

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)



*e*s-Ejective Module*

Mustafa K. Mashre ¹, Wasan Khalid²

^{1,2}Department of Mathematics, University of Baghdad, College of Science. Baghdad, Iraq. Email: mutafa.mashre1903@sc.uobaghdad.edu.iq

ARTICLE INFO

Article history:

Received: 10 /03/2026

Revised form: 11 /05/2026

Accepted : 17 /05/2026

Available online: 30 /09/2026

Keywords:

*e*_essential submodule**e*_singular essential submodule**e*s-ejective module**self e*s-ejective module*

ABSTRACT

Let T and H be nR -modules, H will be called T - e^* -ejective if for each submodule V of T and each homomorphism $w: K \rightarrow H$, there exists a homomorphism $\gamma: T \rightarrow H$ and $C \leq_{e^*s} V$, such that $w(c) = \gamma(c)$, $\forall c \in C$. We define e^* -Goldie-extending module if every e^* -closed submodule A of T , there exists a direct summand S of T such that $A \cap S$ is e^* -essential in A and S , we prove properties of e^* -ejective module and present characterizations of this type of modules.

MSC.

<https://doi.org/10.29304/jqcm.2026.18.32724>

1. Introduction

Let R be a ring with 1, let T be a left unitary R -module and $I(T)$ be the injective hull of T , a sub-module B of T is called small in T denoted by $B \ll T$ if for any sub-module A of T such that $T = B + A$ implies that $T = A$ [3]. It is known that a sub-module B of T is called essential or T is an essential extension of B if for every nonzero sub-module A of T , $B \cap A \neq 0$ [3].

Oscan in [6] introduced cosingular sub-module of T to be the set $Z^*(T) = \{c \in T \mid Rc \ll I(T)\}$. $Z^*(T)$ is called cosingular submodule if $Z^*(T) = T$,

Baanon and Khalid in [1] present a generalization of essential sub-module called e^* -essential sub-module where B is e^* -essential sub-module of T if for any nonzero cosingular sub-module A of T , $B \cap A \neq 0$,

The singular submodule is defined as $Z(T) = \{t \in T : ann(t) \subseteq e^*R\}$. If $Z(T) = T$, T is called a singular module, and if $Z(T) = 0$, T is called nonsingular, [2, p. 30]. In [5] A & W generalized $Z(T)$ depending on the concept of e^* -essential submodules as follows:

$Ze^*(T) = \{n \in T : ann(n) \subseteq e^*R\}$ it was called e^* -singular submodule. If $Ze^*(T) = T$, then T is called e^* -singular module and if $Ze^*(T) = 0$, T is called non e^* -singular module. A new generalization of an essential submodule called e^* -singular essential submodule appeared in [4]. Let B be a submodule of T , then T is e^* -singular essential extension to B , or B is e^* -singular essential in T if $B \cap A \neq 0$, for every nonzero e^* -singular submodule A of T . For short (e^* -S-essential) and denoted by $B \subseteq_{e^*s} T$, [4].

*Corresponding author: Mustafa K. Mashre

Email addresses: mutafa.mashre1903@sc.uobaghdad.edu.iq

Communicated by 'sub editor'

In this article we introduce e^*s -ejective module depend on the concept of e^* -singular essential submodule, characterizations of e^*s -ejective are given. It has been proven that under certain conditions a submodule of e^*s -ejective module be a e^*s -ejective

2. e^*s - essential sub-module.

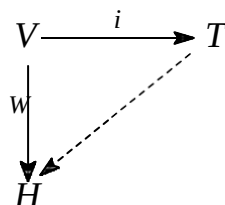
In this section the relation ξ_{e^*s} on the sub-module of T has defined. by using the concept of e^*s -essential submodule has been defined.

Definition 2.1.[5]: Let T be an R -module, a sub-module B of T is said to be e^*s -essential if $B \cap A \neq 0$ for each non zero e^* -singular sub-module A of T denoted by $B \leq_{e^*s} A$.

A left ideal A of aring R is e^*s -essential in R if and only if A is e^*s - essential sub-module.

Definition 2.2: Let T and H be an R -modules, H will be called T - e^*s -ejective if for each submodule V of T and each homomorphism $w: V \rightarrow H$, there exists a homomorphism $\gamma: T \rightarrow H$ and $C \leq_{e^*s} V$, such that $w(c) = \gamma(c), \forall c \in C$

An R -module is called self- e^*s -ejective module if T is T - e^*s -ejective.



