Journal of Al-Qadisiyah for Computer Science and Mathematics
3nd. Sinentific Conference 19-20/ APRIL -2011
Vol 3 No.2 Year 2011

Page 63-72

A new kind of Fuzzy Topological Vector Spaces

SALAH MAHDI ALI

Department of Mathematics, College of C. Sc. and Math.,

AL-Qadisiya University, Iraq.

salah_x9@yahoo.com

Abstract.

In this paper, we introduce a new kind of fuzzy topological vector spaces that is a locally affine fuzzy topological vector space, finological space, $S-space$. Finally, we evidence that $S-space$ has a base at zero.

2010 Mathematics Subject Classification : Primary *54A40, 46S40*
Keywords : affine fuzzy set, $S-$fuzzy set, finological space, $S-$space .

1. Introduction

Fuzzy topological vector space defined and studied by Katsaras [2, 3]. In [3] the author was give a characteristic to a base at zero for a fuzzy vector topology in fuzzy topological vector spaces. Moreover, he was introduced the concept of bounded sets in a fuzzy topological vector space.

In this paper, we introduce a new kind of fuzzy topological vector spaces that is a locally affine fuzzy topological vector space by dependence on a new concept that is affine fuzzy sets. After that we introduce a finological space by dependence on a concept of bounded sets and we study a special kind of finological spaces that is $S-$space and we evidence that $S-$space has a base at zero.

2. Preliminaries

Let X be a non-empty set. A fuzzy set in X is the element of the set I^X of all functions from X into the unit interval I =[0,1]. If $C_\alpha : X \to I$ is a function defined by $C_\alpha(x) = \alpha$ for all $x \in X$, $\alpha \in I$, then C_α is called the constant fuzzy set.

Let X be a vector space over a field F, where F is the space of either the real or the complex numbers. If $A_1, A_2, \ldots, A_n$ are fuzzy sets in X, then the sum $A_1 + A_2 + \cdots + A_n$ (see [2]) is the fuzzy set A in X defined by

$A(x) = \sup\limits_{x_1+x_2+\cdots+x_n=x} \min\{A_1(x_1), A_2(x_2),..., A_n(x_n)\}$. Also, if A is a fuzzy set in X and $\alpha \in F$, then αA is a fuzzy set defined by

$$\alpha A(x) = \begin{cases} A(x/\alpha) & \text{if } \alpha \neq 0 \text{ forall } x \in X \\ 0 & \text{if } \alpha = 0, x \neq 0 \\ \sup\limits_{y \in X} A(y) & \text{if } \alpha = 0, x = 0 \end{cases}.$$

If A is a fuzzy set in X and $x \in X$, then $x + A$ is a fuzzy set in X and defined by $(x + A)(y) = A(y - x)$. A fuzzy set A in X is called a balanced fuzzy set if $\alpha A \subset A$ or $A(\alpha x) \geq A(x)$ for all α with $|\alpha| \leq 1$. For the definition of a fuzzy topology, we will use the one given by Lowen [1] that is a fuzzy topology on a set X we will mean a subset γ of I^X satisfying the following conditions :

(i) γ contains every constant fuzzy set in X ;

(ii) If $A_1, A_2 \in \gamma$, then $A_1 \cap A_2 \in \gamma$;

(iii) If $A_i \in \gamma$ for all $i \in \Lambda$ (Λ any index), then $\bigcup\limits_{i \in \Lambda} A_i \in \gamma$.

The pair (X, γ) is called a fuzzy topological space . If $A \in \gamma$, then A is called an open fuzzy set and A is called a neighborhood of $x \in X$ if there exists an open fuzzy set B with $B \subseteq A$ and $B(x) = A(x) > 0$. Moreover, A is open fuzzy set in a fuzzy topological space X if and only if A is a neighborhood of x for each $x \in X$ with $A(x) > 0$. A fuzzy vector topology on a vector space X over F (see [3]) is a fuzzy topology γ on X such that the two functions

$\mu : (X, \gamma) \times (X, \gamma) \to (X, \gamma)$, such that $\mu(x, y) = x + y$, for all $x, y \in X$;

$\upsilon : (F, \Im_U) \times (X, \gamma) \to (X, \gamma)$, such that $\upsilon(\alpha, x) = \alpha.x$, for $\alpha \in F, x \in X$, are fuzzy continuous when F is equipped with the usual fuzzy topology, ($\Im_U$ is a fuzzy topo-logy generated by the usual topology U on F) and $F \times X, X \times X$ have the correspo-nding product fuzzy topologies. A vector space X equipped with a fuzzy vector topology γ is called a fuzzy topological vector space.

