

Construction of (k, r) -caps in the Projective 3-Space $PG(3, q)$ over the Galois Field $GF(19)$

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ABSTRACT

construct and investigate (k, r) -cap in the projective three-dimensional space $PG(3, 19)$ defined over the finite field $GF(19)$. we present an explicit construction of the maximum complete cap- $(k, 2)$ in $PG(3, 19)$, commonly referred to as $(k, 2)$. By systematically adjoining zero-index points, this construction is extended to obtain maximal complete (k, r) -caps. These results confirm both the existence and completeness of such configurations and provide explicit constructions as a foundation for further developments in finite geometry and applications to coding theory.

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1. Introduction:

In this paper, we present the construction of (k, r) -caps in $PG(3, q)$ over the finite field $GF(q)$, where $q = 19$. Many scholars have investigated the development and characterization of (k, r) -caps in finite geometry $PG(2, q)$ for $2 \leq q \leq 172$. In the current study, we concentrate on finite three-dimensional incidence geometry space $PG(3, q)$ over the finite field $GF(q)$, which can be seen as a higher-dimensional extension of the finite geometry over finite fields. Hirschfeld [1] established the fundamental definitions and principal theorems of geometry over finite fields. Al-Mukhtar [2] examined complete arcs and surfaces in three-dimensional incidence geometry space over finite fields and constructed caps- (k, r) in $PG(3, q)$ for $q=2, 3$, and 5. Kareem [3] provided a construction of a complete (k, r) -caps in $PG(3, 4)$. Abdulla and Kasm [4] applied algebraic geometry techniques in the three-dimensional projective space $PG(3, 7)$.

In this paper, we present the construction of (k, r) -caps in $PG(3, q)$ over the finite field $GF(19)$. The manuscript is structured into five sections. Section 1 introduces the preliminaries of three-dimensional projective space, including key definitions and theorems. Sections 2 through 5 focus on the establishment of a maximum complete (k, r) -caps for $r=2, 3, 4, \dots, 20$. All constructions have been performed manually, without relying on computer software finite field $GF(q)$.

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1. Preliminaries

Definition 1 (Three-Dimensional Finite Projective Structure) [3].

A three-dimensional finite projective structure over a field (q) , denoted by $(PG(3,q))$, is a geometric configuration consisting of points, lines, and planes, together with incidence relations among them, satisfying the following axioms:

1. Every pair of distinct points lies on exactly one line.
2. Any set of three points that do not all lie on the same line, or any line together with a point not on it, uniquely defines a plane.
3. Two distinct lines contained in the same plane intersect in precisely one point.
4. Any line that does not lie entirely within a given plane meets that plane in exactly one point.
5. Any two distinct planes share exactly one common line.

For a finite field $(GF(q))$, where $(p = q^m)$ for some prime (p) and integer $(m \geq 1)$, the structure $(PG(3,q))$ forms a three-dimensional finite projective structure over $(GF(q))$. Each point in $(PG(3,q))$ can be represented by a non-zero quadruple (x_1, x_2, x_3, x_4) , where $x_1, x_2, x_3, x_4 \in GF(q)$, excluding the zero quadruple $(0,0,0,0)$. Two quadruples (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) represent the same point if there exists $\lambda \in GF(q) \setminus \{0\}$

$$(x_1, x_2, x_3, x_4) = \lambda (y_1, y_2, y_3, y_4).$$

Similarly, a plane in $PG(3,q)$ can be described by a non-zero quadruple $[x_1, x_2, x_3, x_4]$, where two quadruples $[x_1, x_2, x_3, x_4]$ and $[y_1, y_2, y_3, y_4]$ represent the same plane if there exists $\lambda \in GF(q) \setminus \{0\}$ such that

$$[x_1, x_2, x_3, x_4] = \lambda [y_1, y_2, y_3, y_4].$$

A point $P(x_1, x_2, x_3, x_4)$ is incident with the plane $\pi[a_1, a_2, a_3, a_4]$ if and only if

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = 0.$$

Definition 2 (Plane π) [1].

