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Affine Fuzzy Set

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Abstract.

In this paper, we introduce a new concept that is the affine fuzzy set and we give some properties of it .

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1. Introduction

There are many kind of fuzzy sets in a fuzzy vector space . Katsaras in [1] was defined a convex fuzzy set, a balanced fuzzy set and absorbing fuzzy set. In this paper we defined and introduced a new kind of fuzzy sets that is affine fuzzy sets by dependence on a concept of affine set which introduced in [2] . Some properties of affine fuzzy sets was prove it in this paper.

2. Preliminaries

Let X be a non-empty set. A fuzzy set in X is the element of the set I^X of all functions from X into the unit interval I =[0,1]. If $C_\alpha : X \to I$ is a function defined by $C_\alpha(x) = \alpha$ for all $x \in X$, $\alpha \in I$, then C_α is called a constant fuzzy set. Let X be a vector space over a field F , where F is the space of either the real or the complex numbers. If A_1 , A_2 , …, A_n are fuzzy sets in X , then the sum $A_1 + A_2 + \cdots + A_n$ (see [1]) is the fuzzy set A in X defined by

$A(x) = \sup_{x_1+x_2+\cdots+x_n=x} \min\{A_1(x_1), A_2(x_2),..., A_n(x_n)\}$. Also, if A is a fuzzy set in X and $\alpha \in F$, then αA is a fuzzy defined by

$$\alpha A(x) = \begin{cases} A(x/\alpha) & if\ \alpha \neq 0\ for\ x \in X \\ 0 & if\ \alpha = 0, x \neq 0 \\ \sup_{y\in X} A(y) & if\ \alpha = 0, x = 0 \end{cases}.$$

Let X, Y be non-empty sets and $f : X \to Y$ be a function. If A_1 and A_2 are fuzzy sets in X, then $f(A_1) \subset f(A_2)$.

Let A be a subset of a vector space X on a field F. Then A is an affine set in X (see [2]) if $\lambda A + (1-\lambda)A \subset A$ for all $\lambda \in F$. The smallest affine set of X which contains a subset A of X is called the affine spanned of A (in simple $aff(A)$) and $aff(A) = \{\sum_{k=1}^{n} \lambda_k x_k : \lambda_k \in F, x_k \in A, \sum_{k=1}^{n} \lambda_k = 1\}$.

3. Main Results

Theorem 3.1. [2]

Let A, B be fuzzy sets in a vector space X over F, and let $\lambda, \alpha \in F$. Then,

(1) $\alpha(\lambda A) = \lambda(\alpha A) = (\alpha\lambda)A$;

(2) If $A \subseteq B$, then $\lambda A \subseteq \lambda B$.

(3) $\alpha(A+B) = \alpha A + \alpha B$

Definition 3.2.

Let X be a vector space over F. A fuzzy set A in X is called an affine fuzzy set if $\lambda A + (1-\lambda)A \subset A$ for each $\lambda \in F$.

Theorem 3.3.

Let A be a fuzzy set in a vector space X over F and $\lambda \in F$, then the following statement are equivalent.

(1) A is an affine fuzzy set.

(2) for all x, y in X, we have $A(\lambda x + (1-\lambda)y) \geq \min\{A(x), A(y)\}$.

Proof :

$(1) \Rightarrow (2)$

Suppose that A is an affine fuzzy set, by (Definition 3.2) we have $\lambda A+(1-\lambda)A\subset A$ for each $\lambda\in F$. Now, for each $x,y\in X$

$A(\lambda x+(1-\lambda)y)\geq(\lambda A+(1-\lambda)A)(\lambda x+(1-\lambda)y)$

$=\sup\limits_{\lambda x+(1-\lambda)y=x_1+y_1}\min\{(\lambda A)(x_1),((1-\lambda)A)(y_1)\}$

$\geq\min\{(\lambda A)(\lambda x),((1-\lambda)A)((1-\lambda)y)\}\geq\min\{A(x),A(y)\}$.

$(2)\Rightarrow(1)$

Let $x\in X$, $(\lambda A+(1-\lambda)A)(x)=\sup\limits_{x_1+x_2=x}\min\{(\lambda A)(x_1),((1-\lambda)A)(x_2)\}$

(a) If $\lambda\neq0$ and $1-\lambda\neq0$, then

$(\lambda A)(x_1)=A(\frac{1}{\lambda}x_1)$ and $((1-\lambda)A)(x_2)=A(\frac{1}{(1-\lambda)}x_2)$

Thus, $(\lambda A+(1-\lambda)A)(x)=\sup\limits_{x_1+x_2=x}\min\{A(\frac{1}{\lambda}x_1),A(\frac{1}{(1-\lambda)}x_2)\}$

But,

$\min\{A(\frac{1}{\lambda}x_1),A(\frac{1}{(1-\lambda)}x_2)\}\leq A(\lambda(\frac{1}{\lambda}x_1)+(1-\lambda)(\frac{1}{(1-\lambda)}x_2))=A(x_1+x_2)=A(x)$.

