}$$

Definition (1.1) [15]: Let $\Psi: \mathbb{C}^3 \times U \to \mathbb{C}$ and $h(z)$ be analytic in U. If $p(z)$ and $\Psi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and if $p(z)$ satisfies the second-order differential superordination:

$$h(z) \prec \Psi(p(z), zp'(z), z^2p''(z); z), \tag{1.3}$$

then $p(z)$ is called a solution of the differential Superordination (1.3). An analytic function $q(z)$ which is called a subordinant of the solutions of the differential Superordination (1.3), or more simply, a subordinant if $q(z) \prec p(z)$ for all the functions $p(z)$ fulfills (1.3). A univalent subordinant $\tilde{q}(z)$ that fulfills $q(z) \prec \tilde{q}(z)$ for all the subordinants $q(z)$ of (1.3) is said to be the best subordinant.

Definition (1.2) [15]: Let $\Psi: \mathbb{C}^3 \times \mathrm{U} \to \mathbb{C}$ and h(z) be univalent in U. If p(z) is analytic in U and satisfies the second-order differential subordination:

$$\Psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.4}$$

then p(z) is called a solution of the differential subordination (1.4), and the univalent function $q(z)$ is called a dominant of the solution of the differential subordination (1.4), or more simply dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.4). A univalent dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominant $q(z)$ of (1.4) is said to be the best dominant.

Using the results, (see [1,2,4,6,5,9,13,14,18,20,22,23,24,25,27]) to obtain adequate criteria for the satisfaction of normalized analytic functions

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U and $q_1(0) = q_2(0) = 1$. Shanmugam et al. [22][23], as well as Goyal et al. [12], sandwich results for analytic function classes were recently obtained. (See also [1,3,4,6,11]).

Elkhatib et al. [19] generalized the Rafid operator [26] as follows: For $p \in \mathbb{N}, 0 \le \delta \le 1, 0 \le \mu < 1$ and $f \in \sum_p$, the generalized Rafid operator $I^{\delta}_{p,\mu}: \sum_p \to \sum_p$ defined by:

$$I^{\delta}_{p,\mu} f(z) = \frac{1}{(p-\mu)^{\delta+1} \, \Gamma(\delta+1)} \int_0^\infty t^{p+\delta} e^{-(\frac{t}{p.\mu})} f(zt) dt. \tag{1.5}$$

Then

$$I^{\delta}_{\rho,\mu} \, f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \Psi(k, \delta, \mu) a_k z^k, \tag{1.6}$$

where $\Psi(k,\delta,\mu)=\frac{(p-\mu)^{k+p}\;\Gamma(k+p+\delta+1)}{\Gamma(\delta+1)}$ in addition, the operator dífferential relation

$$z\left(I_{p,\mu}^{\delta}\,f(z)\right)'=(\delta+1)I_{p,\mu}^{\delta+1}\,f(z)-(p+\delta+1)I_{p,\mu}^{\delta}\,f(z). \tag{1.7}$$

This concept's major aim is to discover suitable conditions for specific normalized analytic functions f to satisfy:

$$q_1(z)\prec\left[z^p\,I_{\rho,\mu}^{\delta}\,f(z)\right]^m\prec q_2(z),$$

and

$$q_1(z)\prec\left[\frac{a\,z^pI_{\rho,\mu}^{\delta+1}\,f(z)+\lambda\,z^pI_{\mu\rho}^{\delta}\,f(z)}{a+\lambda}\right]^{\frac{1}{m}}\prec q_2(z),$$

whenever univalent functions $q_1(z)$ and $q_2(z)$ are given in U with $q_1(z)=q_2(z)=1$, and $m\in\mathbb{C}\backslash\{0\},a,\lambda\in\mathbb{C}^*=\mathbb{C}\backslash\{0\}$ and $a+\lambda\neq0$.

2-Preliminaries

The definitions and lemmas given below will assist us in proving our basic results.