A plane π in $PG(3,q)$ is the set of all points $P(x_1, x_2, x_3, x_4)$ satisfying a linear equation of the form

$u_1 x_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 = 0$, where $u_1, u_2, u_3, u_4 \in GF(q)$. Such a plane is denoted by

$$\pi[u_1, u_2, u_3, u_4]$$

Theorem 1 [2].

Consider the matrix whose rows are the homogeneous coordinate vectors of four distinct points A, B, C, and D in $PG(3,q)$. These points belong to a single plane precisely when this matrix is singular; equivalently, its determinant is equal to zero.

Corollary 1 [2].

Let P_1, P_2, P_3 , and P_4 be four distinct points in $PG(3,q)$. If these points lie on a common line, then the determinant associated with their homogeneous coordinate vectors vanishes, i.e., $\Delta=0$

Theorem 2 [2]. The point $PG(3,q)$ can be represented in unique canonical forms as follows:

$$(1,0,0,0), (U, 1,0,0), (U, V, 1,0), (U, V, B, 1) \text{ for all } V, U, B \text{ in } GF(q).$$

Theorem 3 [2]. The planes of $PG(3,q)$ can be expressed in unique canonical forms as follows:

$$(1,0,0,0), (U, 1,0,0), (U, V, 1,0), (U, V, B, 1) \text{ for all } V, U, B \text{ in } GF(q).$$

Theorem 4 [2].

A three-dimensional finite projective structure $PG(3,q)$ over the finite field $GF(q)$ satisfies the following combinatorial properties:

- a. Each line comprises precisely $(q + 1)$ points, and all point lies on exactly $(q + 1)$ lines.
- b. planes $(q^2 + q + 1)$.
- c. The total number of points in $PG(3,q)$ is $(q^3 + q^2 + q + 1)$, which is equal to the total number of planes in the structure.
- d. Each line lies on exactly $(q+1)$ planes

Theorem 5 [2]. There exists $(q^2 + 1)(q^2 + q + 1)$ of lines in $PG(3,q)$.

Definition 3 [1]:

A cap- (k,r) in $PG(n,q)$, $n \geq 3$, is a set $C = \{P_1, P_2, \dots, P_k\} \subset PG(n,q)$

such that for every line $\ell \subset PG(n,q)$, $|\ell \cap C| \leq r$.

In particular, a $(k,2)$ -cap in $PG(3,q)$ satisfies

$|\ell \cap C| \leq 2 \Rightarrow$ no three points of C are collinear

Definition 4: A (k,r) -cap $C \subset PG(n,q)$ is said to be complete if there does not exist a $(k+1,r)$ -cap $C' \subset PG(n,q)$ such that $C \subset C'$.

Definition 5 [2]:

let C_i represent the number of points of index i in $PG(3,q)$ outside the (k,r) -cap C . The corresponding constants $\{C_i\}$ then satisfy:

- i. $\sum_{\alpha}^{\beta} C_i = q^3 + q^2 + q + 1 - k$
- ii. $\sum_{\alpha}^{\beta} i C_i = \frac{k(k-1)(k-2) \dots (k-n+1)}{n!} \cdot (q^2 + q + 1 - n)$
where $\alpha = \min\{i / C_i \neq 0\}$ $\beta = \max\{i / C_i \neq 0\}$

Remark 1 [3]. A (k,r) -CAP IS COMPLETE IF $\sum C_0 = 0$.

Definition 6 [2].

For a (k,r) -cap $C \subset PG(n,q)$ a line $\ell \not\subset C$ is called an i -secant if

$|\ell \cap C| = i$. In particular:

1. $i=0$: ℓ is an external line,
2. $i=1$: ℓ is a unisecant,
3. $i=2$: ℓ is a bisecant,
4. $i=3$: ℓ is a trisecant.

Remark 1 [3]. A (k,r) -CAP IS maximum IF AND ONLY IF EVERY LINE IN $PG(3,q)$ IS EITHER A 0-SECANT OR AN r -SECANT.