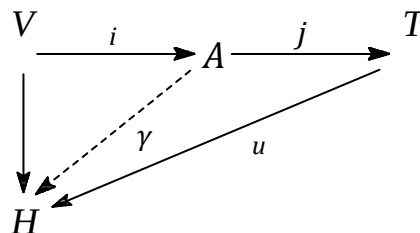
Let T_1, T_2 be two R -modules, we say that T_1 and T_2 are relatively e^*s -ejective modules, if T_i is T_j - e^*s -ejective for every $i, j = 1, 2, i \neq j$.

An R -module H is e^*s -ejective module if H is T - e^*s -ejective, for every R -module T .

Next, we give some properties of this type of modules.

Proposition 2.3: Let T and H be R -modules and B be a submodule of T , if H is T - e^*s -ejective, then H is B - e^*s -ejective.

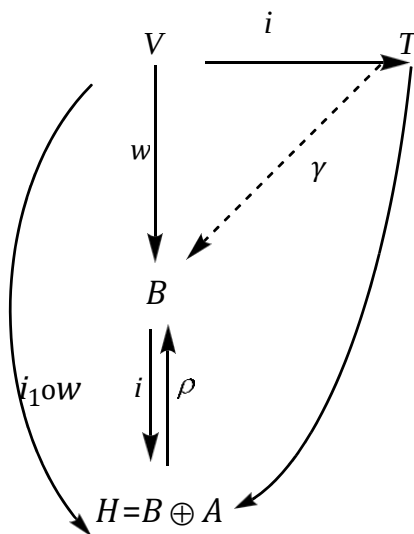
Proof: Let V be a submodule of B and let $w: V \rightarrow H$ be any R -homomorphism, now consider the following diagram:



Where i, j are the inclusion map, since $V \leq B \leq T$, and H is T - e^*s -ejective, then there exist $C \leq_{e^*s} V$ and an R -homomorphism $u: T \rightarrow H$, such that $w(c) = u(c), \forall c \in C$. Take $\gamma = u|_B: B \rightarrow H$, clearly that $w(c) = \gamma(c), \forall c \in C$. Thus H is B - e^*s -ejective.

Proposition 2.4: Let T and H be R -modules and let $H = B \oplus A$, if H is T - e^*s -ejective, then B is T - e^*s -ejective.

Proof: Let $H = B \oplus A$ be T - e^*s -ejective. To show B is T - e^*s -ejective, let V be a submodule of T and $w: V \rightarrow B$ be an R -homomorphism. Now, consider the following diagram:

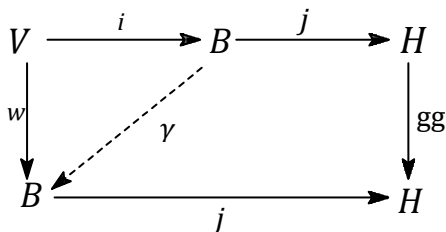


where i, j are the inclusion maps and ρ be the projection map, since H is T -e*s-ejective, then there exists $C \leq_{e*s} V$ and R -homomorphism $u: T \rightarrow H$, such that $u(c) = i_1 \circ w(c), \forall c \in C$. Let $\gamma = \rho \circ u, \gamma(c) = \rho(u(c)) = (\rho \circ i_1 \circ w)(c) = w(c), \forall c \in C$. Thus B is T -e*s-ejective.

The following proposition gives a condition under which a submodule of a self-e*s-ejective module is a self-e*s-ejective

Proposition 2.5: Let H be a self-e*s-ejective module, then every fully invariant submodule B of H is a self-e*s-ejective.