Definition (2.1) [15]: Denote by Q the set of all functions q that are analytic and injective on $\overline{U}\backslash E(q)$, where $\overline{U}=U\cup\{z\in\partial U\}$, and

$$E(q)=\left\{\varepsilon\in\partial U:\lim_{z\to\varepsilon}q(z)=\infty\right\}, \tag{2.1}$$

and are such that $q'(z)\neq0$ for $\varepsilon\in\partial U\setminus E(q)$. Further, let the subclass of Q for which $q(0)=a$ be denoted by $Q(a)$, and $Q(0)=Q_0$, $Q(1)=Q_1=\{q\in Q:q(0)=1\}$.

Lemma (2.1) [11]: Let $q(z)$ be a convex univalent function in U and let $\alpha\in\mathbb{C},\beta\in\mathbb{C}\backslash\{0\}$ with $q(0)=1$ and suppose that

$$Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}>max\left\{0,-Re\left(\frac{\alpha}{\beta}\right)\right\}.$$

If p is analytic in U and

$$\alpha p(z)+\beta zp'(z)\prec\alpha q(z)+\beta zq'(z), \tag{2.2}$$

then $p(z)\prec q(z)$ and q is the best dominant of (2.2)

Lemma (2.2) [11]: Let $q(z)$ be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\theta(w)\neq0$, when $w\in q(U)$. Set $Q(z)=zq'(z)\theta\big(q(z)\big)$ and $h(z)=\phi\big(q(z)\big)+Q(z)$. Suppose that

1. $Q(z)$ is starlike univalent in U,
2. $Re\left\{\frac{zh'(z)}{Q(z)}\right\}>0$ for $z\in U$.

If p is analytic function in U, with $p(0)=q(0)$, $p(U)\subseteq D$ and

$$\theta\big(p(z)\big)+zp'(z)\phi\big(p(z)\big)\prec\theta\big(q(z)\big)+zq'(z)\phi\big(q(z)\big), \tag{2.3}$$

then $p\prec$ q and q is the best dominant of (2.3).

Lemma (2.3) [16]: Let $q(z)$ be a convex univalent in U and $q(0) = 1$. Let $\beta \in \mathbb{C}$, that $Re(\beta) > 0$. If $p(z) \in H[q(0), 1] \cap Q$ and $p(z) + \beta zp'(z)$ is univalent in U, then

$$q(z) + \beta zq'(z) \prec p(z) + \beta zp'(z), \tag{2.4}$$

which implies that q(z) ≺ p(z) and q(z)is the best subordinant of (2.4).

Lemma (2.4) [16]: Let q(z) be a convex univalent function in the unit dick U and let θ and ϕ be analytic in a domain D containing q(U). Suppose that

1. $Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ for z ∈ U,
2. $Q(z) = zq'(z)\phi(q(z))$ is starlike univalent in U.

If $p \in H[q(0), 1] \cap Q$, with $p(U) \subset D$, $\theta(p(z)) + zp'(z)\phi(p(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)), \tag{2.5}$$

then $q \prec p$ and q is the best subordinant of (2.5).

3- Results of Differential Subordinations

Theorem: (3.1) Let $q(z)$ be a convex univalent in the open unit disk U with $q(0) = 1$ and $q'(z) \neq 0$ for all $z \in U$. Let $\varrho, m \in \mathbb{C}\backslash\{0\}$, and suppose that

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > max\left\{0, -Re\left(\frac{m}{\varrho}\right)\right\}. \tag{3.1}$$

If $f \in \sum_p$ satisfies the subordination condition:

$$\psi(z) \prec q(z) + \frac{\varrho}{m} zq'(z), \tag{3.2}$$

where

$$\psi(z) = \varrho(\delta + 1)\left[z^p\, I_{p,\mu}^{\delta} f(z)\right]^m \left[\left(\frac{I_{p,\mu}^{\delta+1} f(z)}{I_{p,\mu}^{\delta} f(z)} - 1\right)\right] + \left[z^p\, I_{p,\mu}^{\delta} f(z)\right]^m, \tag{3.3}$$

then

$$\left[z^p\, I_{p,\mu}^{\delta} f(z)\right]^m \prec q(z), \tag{3.4}$$

where the best dominating is $q(z)$.