Theorem 6 [1]. A maximum $(k, 2)$ -cap in $PG(3, q)$ is an ovaloid.

2- The $(k, 2)$ –caps in $PG(3,19)$:

$PG(3,19)$ contains 7240 points and 7240 planes such that each point is on 381 planes and every plane contains 381 points, and every line contains 20 points and it is the intersection of 20 planes, (See table 1 and 2). In Table 1, the set

A	1	2	21	382	763
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is chosen, representing the unit and reference points

$$1(1,0,0,0), 2(0,1,0,0), 21(0,0,1,0), 382(0,0,0,1), 763(1,1,1,1),$$

This set forms a $(5, 2)$ -cap since no three points of A are collinear, as shown in Table 2.

However, A is not complete, since there exist points of index zero that are not included in the cap.

383	-	762	764	-	1103	1104	...	1464	1465	...	1825	1826
...	2186	2187	...	2547	2548	...	2908	2909	...	3269	3270	...
3630	3631	...	3991	3992	...	4352	4353	...	4713	4714	...	5074
5075	...	5435	5436	...	5796	5797	...	6157	6158	...	6518	6519
...	6879	6880	...	7240								

B then, some of these points can be added to A to form a complete $(383,2)$ -cap B

L1																		n
40	59	78	97	116	135	154	173	192	211	230	249	268	287	306	325	344	363	378
22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	
20	58	76	94	112	130	148	166	184	202	220	238	256	274	292	310	328	346	
11	49	77	86	114	123	151	160	188	197	225	234	262	271	299	308	336	345	
8	46	71	96	102	127	152	158	183	208	214	239	264	270	295	320	326	351	
16	54	68	82	115	129	143	157	190	204	218	232	265	279	293	307	340	354	
17	55	70	85	100	134	149	164	179	194	228	243	258	273	288	322	337	352	
5	43	65	87	109	131	153	156	178	200	222	244	266	269	291	313	335	357	
10	48	75	83	110	118	145	172	180	207	215	242	250	277	304	312	339	347	
9	47	73	80	106	132	139	165	191	198	224	231	257	283	290	316	342	349	
4	42	63	84	105	126	147	168	189	210	212	233	254	275	296	317	338	359	
19	57	74	91	108	125	142	159	176	193	229	246	263	280	297	314	331	348	
14	52	64	95	107	119	150	162	174	205	217	248	260	272	303	315	327	358	
13	51	62	92	103	133	144	155	185	196	226	237	267	278	289	319	330	360	
18	56	72	88	104	120	136	171	187	203	219	235	251	286	302	318	334	350	
6	44	67	90	113	117	140	163	186	209	213	236	259	282	305	309	332	355	
7	45	69	93	98	122	146	170	175	199	223	247	252	276	300	324	329	353	
15	53	66	79	111	124	137	169	182	195	227	240	253	285	298	311	343	356	
12	50	60	89	99	128	138	167	177	206	216	245	255	284	294	323	333	362	
3	41	61	81	101	121	141	161	181	201	221	241	261	281	301	321	341	361	

Then $B = A \cup L_1 =$

B																				n
1	40	59	78	97	116	135	154	173	192	211	230	249	268	287	306	325	344	363		383
2	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39		
21	20	58	76	94	112	130	148	166	184	202	220	238	256	274	292	310	328	346	364	
382	11	49	77	86	114	123	151	160	188	197	225	234	262	271	299	308	336	345	373	
763	8	46	71	96	102	127	152	158	183	208	214	239	264	270	295	320	326	351	376	
	16	54	68	82	115	129	143	157	190	204	218	232	265	279	293	307	340	354	368	
	17	55	70	85	100	134	149	164	179	194	228	243	258	273	288	322	337	352	367	
	5	43	65	87	109	131	153	156	178	200	222	244	266	269	291	313	335	357	379	
	10	48	75	83	110	118	145	172	180	207	215	242	250	277	304	312	339	347	374	
	9	47	73	80	106	132	139	165	191	198	224	231	257	283	290	316	342	349	375	
	4	42	63	84	105	126	147	168	189	210	212	233	254	275	296	317	338	359	380	
	19	57	74	91	108	125	142	159	176	193	229	246	263	280	297	314	331	348	365	
	14	52	64	95	107	119	150	162	174	205	217	248	260	272	303	315	327	358	370	
	13	51	62	92	103	133	144	155	185	196	226	237	267	278	289	319	330	360	371	
	18	56	72	88	104	120	136	171	187	203	219	235	251	286	302	318	334	350	366	
	6	44	67	90	113	117	140	163	186	209	213	236	259	282	305	309	332	355	378	
	7	45	69	93	98	122	146	170	175	199	223	247	252	276	300	324	329	353	377	
	15	53	66	79	111	124	137	169	182	195	227	240	253	285	298	311	343	356	369	
	12	50	60	89	99	128	138	167	177	206	216	245	255	284	294	323	333	362	372	
	3	41	61	81	101	121	141	161	181	201	221	241	261	281	301	321	341	361	381	