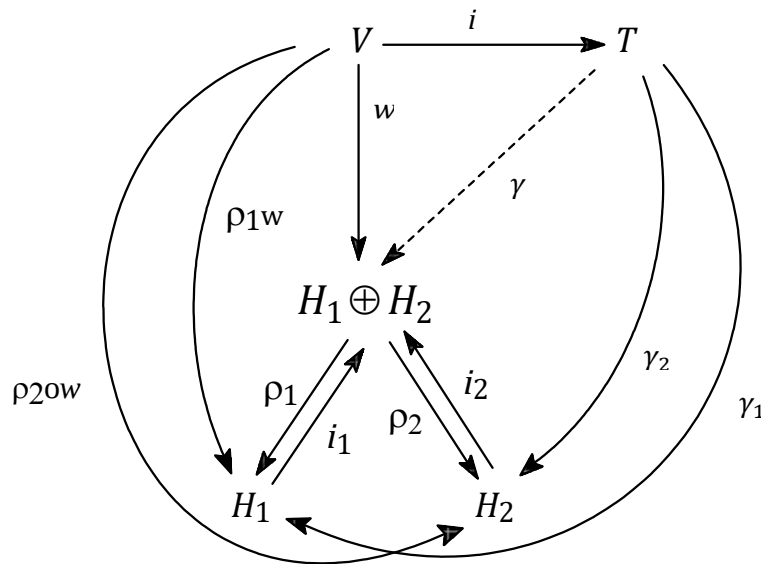
Proof: Suppose that H is a self-e*s-ejective, and B is a fully invariant submodule of H , let V be a submodule of B and let $w: V \rightarrow B$ be an R -homomorphism. Consider the following diagram:



where i, j be the inclusion map, since $V \leq B \leq H$ and H is a self-e*s-ejective module. Then there exists $C \leq_{e*s} V$ and an R -homomorphism $u: H \rightarrow H$, then $u(c) = j \circ w(c), \forall c \in C$, since B is fully invariant, then $u(B) \leq B$, put $\gamma = u|_B: B \rightarrow B$, R -homomorphism. Then $\gamma(x) = \gamma|_B(x) = j \circ w(c) = j(w(c)) = w(c), \forall c \in C$, then B is a self-e*s-ejective.

Proposition 2.6: Let H_1, H_2 and T be an R -modules, if H_1 and H_2 are T -e*s-ejective. Then $H_1 \oplus H_2$ is T -e*s-ejective.

Proof: Suppose that H_1 and H_2 are T -e*s-ejective modules. Let V be a submodule of T and let $w: V \rightarrow H_1 \oplus H_2$ be an R -homomorphism. Consider the following diagram:



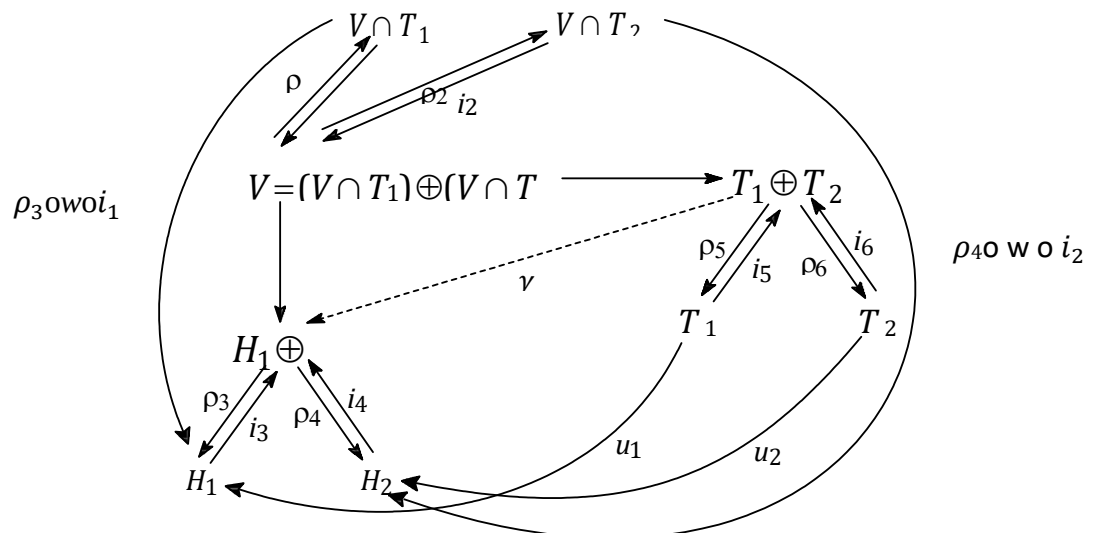
where i, i_1, i_2 be the inclusion maps and ρ_1, ρ_2 be the projection maps. Since H_1 and H_2 are T -e*s-ejective,