Proof. Putting

$$p(z) = \left[z^p\, I_{p,\mu}^{\delta} f(z)\right]^m, \tag{3.5}$$

then the function $p(z)$ is analytic in U and $p(0) = 1$ as a result of differentiating (3.5) with respect to z and then using the identity (1.7) in the resultant equation.

$$\psi(z) = \varrho(\delta + 1)\left[z^p\, I_{p,\mu}^{\delta} f(z)\right]^m \left[\left(\frac{I_{p,\mu}^{\delta+1} f(z)}{I_{p,\mu}^{\delta} f(z)} - 1\right)\right] + \left[z^p\, I_{p,\mu}^{\delta} f(z)\right]^m$$

$$= p(z) + \frac{\varrho}{m} zp'(z).$$

Thus the subordination (3.2) is equivalent to

$$p(z)+\frac{\varrho}{m}zp'(z)\prec q(z)+\frac{\varrho}{m}zq'(z).$$

An application of Lemma (2.1) with $\beta=\frac{\varrho}{\mathrm{m}}$, $\alpha=1$,we obtain (3.4).

Putting $\mathrm{q(z)}=\left(\frac{1+\mathrm{Az}}{1+\mathrm{Bz}}\right)(-1\le \mathrm{B}<\mathrm{A}\le 1)$ in theorem (3.1), we obtain the following corollary:

Corollary (3.1): Let $m,\varrho\in\mathbb{C}\backslash\{0\}$ and $(-1\le \mathrm{B}<\mathrm{A}\le 1)$. Suppose that

$$Re\left\{\frac{1-Bz}{1+Bz}\right\}>max\left\{0,-Re\left(\frac{m}{\varrho}\right)\right\}$$

If $f\in\sum_p$ satisfies the subordination condition:

$$\psi(\mathrm{z})\prec\left(\frac{1+\mathrm{Az}}{1+\mathrm{Bz}}\right)+\left(\frac{\varrho}{\mathrm{m}}\right)\frac{z(A-B)}{(1+Bz)^2},$$

when $\psi(\mathrm{z})$ given by (3.3), then

$$\left[z^p\, I_{\mu,p}^{\delta}f(z)\right]^m\prec\left(\frac{1+Az}{1+Bz}\right),$$

where the best dominating is $\left(\frac{1+\mathrm{Az}}{1+\mathrm{Bz}}\right)$.
In corollary (3.1), we can get following result with $A=1$ and $B=-1$.

Corollary (3.2): Let $m,\varepsilon\in\mathbb{C}\backslash\{0\}$ and Suppose that

$$Re\left\{\frac{1+z}{1-z}\right\}>max\left\{0,-Re\left(\frac{m}{\varrho}\right)\right\}.$$

If $\mathrm{f}\in\sum_p$ fulfill the following subordination condition:

$$\psi(z)\prec\left(\frac{1+z}{1-z}\right)+\left(\frac{m}{\varrho}\right)\frac{2z}{(1-z)^2},$$

when $\psi(z)$ is given by (3.3), then

$$\left[z^p\, I_{\mu,p}^{\delta}f(z)\right]^m\prec\left(\frac{1+z}{1-z}\right),$$

and $\left(\frac{1+\mathrm{z}}{1-\mathrm{z}}\right)$ is the best dominant.

Theorem (3.2) : Let $q(z)$ be a convex univalent in the open unit disk U with $q(0)=1$, and $q'(z)\neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Let $\xi,a,\lambda,\mu\in\mathbb{C}^*,\sigma,u\in\mathbb{C}$ with $a+\lambda\neq 0$, $\frac{az^p\, I_{\mu,p}^{\delta+1}f(z)+\lambda z^p\, I_{\mu,p}^{\delta}f(z)}{a+\lambda}\neq 0$, $z\in U$, and suppose that q satisfy the following condition

$$Re\left\{1+\frac{2\sigma}{\xi}(q(z))^2+z\frac{q''(z)}{q'(z)}-z\frac{q'(z)}{q(z)}\right\}>0, \tag{3.6}$$

and $f\in\sum_p$ if satisfies:

$$H(z)\prec\sigma\big(q(z)\big)^2-u+\xi z\frac{q'(z)}{q(z)}, \tag{3.7}$$

where

$$H(z)=\sigma\left[\frac{az^p\, I_{\mu,p}^{\delta+1}f(z)+\lambda z^p\, I_{\mu,p}^{\delta}f(z)}{a+\lambda}\right]^{\frac{2}{m}}$$
$$-u+\xi\left(\frac{1}{m}\right)+(1+\delta)\left[\left(\frac{a\, I_{\mu,p}^{\delta+2}f(z)+\lambda\, I_{\mu,p}^{\delta+1}f(z)}{a\, I_{\mu,p}^{\delta+1}f(z)+\lambda\, I_{\mu,p}^{\delta}f(z)}-1\right)\right], \tag{3.8}$$

then

$$\left[\frac{az^p\, I_{\mu,p}^{\delta+1}f(z)+\lambda z^p\, I_{\mu,p}^{\delta}f(z)}{a+\lambda}\right]^{\frac{1}{m}}\prec q(z), \tag{3.9}$$

where the best dominating is $q(z)$.

Proof. Define the analytic function $p(z)$:

$$p(z)=\left[\frac{az^p\, I_{\mu,p}^{\delta+1}f(z)+\lambda z^p\, I_{\mu,p}^{\delta}f(z)}{a+\lambda}\right]^{\frac{1}{m}}, \tag{3.10}$$

then the function $p(z)$ is analytic in U and $p(0)=1$. By differentiating (3.10) with respect to z, and using the identity (1.7) in the resulting equation, we get

$$\frac{zp'(z)}{p(z)}=\frac{1}{m}(\delta+1)\left[\frac{a\, I_{\mu,p}^{\delta+2}f(z)\,+\,\lambda\, I_{\mu,p}^{\delta+1}f(z)}{a\, I_{\mu,p}^{\delta+1}f(z)\,+\,\lambda\, I_{\mu,p}^{\delta}f(z)}-1\right]. \tag{3.11}$$

By setting $\theta(w)=\sigma w^2-u$ and $\phi(w)=\frac{\xi}{w}, w\neq 0$ we see that $\theta(w)$ is analytic function in $\mathbb{C}$, and $\phi(w)$ is analytic in $\mathbb{C}\backslash\{0\}$ and $\phi(w)\neq 0, w\in\mathbb{C}\backslash\{0\}$. Also, we get

$$Q(z)=zq'(z)\phi(w)=\xi z\frac{q'(z)}{q(z)},$$

and

$$A(z)=\ \theta(q(z))+Q(z)=\sigma\big(q(z)\big)^2-u+\xi z\frac{q'(z)}{q(z)},$$

we find that $Q(z)$ is starlike univalent in U, we have

$$\grave{A}(z)=2\sigma\, q(z)q'(z)+\xi z\frac{q''(z)}{q'(z)}-\xi z\left(\frac{q'(z)}{q(z)}\right)^2+\xi\frac{q'(z)}{q(z)},$$

hence that

$$Re\left\{\frac{zA'(z)}{Q(z)}\right\}=Re\left\{1+\frac{2\sigma}{\xi}(q(z))^2+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0.$$

By using (3.11), we obtain

$$\sigma\big(p(z)\big)^2-u+\xi z\frac{p'(z)}{p(z)}=\sigma\left[\frac{az^p\, I_{\mu,p}^{\delta+1}f(z)+\lambda z^p\, I_{\mu,p}^{\delta}f(z)}{a+\lambda}\right]^{\frac{2}{m}}-u+\xi\left(\frac{1}{m}\right)(\delta+1)\left[\left(\frac{aI_{\mu,p}^{\delta+2}f(z)+\lambda\, I_{\mu,p}^{\delta+1}f(z)}{a\, I_{\mu,p}^{\delta+1}f(z)+\lambda\, I_{\mu,p}^{\delta}f(z)}-1\right)\right].$$

By using (3.8), we have

$$\sigma\big(p(z)\big)^2-u+\xi z\frac{p'(z)}{p(z)}=\sigma\big(q(z)\big)^2-u+\xi z\frac{q'(z)}{q(z)},$$

we can infer that subordination (3.7) implies that $p(z)\prec q(z)$, and that the function $q(z)$ is the best domain by using Lemma (2.2).

Putting $q(z)=\left(\frac{1+Az}{1+Bz}\right)(-1\leq B<A\leq 1)$, in Theorem (3.2), the condition (3.6) becomes

$$Re\left\{1+\frac{2\sigma}{\xi}\left(\frac{1+Az}{1+Bz}\right)^2+\frac{z(A-B)}{(1+Az)(1+Bz)}-\frac{2zB}{1+Bz}\right\}>0, \tag{3.12}$$

as a result, we may deduce the following conclusion.