such that B is a maximal $(383,2)$ -cap in $PG(3,19)$. Because all line meets B in either 0 or 2 points, B is termed an literature. Let B be a $(383,2)$ -cap. The points of index zero are defined as

3- The $(k,3)$ -caps in $PG(3,19)$:

Distinct $(k,3)$ -caps can be obtained by adjoining certain points of index zero to B, namely

L2																		n
401	420	439	458	477	496	515	534	553	572	591	610	629	648	667	686	705	724	702
743	1104	1465	1826	2187	2548	2909	3270	3631	3992	4353	4714	5075	5436	5797	6158	6519	6880	
383	384	385	386	387	388	389	390	391	392	393	394	395	396	397	398	399	400	
419	437	455	473	491	509	527	545	563	581	599	617	635	653	671	689	707	725	
410	438	447	475	484	512	521	549	558	586	595	623	632	660	669	697	706	734	
407	432	457	463	488	513	519	544	569	575	600	625	631	656	681	687	712	737	
415	429	443	476	490	504	518	551	565	579	593	626	640	654	668	701	715	729	
416	431	446	461	495	510	525	540	555	589	604	619	634	649	683	698	713	728	
404	426	448	470	492	514	517	539	561	583	605	627	630	652	674	696	718	740	
409	436	444	471	479	506	533	541	568	576	603	611	638	665	673	700	708	735	
408	434	441	467	493	500	526	552	559	585	592	618	644	651	677	703	710	736	
403	424	445	466	487	508	529	550	571	573	594	615	636	657	678	699	720	741	
418	435	452	469	486	503	520	537	554	590	607	624	641	658	675	692	709	726	
413	425	456	468	480	511	523	535	566	578	609	621	633	664	676	688	719	731	
412	423	453	464	494	505	516	546	557	587	598	628	639	650	680	691	721	732	
417	433	449	465	481	497	532	548	564	580	596	612	647	663	679	695	711	727	
405	428	451	474	478	501	524	547	570	574	597	620	643	666	670	693	716	739	
406	430	454	459	483	507	531	536	560	584	608	613	637	661	685	690	714	738	
414	427	440	472	485	498	530	543	556	588	601	614	646	659	672	704	717	730	
411	421	450	460	489	499	528	538	567	577	606	616	645	655	684	694	723	733	
402	422	442	462	482	502	522	542	562	582	602	622	642	662	682	702	722	742	
744	1106	1468	1830	2192	2554	2916	3278	3640	4002	4364	4726	5088	5450	5812	6174	6536	6898	
745	1108	1471	1834	2197	2560	2923	3286	3649	3993	4356	4719	5082	5445	5808	6171	6534	6897	
746	1110	1474	1838	2202	2566	2911	3275	3639	4003	4367	4731	5076	5440	5804	6168	6532	6896	
747	1112	1477	1842	2188	2553	2918	3283	3648	3994	4359	4724	5089	5454	5800	6165	6530	6895	
748	1114	1480	1827	2193	2559	2925	3272	3638	4004	4370	4717	5083	5449	5815	6162	6528	6894	
749	1116	1483	1831	2198	2565	2913	3280	3647	3995	4362	4729	5077	5444	5811	6159	6526	6893	
750	1118	1467	1835	2203	2552	2920	3288	3637	4005	4354	4722	5090	5439	5807	6175	6524	6892	
751	1120	1470	1839	2189	2558	2927	3277	3646	3996	4365	4715	5084	5453	5803	6172	6522	6891	
752	1122	1473	1843	2194	2564	2915	3285	3636	4006	4357	4727	5078	5448	5799	6169	6520	6890	
753	1105	1476	1828	2199	2551	2922	3274	3645	3997	4368	4720	5091	5443	5814	6166	6537	6889	
754	1107	1479	1832	2204	2557	2910	3282	3635	4007	4360	4732	5085	5438	5810	6163	6535	6888	
755	1109	1482	1836	2190	2563	2917	3271	3644	3998	4371	4725	5079	5452	5806	6160	6533	6887	
756	1111	1466	1840	2195	2550	2924	3279	3634	4008	4363	4718	5092	5447	5802	6176	6531	6886	
757	1113	1469	1844	2200	2556	2912	3287	3643	3999	4355	4730	5086	5442	5798	6173	6529	6885	
758	1115	1472	1829	2205	2562	2919	3276	3633	4009	4366	4723	5080	5437	5813	6170	6527	6884	
759	1117	1475	1833	2191	2549	2926	3284	3642	4000	4358	4716	5093	5451	5809	6167	6525	6883	
760	1119	1478	1837	2196	2555	2914	3273	3632	4010	4369	4728	5087	5446	5805	6164	6523	6882	
761	1121	1481	1841	2201	2561	2921	3281	3641	4001	4361	4721	5081	5441	5801	6161	6521	6881	