then there exists $C_1 \leq_{e*s} V$, $C_2 \leq_{e*s} V$ and an R -homomorphisms $\gamma_1: T \rightarrow H_1$ and $\gamma_2: T \rightarrow H_2$, such that $\gamma_1(c) = (\rho_1 \circ w)(c)$, $\forall c \in C_1$ and $\gamma_2(c) = (\rho_2 \circ w)(c)$, $\forall c \in C_2$. Put $C = C_1 \cap C_2$, then it is clear that $C \leq_{e*s} V$, define $\gamma: T \rightarrow H_1 \oplus H_2$ by: $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, $\forall t \in T$. Let $c \in C$, then: $\gamma(c) =$

$$(\gamma_1(c), \gamma_2(c)) = ((\rho_1 \circ w)(c), (\rho_2 \circ w)(c)) = (\rho_1(w(c)), \rho_2(w(c))) = (\rho_1(w(c_1, c_2)), \rho_2(w(c_1, c_2))) = w(c_1, c_2) = w(c), \forall c \in C, c_1 \in H_1, c_2 \in H_2.$$

Proposition 2.7: Let H_1 be T_1 -e*s-ejective and H_2 be T_2 -ejective, if $T_1 \oplus T_2$ is a duo module, then $H_1 \oplus H_2$ is $T_1 \oplus T_2$ -e*s-ejective.

Proof: Suppose that H_1 is T_1 -e*s-ejective and H_2 is T_2 -e*s-ejective and let V be a submodule of $T_1 \oplus T_2$. Let $w: V \rightarrow H_1 \oplus H_2$ be an R -homomorphism, since $T_1 \oplus T_2$ is a duo module, then $V = (V \cap T_1) \oplus (V \cap T_2)$, [7] Now consider the following diagram:



where i_1, i_2, i_3, i_4, i_5 and i_6 are the inclusion maps and $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5$ and ρ_6 are the projection maps.

Since $V \cap T_1$ is a submodule of T_1 , $\rho_3 \circ w \circ i_1: V \cap T_1 \rightarrow H_1$ is an R -homomorphism and H_1 is T_2 -

e^*s -ejective, then there exists $C_1 \leq_{e^*s} V \cap T_1$ and an R -homomorphism $u_1: T_1 \rightarrow H_1$, such that

$u_1(x) = (\rho_3 \circ w \circ i_1)(x), \forall x \in X$. Since $V \cap T_2$ is a submodule of T_2 , $\rho_4 \circ w \circ i_2: V \cap T_2 \rightarrow H_2$ is

an R -homomorphism and H_2 is T_2 - e^*s -ejective, then there exist $u_2: T_2 \rightarrow H_2$, such that $u_2(c) = (\rho_4 \circ w \circ i_2)(c), \forall c \in C$. Take $C = C_1 \oplus C_2, C \leq_{e^*s} V$ by [5]. Let $c \in C$, then $c = (c_1, c_2)$, where $c_1 \in C_1, c_2 \in C_2$. Define $\gamma: T_1 \oplus T_2 \rightarrow H_1 \oplus H_2$ by: $\gamma(c) = \gamma(c_1, c_2) = (u_1(c_1), u_2(c_2)) = ((\rho_3 \circ w \circ i_1)(c_1), (\rho_4 \circ w \circ i_2)(c_2)) = (\rho_3 w(c_1), \rho_4 w(c_2)) = w(c_1, c_2) = w(c), \forall c \in C$.

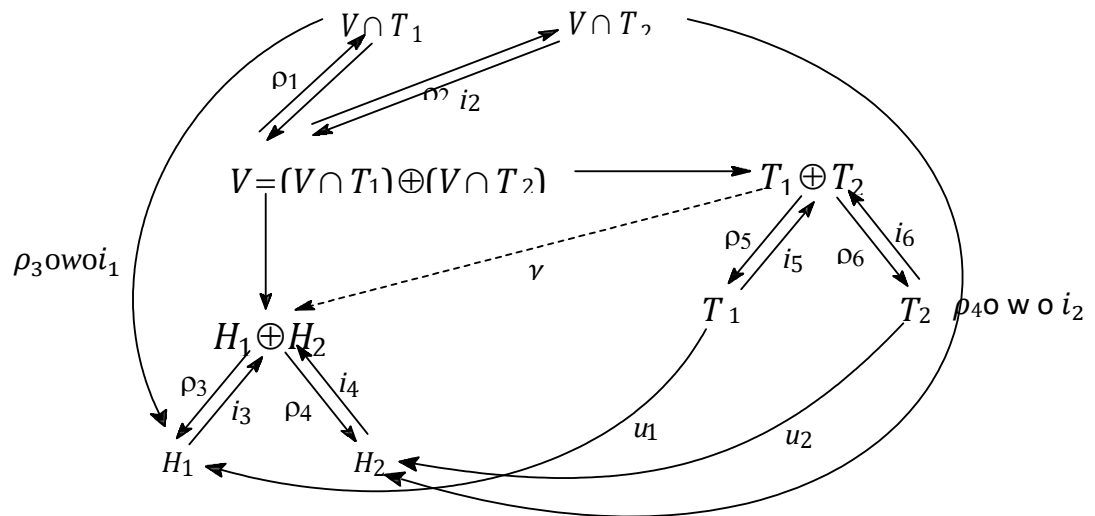
It is known that a module T is called distributive if $B \cap (A \oplus D) = (B \cap A) \oplus (B \cap D)$ for any B, A, D , submodule of T . [4]