Corollary (3.3) : Let $(-1\leq B<A\leq 1)$, $\xi,a,\lambda,m\in\mathbb{C}^*$, $\sigma,u\in\mathbb{C}$, assume that (3.12) holds. If $f\in\sum_p$ and

$$H(z)\prec\sigma\left(\frac{1+Az}{1+Bz}\right)^2-u+\xi\frac{z(A-B)}{(1+Az)(1+Bz)},$$

where $H(z)$ is defined in (3.8), then

$$\left[\frac{az^p\, I_{\mu,p}^{\delta+1}f(z)+\lambda\ z^p\, I_{\mu,p}^{\delta}f(z)}{a+\lambda}\right]^{\frac{1}{m}}\prec\left(\frac{1+Az}{1+Bz}\right),$$

and $\left(\frac{1+Az}{1+Bz}\right)$ is the best dominant.

Taking the function $q(z)=\left(\frac{1+z}{1-z}\right)$, in Theorem(3.2), the condition (3.6) becomes

$$Re\left\{1+\frac{2\sigma}{\xi}\left(\frac{1+z}{1-z}\right)^2+\frac{2z}{1-z^2}+\frac{2z}{1-z}\right\}>0. \tag{3.13}$$

As a result, we may deduce the following conclusion.

Corollary (3.4) : Let $\xi, a, \lambda, m \in \mathbb{C}^*$, $\sigma, u \in \mathbb{C}$. Assume that (3.13) holds. If $f \in \sum_p$ and

$$H(z) \prec \sigma\left(\frac{1+z}{1-z}\right)^2-u+\xi\frac{2z}{1-z^2},$$

where $H(z)$ is defined in (3.8), then

$$\left[\frac{az^p\, \mathrm{I}_{\mu,p}^{\delta+1}\mathrm{f}(z)+\lambda\, z^p\, \mathrm{I}_{\mu,p}^{\delta}\mathrm{f}(z)}{a+\lambda}\right]^{\frac{1}{m}} \prec \left(\frac{1+z}{1-z}\right),$$

and $\left(\frac{1+z}{1-z}\right)$ is the best dominant.

4- Results of Differential Superordinations:

Theorem 4.1: Assume that the function $q(z)$ is a convex univalent in U with $q(0)=1, m \in \mathbb{C}\backslash\{0\}$, $Re\{\varrho\}>0$, if $f \in \sum_p$, such that

$$\left[z^p\, I_{\mu,p}^{\delta} f(z)\right]^m \in H[q(0),1] \cap Q. \tag{4.1}$$

If the function ψ(z) in (3.3) is univalent and the superordination criterion is fulfilled:

$$q(z)+\frac{\varrho}{m}zq'(z) \prec \psi(z), \tag{4.2}$$

holds, then

$$q(z) \prec \left[z^p\, I_{\mu,p}^{\delta} f(z)\right]^m, \tag{4.3}$$

where the best subordinant is $q(z)$.

Proof. Define a function $p(z)$ by

$$p(z)=\left[z^p\, I_{\mu,p}^{\delta} f(z)\right]^m. \tag{4.4}$$

Differentiating (4.4) with respect to z, we get

$$\frac{zp'(z)}{p(z)}=m\left[\frac{z\left(I_{\mu,p}^{\delta} f(z)\right)'+p\, I_{\mu,p}^{\delta} f(z)}{I_{\mu,p}^{\delta} f(z)}\right]. \tag{4.5}$$

A simple computation and using (1.7), from (4.5), we will get

$$\psi(z)=\varrho(\delta+1)\left[z^p\, I_{\mu,p}^{\delta} f(z)\right]^m\left[\left(\frac{I_{\mu,p}^{\delta+1} f(z)}{I_{\mu,p}^{\delta} f(z)}-1\right)\right]+\left[z^p\, I_{\mu,p}^{\delta} f(z)\right]^m = p(z)+\frac{\varrho}{m}zp'(z).$$

Now, by using Lemma (2.3), we get the desired result.