Then $C = B \cup L_2 =$

C																			n
1	40	59	78	97	116	135	154	173	192	211	230	249	268	287	306	325	344	363	
2	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	
21	20	58	76	94	112	130	148	166	184	202	220	238	256	274	292	310	328	346	364
382	11	49	77	86	114	123	151	160	188	197	225	234	262	271	299	308	336	345	373
763	8	46	71	96	102	127	152	158	183	208	214	239	264	270	295	320	326	351	376
	16	54	68	82	115	129	143	157	190	204	218	232	265	279	293	307	340	354	368
	17	55	70	85	100	134	149	164	179	194	228	243	258	273	288	322	337	352	367
	5	43	65	87	109	131	153	156	178	200	222	244	266	269	291	313	335	357	379
	10	48	75	83	110	118	145	172	180	207	215	242	250	277	304	312	339	347	374
	9	47	73	80	106	132	139	165	191	198	224	231	257	283	290	316	342	349	375
	4	42	63	84	105	126	147	168	189	210	212	233	254	275	296	317	338	359	380
	19	57	74	91	108	125	142	159	176	193	229	246	263	280	297	314	331	348	365
	14	52	64	95	107	119	150	162	174	205	217	248	260	272	303	315	327	358	370
	13	51	62	92	103	133	144	155	185	196	226	237	267	278	289	319	330	360	371
	18	56	72	88	104	120	136	171	187	203	219	235	251	286	302	318	334	350	366
	6	44	67	90	113	117	140	163	186	209	213	236	259	282	305	309	332	355	378
	7	45	69	93	98	122	146	170	175	199	223	247	252	276	300	324	329	353	377
	15	53	66	79	111	124	137	169	182	195	227	240	253	285	298	311	343	356	369
	12	50	60	89	99	128	138	167	177	206	216	245	255	284	294	323	333	362	372
	3	41	61	81	101	121	141	161	181	201	221	241	261	281	301	321	341	361	381
	401	420	439	458	477	496	515	534	553	572	591	610	629	648	667	686	705	724	
	743	1104	1465	1826	2187	2548	2909	3270	3631	3992	4353	4714	5075	5436	5797	6158	6519	6880	
	383	384	385	386	387	388	389	390	391	392	393	394	395	396	397	398	399	400	
	419	437	455	473	491	509	527	545	563	581	599	617	635	653	671	689	707	725	
	410	438	447	475	484	512	521	549	558	586	595	623	632	660	669	697	706	734	
	407	432	457	463	488	513	519	544	569	575	600	625	631	656	681	687	712	737	
	415	429	443	476	490	504	518	551	565	579	593	626	640	654	668	701	715	729	
	416	431	446	461	495	510	525	540	555	589	604	619	634	649	683	698	713	728	
	404	426	448	470	492	514	517	539	561	583	605	627	630	652	674	696	718	740	
	409	436	444	471	479	506	533	541	568	576	603	611	638	665	673	700	708	735	
	408	434	441	467	493	500	526	552	559	585	592	618	644	651	677	703	710	736	
	403	424	445	466	487	508	529	550	571	573	594	615	636	657	678	699	720	741	
	418	435	452	469	486	503	520	537	554	590	607	624	641	658	675	692	709	726	