By the same argument in Proposition 2.7, one can prove the following proposition:

Proposition 2.8: Let H_1 be T_1 - e^*s -ejective and H_2 be T_2 - e^*s -ejective if $T_1 \oplus T_2$ be a distributive module, then $H_1 \oplus H_2$ is $T_1 \oplus T_2$ - e^*s -ejective.

Proposition 2.9: Let H_1 be T_1 - e^*s -ejective and H_2 be T_2 - e^*s -ejective if $\text{ann}(T_1) + \text{ann}(T_2) = R$, then $H_1 \oplus H_2$ is $T_1 \oplus T_2$ - e^*s -ejective.

Proof: Suppose that H_1 is T_1 - e^*s -ejective and H_2 is T_2 - e^*s -ejective. Let V be a submodule of $T_1 \oplus T_2$ and $w: V \rightarrow H_1 \oplus H_2$ be an R -homomorphism, since $\text{ann}(T_1) + \text{ann}(T_2) = R$, then $V = B \oplus A$, where B is a submodule of T_1 and A is a submodule of T_2 . Now, consider the following diagram:



where i_1, i_2, i_3, i_4, i_5 and i_6 are the inclusion maps and $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5$ and ρ_6 are the projection maps,

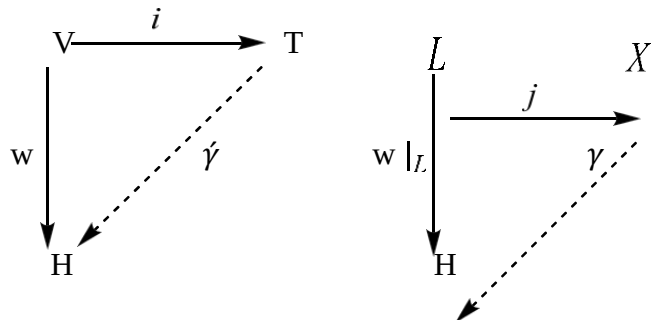
since B is a submodule of T_1 , $\rho_3 \circ w \circ i_3: B \rightarrow H_1$ is an R -homomorphism and H_1 is T_2 - e^*s -ejective,

then there exists $C_1 \leq_{e^*s} B$ and an R -homomorphism $u_1: T_1 \rightarrow H_1$, such that $u_1(c) = (\rho_3 \circ w \circ i_3)(c), \forall c \in C_1$. Since A is a submodule of T_2 , $\rho_4 \circ w \circ i_2: A \rightarrow H_2$ is an R -homomorphism and H_2 is T_2 - e^*s -ejective, then there exists $C_2 \leq_{e^*s} A$ and an R -homomorphism $u_2: T_2 \rightarrow H_2$, such that $u_2(c) = (\rho_4 \circ w \circ i_2)(c), \forall c \in C_2$. Take $C = C_1 \oplus C_2$, then $C \leq_{e^*s} V$. Let $c \in C$, then $c = (c_1, c_2), c_1 \in C_1, c_2 \in C_2$, define $p: T_1 \oplus T_2 \rightarrow H_1 \oplus H_2$ by: $\gamma(c) = \gamma(c_1, c_2) = (u_1(c_1), u_2(c_2)) = ((\rho_3 \circ w \circ i_3)(c_1), (\rho_4 \circ w \circ i_2)(c_2)) = (\rho_3 w(c_1), \rho_4 w(c_2)) = w(c_1, c_2) = w(c), \forall c \in C$.

Proposition 2.10: Let T and H be R -modules, then H is T -e*s-ejective if and only if there exists $X \leq_{e*s} T$, such that H is C -e*s-ejective and for any R -homomorphism $\gamma': T \rightarrow H$, such that $\gamma'(a) = \gamma(a), \forall a \in A$.