Taking $q(z)=\left(\frac{1+Az}{1+Bz}\right)$, $(-1 \le B < A \le 1)$, we obtain the following conclusion from Theorem (4.1) .

Corollary (4.1): Let $m \in \mathbb{C}\backslash\{0\}$, $Re\{\varrho\}>0$,and $(-1 \le B < A \le 1)$, such that

$$\left[z^p\, I_{\mu,p}^{\delta} f(z)\right]^m \in H[q(0),1] \cap Q.$$

If ψ(z) in (3.3) is univalent in U, and $f \in \sum_p$ fulfills the superordination condition,

$$\left(\frac{1+Az}{1+Bz}\right) + \left(\frac{\varrho}{\mathrm{m}}\right)\frac{z(A-B)}{(1+Bz)^2} \prec \psi(z),$$

then

$$\left(\frac{1+Az}{1+Bz}\right) \prec \left[z^p\, I_{\mu,p}^{\delta} f(z)\right]^m.$$

The best subordinant is the function $\left(\frac{1+Az}{1+Bz}\right)$.

Theorem (4.2): Let $q(z)$ be a convex univalent function in the open unit disk U with $q(0) = 1$, and $q'(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Let $\xi, a, \lambda, m \in \mathbb{C}^*, \sigma, u \in \mathbb{C}$ with $a+\lambda \neq 0$, $\frac{az^p\, I_{\mu,p}^{\delta+1} f(z) + \lambda z^p\, I_{\mu,p}^{\delta} f(z)}{a+\lambda} \neq 0$, $z \in U$. Suppose that q satisfy the following condition

$$Re\left\{\frac{2\sigma}{\xi}(q(z))^2 q'(z)\right\} > 0.$$

Let $f \in \sum_p$ and satisfy the next condition

$$\left[\frac{az^p\, I_{\mu,p}^{\delta+1} f(z) + \lambda z^p\, I_{\mu,p}^{\delta} f(z)}{a+\lambda}\right]^{\frac{1}{m}} \in [q(0),1] \cap Q. \tag{4.6}$$

If the function H(z) is given by (3.8), is univalent in U,

$$\sigma\big(q(z)\big)^2 - u + \xi z\frac{q'(z)}{q(z)} \prec H(z), \tag{4.6a}$$

then

$$q(z) \prec \left[\frac{az^p\, I_{\mu,p}^{\delta+1} f(z) + \lambda\, z^p\, I_{\mu,p}^{\delta} f(z)}{a+\lambda}\right]^{\frac{1}{m}}, \tag{4.7}$$

where the best subordinant is $q(z)$.

Proof. Let $p(z)$ defined on U by (3.10). After that, a calculation reveals that

$$\frac{zp'(z)}{p(z)} = \left(\frac{1}{m}\right)(1+\delta)\left[\left(\frac{aI_{\mu,p}^{\delta+2} f(z) + \lambda\, I_{\mu,p}^{\delta+1} f(z)}{a\, I_{\mu,p}^{\delta+1} f(z) + \lambda\, I_{\mu,p}^{\delta} f(z)} - 1\right)\right]. \tag{4.8}$$

Setting $\theta(\mathrm{w}) = \sigma \mathrm{w}^2 - \mathrm{u}$ and $\phi(\mathrm{w}) = \frac{\xi}{\mathrm{w}}, \mathrm{w} \neq 0$, it can be easily observed that the $\theta(\mathrm{w})$ is analytic function in $\mathbb{C}$ and $\phi(\mathrm{w})$ is analytic in $\mathbb{C}\backslash\{0\}$, that $\phi(\mathrm{w}) \neq 0, (\mathrm{w} \in \mathbb{C}\backslash\{0\})$. Also, we get

$$Q(z) = zq'(z)\phi(\mathrm{w}) = \xi z\frac{q'(z)}{q(z)},$$

it was discovered that $Q(z)$ is starlike univalent in U.

Because $q(z)$ is convex, we may deduce that

$$Re\left\{\frac{\theta'(\mathrm{q}(\mathrm{z}))}{\phi(\mathrm{q}(\mathrm{z}))}\right\} = Re\left\{\frac{2\sigma}{\xi}(q(z))^2 q'(z)\right\} > 0.$$

By making use (4.8) the hypothesis (4.6a) can be equivalently

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)).$$

The proof is therefore completed by utilizing the Lemma (2.4) .