	413	425	456	468	480	511	523	535	566	578	609	621	633	664	676	688	719	731	
	412	423	453	464	494	505	516	546	557	587	598	628	639	650	680	691	721	732	
	417	433	449	465	481	497	532	548	564	580	596	612	647	663	679	695	711	727	
	405	428	451	474	478	501	524	547	570	574	597	620	643	666	670	693	716	739	
	406	430	454	459	483	507	531	536	560	584	608	613	637	661	685	690	714	738	
	414	427	440	472	485	498	530	543	556	588	601	614	646	659	672	704	717	730	
	411	421	450	460	489	499	528	538	567	577	606	616	645	655	684	694	723	733	
	402	422	442	462	482	502	522	542	562	582	602	622	642	662	682	702	722	742	
	744	1106	1468	1830	2192	2554	2916	3278	3640	4002	4364	4726	5088	5450	5812	6174	6536	6898	
	745	1108	1471	1834	2197	2560	2923	3286	3649	3993	4356	4719	5082	5445	5808	6171	6534	6897	
	746	1110	1474	1838	2202	2566	2911	3275	3639	4003	4367	4731	5076	5440	5804	6168	6532	6896	
	747	1112	1477	1842	2188	2553	2918	3283	3648	3994	4359	4724	5089	5454	5800	6165	6530	6895	
	748	1114	1480	1827	2193	2559	2925	3272	3638	4004	4370	4717	5083	5449	5815	6162	6528	6894	
	749	1116	1483	1831	2198	2565	2913	3280	3647	3995	4362	4729	5077	5444	5811	6159	6526	6893	
	750	1118	1467	1835	2203	2552	2920	3288	3637	4005	4354	4722	5090	5439	5807	6175	6524	6892	
	751	1120	1470	1839	2189	2558	2927	3277	3646	3996	4365	4715	5084	5453	5803	6172	6522	6891	
	752	1122	1473	1843	2194	2564	2915	3285	3636	4006	4357	4727	5078	5448	5799	6169	6520	6890	
	753	1105	1476	1828	2199	2551	2922	3274	3645	3997	4368	4720	5091	5443	5814	6166	6537	6889	
	754	1107	1479	1832	2204	2557	2910	3282	3635	4007	4360	4732	5085	5438	5810	6163	6535	6888	
	755	1109	1482	1836	2190	2563	2917	3271	3644	3998	4371	4725	5079	5452	5806	6160	6533	6887	
	756	1111	1466	1840	2195	2550	2924	3279	3634	4008	4363	4718	5092	5447	5802	6176	6531	6886	
	757	1113	1469	1844	2200	2556	2912	3287	3643	3999	4355	4730	5086	5442	5798	6173	6529	6885	
	758	1115	1472	1829	2205	2562	2919	3276	3633	4009	4366	4723	5080	5437	5813	6170	6527	6884	
	759	1117	1475	1833	2191	2549	2926	3284	3642	4000	4358	4716	5093	5451	5809	6167	6525	6883	
	760	1119	1478	1837	2196	2555	2914	3273	3632	4010	4369	4728	5087	5446	5805	6164	6523	6882	
	761	1121	1481	1841	2201	2561	2921	3281	3641	4001	4361	4721	5081	5441	5801	6161	6521	6881	

Let C be a complete $(743,3)$ -cap in $PG(3,p)$. So $(C_0=0)$, C is a maximum complete $(k,3)$ -cap. Proceeding similarly for higher indices, we eventually obtain the full construction of the cap.