(b) If $\lambda\neq0$ and $1-\lambda=0$, then $(\lambda A)(x_1)=A(x_1)$ and

$$((1-\lambda)A)(x_2)=\begin{cases}0 & ,x_2\neq0\\ \sup\limits_{z\in X}A(z) & ,x_2=0\end{cases}$$

(i) If $x_2\neq0$, then $((1-\lambda)A)(x_2)=0$ and $(\lambda A+(1-\lambda)A)(x)=\sup\limits_{x_1+x_2=x}\min\{A(x_1),0\}=0$

Since $((1-\lambda)A)(x_2)\geq A(x_2)$, then $A(x_2)=0$ and we get $\min\{A(x_1),A(x_2)\}=0$.

(ii) If $x_2=0$, then $((1-\lambda)A)(x_2)=\sup\limits_{z\in X}A(z)$ and

$(\lambda A+(1-\lambda)A)(x)=\sup\limits_{x_1=x}\min\{\lambda A(x_1),\sup\limits_{z\in X}A(z)\}\leq\sup\limits_{x_1=x}A(x_1)=A(x)$.

(c) If $\lambda=0$ and $1-\lambda\neq0$, by the same way in (b).

(d) if $\lambda=0$ and $1-\lambda=0$, its impossible.

Theorem 3.4.

Let X be a vector space over a field F and $x_\circ\in X$, then

(1) the constant fuzzy set C_α , $\alpha\in I$ is an affine fuzzy set.

(2) the characteristic function of $\{x_\circ\}$(in symbol $\chi_{\{x_\circ\}}$) is an affine fuzzy set.

Proof :

(1) clear.

(2) Let $x, y \in X$, $\lambda \in F$ and suppose that

$\chi_{\{x_\circ\}}(\lambda x + (1-\lambda)y) < \min\{\chi_{\{x_\circ\}}(x), \chi_{\{x_\circ\}}(y)\}$

Then $\chi_{\{x_\circ\}}(\lambda x + (1-\lambda)y) = 0$ and $\min\{\chi_{\{x_\circ\}}(x), \chi_{\{x_\circ\}}(y)\} = 1$. Thus,

$\lambda x + (1-\lambda)y \neq x_\circ$ and $x = y = x_\circ$ implies that $x_\circ \neq x_\circ$ which is impossible. Hence,

$\chi_{\{x_\circ\}}(\lambda x + (1-\lambda)y) \geq \min\{\chi_{\{x_\circ\}}(x), \chi_{\{x_\circ\}}(y)\}$. Using (Theorem 3.3), we get the result.

Theorem 3.5.

Let A be a fuzzy set in a vector space X over F . Then, A is an affine fuzzy set in X if and only if $A_{[\alpha]} = \{x \in X : A(x) \geq \alpha\}$ is an affine set in X for all $0 \leq \alpha \leq 1$.

Proof :

Suppose that A is an affine fuzzy set in X . Let $x, y \in A_{[\alpha]}$, then $A(x) \geq \alpha$ and $A(y) \geq \alpha$, thus, $\min\{A(x), A(y)\} \geq \alpha$. Since A is an affine fuzzy set in X , then $A(\lambda x + (1-\lambda)y) \geq \min\{A(x), A(y)\} \geq \alpha$, for all $\lambda \in F$. Thus, $\lambda x + (1-\lambda)y \in A_{[\alpha]}$. By other word $A_{[\alpha]}$ is an affine set in X .

The converse :

Suppose that $A_{[\alpha]}$ is an affine set in X for all $\alpha \in [0,1]$ and let $x, y \in X$, $\lambda \in F$.

Let $\alpha = \min\{A(x), A(y)\}$, then $\alpha \in [0,1]$. Now, $A(x) \geq \min\{A(x), A(y)\} = \alpha$ and $A(y) \geq \min\{A(x), A(y)\} = \alpha$

That is mean $x, y \in A_{[\alpha]}$. But $A_{[\alpha]}$ is an affine set , then $\lambda x + (1-\lambda)y \in A_{[\alpha]}$. Thus, $A(\lambda x + (1-\lambda)y) \geq \alpha = \min\{A(x), A(y)\}$. From (Theorem 3.3.), A is an affine fuzzy set in X .