5- Sandwich Results:

By combining Theorems 3.2 and 4.2, we have the following sandwich theorem:

Theorem (5.1) : Let q_i be two univalent convex functions in U, with $q_i(0) = 1, q_i'(z) \neq 0, (i = 1, 2)$. Assume that q_1 and q_2 satisfy the conditions (3.7) and (4.7), respectively.

If $f \in \sum_p$, and suppose that f satisfies the next condition:

$$\left[\frac{az^p\, I_{\mu,p}^{\delta+1} f(z) + \lambda z^p\, I_{\mu,p}^{\delta} f(z)}{a+\lambda}\right]^{\frac{1}{m}} \in H[q(0),1] \cap Q,$$

and $\frac{az^p\, I_{\mu,p}^{\delta+1} f(z) + \lambda z^p\, I_{\mu,p}^{\delta} f(z)}{a+\lambda} \neq 0$, and H(z) is univalent in U, and given by (3.8), then

$$\sigma(q_1(z))^2 - u + \xi z\frac{q_1'(z)}{q_1(z)} \prec H(z) \prec \sigma(q_2(z))^2 - u + \xi z\frac{q_2'(z)}{q_2(z)}, \quad (5.1)$$

implies

$$q_1(z) \prec \left[\frac{az^p\, I_{\mu,p}^{\delta+1} f(z) + \lambda z^p\, I_{\mu,p}^{\delta} f(z)}{a+\lambda}\right]^{\frac{1}{m}} \prec q_2(z),$$

where the best subordinant and the best dominant q_1 and q_2, respectively.

By combining Theorems 3.1 and 4.1, we have the following sandwich theorem:

Theorem (5.2) : Let q_1 *and* q_2 be convex univalent functions in U with $q_1(0) = q_2(0) = 1$ and q_2 satisfies (3.1). Suppose that $m \in \mathbb{C}\backslash\{0\}$, $Re\{\varrho\} > 0$. If $f \in \sum_p$, such that

$$\left[z^p\, I_{\mu,p}^{\delta} f(z)\right]^m \in H[q(0),1] \cap Q,$$

and the univalent function ψ(z), defined by (3.3), satisfies

$$q_1(z) + \frac{\varrho}{m} zq_1'(z) \prec \psi(z) \prec q_2(z) + \frac{\varrho}{m} zq_2'(z), \quad (5.2)$$

then

$$q_1(z) \prec \left[z^p\, I_{\mu,p}^{\delta} f(z)\right]^m \prec q_2(z),$$

where q_1 and q_2 are the best subordinant and the best dominant respectively.

References

[1] R. Abd Al-Sajjad and W. G. Atshan, Certain analytic function sandwich theorems involving operator defined by Mittag-Leffler function, AIP Conference Proceedings, 2398(2022), 060065, 1-8.

[2] S. A. Al-Ameedee, W. G. Atshan and F. A. Al-Maamori , On sandwich results of univalent functions defined by a linear operator, Journal of Interdisciplinary Mathematics, 23(4)(2020), 803-809.

[3] S. A. Al-Ameedee, W. G. Atshan and F. A. Al-Maamori, Some new results of differential subordinations for higher- order derivatives of multivalent functions, Journal of Physics: Conference Series, 1804 (2021) 012111, 1-11.

[4] R. M. Ali, V. Ravichandran, M. H. Khan and K. G. Subramanian, Differential sandwich theorems for certain analytic functions, Far East J. Math. Sci., 15(2004), 87– 94.

[5] W. G. Atshan and A. A. R. Ali, On sandwich theorems results for certain univalent functions defined by generalized operators, Iraqi Journal of Science, 62(7) (2021), pp: 2376-2383.

[6] W. G. Atshan and A. A. R. Ali, On some sandwich theorems of analytic functions involving Noor –Sâlâgean operator, Advances in Mathematics: Scientific Journal, 9(10)(2020), 8455-8467.

[7] W. G. Atshan and R. A. Hadi, Some differential subordination and superordination results of p-valent functions defined by differential operator, Journal of Physics: Conference Series, 1664 (2020) 012043, 1-15.