Proof: $\Rightarrow$) Take $X = T$, we get the result.

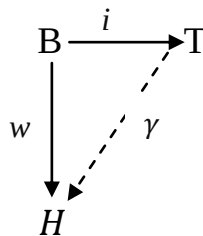
$\Leftarrow$) Let V be a submodule of T and let $w: V \rightarrow H$ be any R -homomorphism. Put $L = V \cap X$, consider the following diagram:



where i, j are the inclusion maps. Since H is X -e*s-ejective, then there exists $A \leq_{e*s} L$, and an R -homomorphism $\gamma: X \rightarrow H$, such that $w|_A = \gamma|_A$. By assumption, there exists $B \leq_{e*s} C$ and an R -homomorphism $\gamma': T \rightarrow H$, such that $\gamma'(a) = \gamma(a), \forall a \in A$. Since $X \leq T$, then $L = V \cap X \leq V \cap T = V$, but $A \leq_{e*s} L$, then $A \leq_{e*s} V$, by [5]. Put $A' = A \cap B \leq_{e*s} V \cap C \leq_{e*s} V$, since $\gamma'(c) = \gamma(c), \forall c \in B$ and $\gamma(c) = w(c), \forall c \in A$, then $\gamma'(c) = w(c), \forall c \in B'$, then H is T -e*s-ejective.

Proposition 2.11: Let T and H be R -modules and B has no proper e^* -essential submodule of T , if H is T -e*s-ejective, then for each homomorphism $w: B \rightarrow H$, there exists a homomorphism $\gamma: T \rightarrow H$ such that $w(b) = \gamma(b), \forall b \in B$.

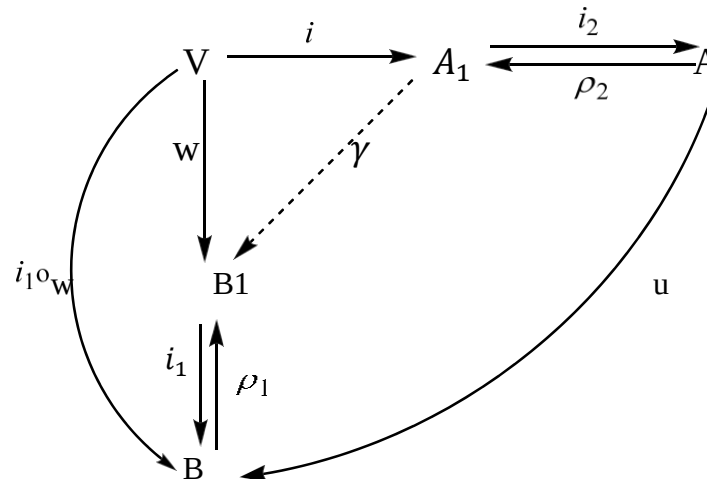
Proof: $\Rightarrow$) Let $w: B \rightarrow H$ be an R -homomorphism. Consider the following diagram:



where i is the inclusion map since H is T -e*s-ejective, then there exists $C \leq_{e*s} B$ and homomorphism $\gamma: T \rightarrow H$, such that $w(c) = \gamma(c), \forall c \in C$. Since B has no proper e^* -essential submodule, then $B = C$.

Proposition 2.12: Let B and A be R -modules, such that B is A -e*s-ejective. If B_1 is a direct summand of B and A_1 is a direct summand of A , then B_1 is A_1 -e*s-ejective.

Proof: Let V be a submodule of A_1 and let $w: V \rightarrow B_1$ be an R -homomorphism. Consider the following diagram:

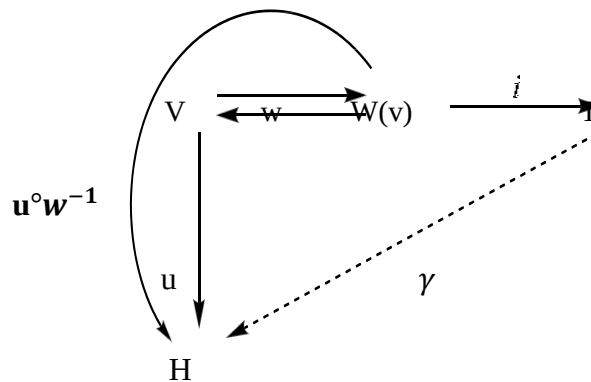


where i, i_1, i_2 , are the inclusion maps, and ρ_1, ρ_2 are the projection maps, since V is a submodule of B , $i_1 \circ w: V \rightarrow B$ is an R -homomorphism and B is A -e*s-ejective, then there exists $C \leq_{e*s} V$ and an R -homomorphism $u: A \rightarrow B$, such that $u(x) = (i_1 \circ w)(c), \forall c \in C$.