Theorem 3.6.

Let A and B be affine fuzzy sets in a vector space X over F , and let $\alpha \in F$. Then, αA, $A \cap B$ and $A + B$ are affine fuzzy sets in X .

Proof :

(1) To prove αA is an affine fuzzy set in X. From (Theorem 3.1.(1),(3)) , for $\lambda \in F$
$\lambda(\alpha A)+(1-\lambda)(\alpha A)=\alpha(\lambda A)+\alpha((1-\lambda)A)=\alpha(\lambda A+(1-\lambda)A)\subset \alpha A.$

(2) To prove $A\cap B$ is an affine fuzzy set in X. Let $\lambda \in F$ and Suppose that for all $x, y \in X$, let

$g_1 : X \to X,\ g_1(x)=\lambda x;$
$g_2 : X \to X,\ g_2(x)=(1-\lambda)x;$
$g_3 : X \to X,\ g_3(x)=x;$

are functions. Since A and B are affine fuzzy sets in X, then $\lambda A+(1-\lambda)A\subset A$ and $\lambda B+(1-\lambda)B\subset B$. In other words $g_1(A)+g_2(A)\subset g_3(A)$ and $g_1(B)+g_2(B)\subset g_3(B)$. Now, $g_1(A\cap B)\subset g_1(A)$ and $g_2(A\cap B)\subset g_2(A)$. Likewise $g_1(A\cap B)\subset g_1(B)$ and $g_2(A\cap B)\subset g_2(B)$. Thus, $g_1(A\cap B)+g_2(A\cap B)\subset g_1(A)+g_2(A)\subset g_3(A)$ and $g_1(A\cap B)+g_2(A\cap B)\subset g_1(B)+g_2(B)\subset g_3(B)$. Subsequently
$\lambda(A\cap B)+(1-\lambda)(A\cap B)=g_1(A\cap B)+g_2(A\cap B)\subset g_3(A)\cap g_3(B)=A\cap B$.

(3) To prove $A+B$ is an affine fuzzy set in X. By using the same style in (2). For $\lambda \in F$

$$\begin{aligned}\lambda(A+B)+(1-\lambda)(A+B)&=\lambda A+(1-\lambda)A+\lambda B+(1-\lambda)B\\ &=g_1(A)+g_2(A)+g_1(B)+g_2(B)\\ &\subset g_3(A)+g_3(B)=A+B.\end{aligned}$$

Definition 3.7.

Let A be a fuzzy set in a vector space X over F, then the smallest affine fuzzy set which contains A is called the affine spanned of A (in symbol $Span(A)$).

Theorem 3.8.

Let A be a fuzzy set of a vector space X over F. Then the affine spanned of A ($Span(A)$) is the set $B=\cup\{\ \mu_j=\sum_{i=1}^{n}\lambda_{ij}A:\lambda_{ij}\in F,\ \sum_{i=1}^{n}\lambda_{ij}=1\}$.

Proof :

It is clear that $A\subset B\subset Span(A)$. We will finish the proof by showing that B is an affine fuzzy set. Let now $\lambda \in F$, $\mu_{k_1},\mu_{k_2}\subset B$, then there is

$\lambda_{ik_1}, \lambda_{jk_2} \in F,\ \begin{matrix} i = 1,\dots,n \\ j = 1,\dots,m \end{matrix}$ such that $\mu_{k_1} = \sum_{i=1}^{n} \lambda_{ik_1} A$, $\mu_{k_2} = \sum_{j=1}^{m} \lambda_{jk_2} A$ and $\sum_{i=1}^{n} \lambda_{ik_1} = 1$, $\sum_{j=1}^{m} \lambda_{jk_2} = 1$.

Now,

$$\lambda\mu_{k_1} + (1-\lambda)\mu_{k_2} = \lambda\left(\sum_{i=1}^{n} \lambda_{ik_1} A\right) + (1-\lambda)\left(\sum_{j=1}^{m} \lambda_{jk_2} A\right)$$

$$= \lambda\lambda_{1k_1} A + \cdots + \lambda\lambda_{nk_1} A + (1-\lambda)\lambda_{1k_2} A + \cdots + (1-\lambda)\lambda_{mk_2} A \subset B$$

Moreover, $\sum_{i=1}^{n} \lambda\lambda_{ik_1} + \sum_{j=1}^{m} (1-\lambda)\lambda_{jk_2} = 1$. It follows that $\lambda B + (1-\lambda)B \subset B$, which complete the proof.

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