20. The $(k,20)$ -caps in $PG(3,19)$:

Complete $(k,20)$ -caps can be constructed by adding points of index zero to a set S . Let C be a complete $(7240,20)$ -cap. So $(C_0=0)$, C is a maximum complete $(k,20)$ -cap.

W															n
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	7240
31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	
46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	
76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	
91	92	93	94	95	96	97	98	99	100	101	102	103	104	105	
106	107	108	109	110	111	112	113	114	115	116	117	118	119	120	
121	122	123	124	125	126	127	128	129	130	131	132	133	134	135	
136	137	138	139	140	141	142	143	144	145	146	147	148	149	150	
151	152	153	154	155	156	157	158	159	160	161	162	163	164	165	
166	167	168	169	170	171	172	173	174	175	176	177	178	179	180	
181	182	183	184	185	186	187	188	189	190	191	192	193	194	195	
196	197	198	199	200	201	202	203	204	205	206	207	208	209	210	
211	212	213	214	215	216	217	218	219	220	221	222	223	224	225	
226	227	228	229	230	231	232	233	234	235	236	237	238	239	240	
241	242	243	244	245	246	247	248	249	250	251	252	253	254	255	
256	257	258	259	260	261	262	263	264	265	266	267	268	269	270	
271	272	273	274	275	276	277	278	279	280	281	282	283	284	285	
286	287	288	289	290	291	292	293	294	295	296	297	298	299	300	
301	302	303	304	305	306	307	308	309	310	311	312	313	314	315	
316	317	318	319	320	321	322	323	324	325	326	327	328	329	330	
331	332	333	334	335	336	337	338	339	340	341	342	343	344	345	
346	347	348	349	350	351	352	353	354	355	356	357	358	359	360	
361	362	363	364	365	366	367	368	369	370	371	372	373	374	375	
376	377	378	379	380	381	382	383	384	385	386	387	388	389	390	
391	392	393	394	395	396	397	398	399	400	401	402	403	404	405	
406	407	408	409	410	411	412	413	414	415	416	417	418	419	420	
421	422	423	424	425	426	427	428	429	430	431	432	433	434	435	
436	437	438	439	440	441	442	443	444	445	446	447	448	449	450	
451	452	453	454	455	456	457	458	459	460	461	462	463	464	465	
466	467	468	469	470	471	472	473	474	475	476	477	478	479	480	
481	482	483	484	485	486	487	488	489	490	491	492	493	494	495	
496	497	498	499	500	501	502	503	504	505	506	507	508	509	510	
511	512	513	514	515	516	517	518	519	520	521	522	523	524	525	
526	527	528	529	530	531	532	533	534	535	536	537	538	539	540	
541	542	543	544	545	546	547	548	549	550	551	552	553	554	555	
556	557	558	559	560	561	562	563	564	565	566	567	568	569	570	
571	572	573	574	575	576	577	578	579	580	581	582	583	584	585	
586	587	588	589	590	591	592	593	594	595	596	597	598	599	600	
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7216	7217	7218	7219	7220	7221	7222	7223	7224	7225	7226	7227	7228	7229	7230	
7231	7232	7233	7234	7235	7236	7237	7238	7239	7240						

Conclusion

This study established the explicit construction of (k, r) -caps in the projective 3-space $PG(3, 19)$ over the finite field $GF(19)$. Starting from the maximal complete $(383, 2)$ –cap (ovaloid), a systematic method of adjoining zero-index points was developed to extend it into maximal complete (k, r) -caps for all $3 \leq r \leq 20$. These results highlight the rich combinatorial structure of projective 3-space and provide explicit configurations with potential applications in finite geometry and coding theory.

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