[8] W. G. Atshan and S. R. Kulkarni, On application of differential subordination for certain subclass of meromorphically p-valent functions with positive coefficients defined by linear operator, Journal of Inequalities in Pure and Applied Mathematics, 10(2)(2009), Article 53, 11 pp.

[9] W. G. Atshan, A. H. Battor and A. F. Abaas, Some sandwich theorems for meromorphic univalent functions defined by new integral operator, Journal of Interdisciplinary Mathematics, 24(3) (2021), 579-591.

[10] T. Bulboacã, Classes of first – order differential superordinations, Demonstratio Mathematica, 35(2) (2002), 287-292 .

[11] T. Bulboaca, Differential Subordinations and Superordinations, Recent Results, House of Scientific Book Publ., Cluj-Napoca, (2005).

[12] S. P. Goyal, P. Goswami and H. Silverman, Subordination and superordination results for a class of analytic multivalent functions, Int. J. Math. Math. Sci. (2008), Article ID 561638, 1– 12.

[13] I. A. Kadum, W. G. Atshan and A. T. Hameed, Sandwich theorems for a new class of complete homogeneous symmetric functions by using cyclic operator,Symmetry,14(10)(2022),2223,1-16.

[14] B. K. Mihsin, W. G. Atshan and S. S. Alhily, On new sandwich results of univalent functions defined by a linear operator, Iraqi Journal of Science, 63(12) (2022), pp: 5467-5475.

[15] S. S. Miller and P. T. Mocanu, Differential subordinations: Theory and Applications, Series on Monographs and Text Books in Pure and Applied Mathematics, 225, Marcel Dekker, New York and Basel, (2000).

[16] S. S. Miller and P. T. Mocanu, Subordinants of differential superordinations, Complex Var. Theory Appl., 48(2003), 815 − 826.

[17] T. Rosy, and S. Varma, On a subclass of meromorphic functions defined by Hilbert space operator, Geometry, 2013, Article ID 671826, 4 pages.

[18] M. A. Sabri, W. G. Atshan and E. El-Seidy, On sandwich-type results for a subclass of certain univalent functions using a new Hadamard product operator, Symmetry, 14(5)(2022),931,1-11.

[19] A. F. Elkhatib, A.F. Hashens, A. O. Mostafa and M. M. Tharwat, Generalization of the Rafid operator and its symmetric role in meromorphic function theory with electrostatic applications, Symmetry, (2025), 17, 1837.

[20] F. O. Salman and W. G. Atshan, New results on integral operator for a subclass of analytic functions using differential subordinations and superordinations, Symmetry, 15(2)(2023), 1-10.

[21] N. Seenivasagan, Differential Subordination and Superordination for Analytic and Meromorphic Functions Defined by Linear Operator, Doctoral Dissertation, University Sains Malaysia, (2007).

[22] T. N. Shanmugam, V. Ravichandran and S. Sivasubramanian, Differential sandwich theorems for subclasses of analytic functions, Aust. J. Math. Anal . Appl., 3 (2006), Article 8, 1– 11.

[23] T. N. Shanmugam, S. Shivasubramanian and H. Silverman, On sandwich theorems for some classes of analytic functions, Int. J. Math. Math. Sci. , (2006), Article ID 29684, 1 − 13.

[24] S. D. Theyab, W. G. Atshan, A. A. Lupas and H. K. Abdullah, New results on higher – order differential subordination and superordination for univalent analytic functions using a new operator, Symmetry, 14(8)(2022), 1576, 1-12.

[25] S. D. Theyab, W. G. Atshan and H. K. Abdullah, On some sandwich results of univalent functions related by differential operator, Iraqi Journal of Science, 63(11)(2022), pp: 4928-4936.

[26] W. G. Atshan and R. H. Buti, Fractional calculus of a class of univalent functions with negative coefficients defined by Hadamard product with Rafid operator, European Journal of Pure and Applied Mathematics, 4(2) (2011), 162-173.

[27] A. K. Wanas, Q. A. Shakir and A. Catas, Coefficient estimates and symmetry analysis for certain families of bi-univalent functions defined by the q-Bernoulli polynomial, Symmetry, 17 (2025), Art. ID 1532, 1-20.