Take $\gamma = \rho_1 \circ u \circ i_2: B_1 \rightarrow A_1$ is an R -homomorphism, such that: $\gamma(c) = (\rho_1 \circ u \circ i_2)(c) = (\rho_1 \circ u)(c) = \rho_1(u(c)) = \rho_1(i_1 \circ w)(c) = w(c), \forall c \in C$.

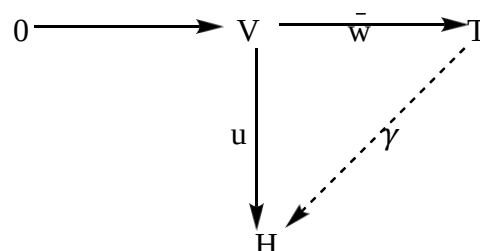
Proposition 2.13: Let H and T be R -modules, then H is H -e*s-ejective if and only if for each monomorphism $w: V \rightarrow T$ and each homomorphism $u: V \rightarrow H$, where V is any R -module there exists $C \leq_{e*s} V$ and an R -homomorphism $\gamma: T \rightarrow H$, such that $\gamma \circ w(c) = u(c), \forall c \in C$.

Proof: $\Rightarrow$): Suppose that H is T -e*s-ejective, let $w: V \rightarrow T$ be an R -homomorphism and $u: V \rightarrow H$ be an R -homomorphism. Consider the following diagram:



where i is the inclusion map, since H is T -e*s-ejective with $w(V) \leq T$ and $u \circ w^{-1}: (V) \rightarrow H$, there exists $L \leq_{e*s} (V)$ and an R -homomorphism $\gamma: T \rightarrow H$, such that $\gamma(c) = u \circ w^{-1}(c), \forall c \in L$. Since $L \leq_{e*s} (V)$ and w is an R -homomorphism, then $C = w^{-1}(L) \leq_{e*s} V$ [4]. Then $\gamma \circ w(c) = u(c), \forall x \in X$, then we get the result.

($\Leftarrow$): Let H be a submodule of T and let $u: V \rightarrow H$ be an R -homomorphism. Consider the following diagram:



where w be the inclusion map, by our assumption there exists $C \leq_{e^*s} V$ and an R -homomorphism $\gamma: T \rightarrow H$, such that $\gamma(c) = u(c)$, $\forall c \in X$, then H is T - e^*s -ejective.

Conclusion:

- e^*s -ejective modules were introduced depending on the concept of e^* -singular submodule.
- The direct of T - e^*s -ejective module is T - e^*s -ejective. The direct sum of e^*s -ejective is e^*s -ejective.
- Every fully invariant of self e^*s -ejective is self e^*s -ejective.
- Under certain conditions H_1 is T_1 - e^*s -ejective and H_2 is T_2 - e^*s -ejective if $T_1 \oplus T_2$ be module, then $H_1 \oplus H_2$ is $T_1 \oplus T_2$ - e^*s -ejective.

References.

- [1] H.R Baonon and W. Khaled, " e^* -essential submodule," Euro Pean Journal of Pure and Applied Mathematics Vol.15, No. pp. 224-228, 2022.
- [2] K. Goodearl, Ring Theory: Nonsingular Rings and Modules, Marcel, Dekker, New York, 1976.
- [3] F.Kasch, modules and rings, London, Academic, press, 1982.
- [4] V. Erdogdu, Distributive Modules, Can. Math. Bull, 30, pp. 248-254, 1987.
- [5] A. Kabban, Wasan Khalid, " e^* -singular essential submodules," Journal of Theory, Mathematics and Physics Vol. 5, No. 01, pp. 31-39, 2026.
- [6] A.G. Öscan, modules with small cyclic submodules in their injective hulls, comm, Algebra Vol 4, No3. P.1575-1589, 2002.
- [7] A.G. Öscan, A.Harmanes and P.F. Smith Duo modules, Glasgow math J. 48. pp. 533-545. 2006.