3. Main Results

Theorem 3.1. [3]

Let Φ be a family of balanced fuzzy sets in a vector space X over F . Then Φ is a base at zero for a fuzzy vector topology if, and only if Φ satisfies the following conditions :

(1) $A(0) > 0$, for each $A \in \Phi$;

(2) for each non-zero constant fuzzy set C_β and any $\alpha \in (0,\beta)$ there exists $A \in \Phi$ with $A \subseteq C_\beta$ and $A(0) > \alpha$;

(3) If $A,B \in \Phi$ and $\alpha \in (0,\min\{A(0),B(0)\})$, then there exists $D \in \Phi$ with $D \subseteq A \cap B$ and $D(0) > \alpha$;

(4) If $A \in \Phi$ and t a non-zero scalar, then for each $\alpha \in (0,A(0))$ there exists $B \in \Phi$ with $B \subseteq tA$ and $B(0) > \alpha$;

(5) Let $A \in \Phi$ and $\alpha \in (0,A(0))$. Then, there exists $B \in \Phi$ such that $B(0) > \alpha$ and $B + B \subseteq A$;

(6) Let $A \in \Phi$ and $x_\circ \in X$. If $\alpha \in (0,A(0))$, then there exists a positive number s such that $A(tx_\circ) > \alpha$, for all scalar $t \in R$ with $|t| \le s$;

(7) For each $A \in \Phi$ there exists a fuzzy set B in X with $B \subseteq A$, $B(0) = A(0)$ such that for each $x \in X$ with $B(x) > 0$ and if $n \in (0,B(x))$, there exists $D \in \Phi$ with $D \subseteq -x + B$ and $D(0) > n$.

Definition 3.2.

Let X be a vector space over F. A fuzzy set A in X is called an affine fuzzy set if $\lambda A + (1-\lambda)A \subset A$ for each $\lambda \in F$.

Theorem 3.3.

Let A be a fuzzy set in a vector space X over F and $\lambda \in F$, then the following statement are equivalent.

(1) A is an affine fuzzy set.

(2) for all x, y in X, we have $A(\lambda x + (1-\lambda)y) \ge \min\{A(x), A(y)\}$.

Proof

$(1) \Rightarrow (2)$

Suppose that A is an affine fuzzy set, by (Definition 3.2) we have $\lambda A + (1-\lambda)A \subset A$ for each $\lambda \in F$. Now, for each $x, y \in X$

$$A(\lambda x + (1-\lambda)y) \ge (\lambda A + (1-\lambda)A)(\lambda x + (1-\lambda)y)$$
$$= \sup_{\lambda x + (1-\lambda)y = x_1 + y_1} \min\{(\lambda A)(x_1), ((1-\lambda)A)(y_1)\}$$
$$\ge \min\{(\lambda A)(\lambda x), ((1-\lambda)A)((1-\lambda)y)\} \ge \min\{A(x), A(y)\}.$$

$(2) \Rightarrow (1)$

Let $x \in X$, $(\lambda A + (1-\lambda)A)(x) = \sup_{x_1 + x_2 = x} \min\{(\lambda A)(x_1), ((1-\lambda)A)(x_2)\}$

(c) If $\lambda \ne 0$ and $1-\lambda \ne 0$, then

$(\lambda A)(x_1) = A(\frac{1}{\lambda}x_1)$ and $((1-\lambda)A)(x_2) = A(\frac{1}{(1-\lambda)}x_2)$

Thus, $(\lambda A + (1-\lambda)A)(x) = \sup\limits_{x_1+x_2=x} \min\{A(\frac{1}{\lambda}x_1), A(\frac{1}{(1-\lambda)}x_2)\}$

But,

$\min\{A(\frac{1}{\lambda}x_1), A(\frac{1}{(1-\lambda)}x_2)\} \le A(\lambda(\frac{1}{\lambda}x_1) + (1-\lambda)(\frac{1}{(1-\lambda)}x_2)) = A(x_1+x_2) = A(x)$.

(d) If $\lambda \ne 0$ and $1-\lambda = 0$, then $(\lambda A)(x_1) = A(x_1)$ and

$$((1-\lambda)A)(x_2) = \begin{cases} 0 & , x_2 \ne 0 \\ \sup\limits_{z\in X} A(z) & , x_2 = 0 \end{cases}$$

(i) If $x_2 \ne 0$, then $((1-\lambda)A)(x_2) = 0$ and $(\lambda A + (1-\lambda)A)(x) = \sup\limits_{x_1+x_2=x} \min\{A(x_1), 0\} = 0$

Since $((1-\lambda)A)(x_2) \ge A(x_2)$, then $A(x_2) = 0$ and we get $\min\{A(x_1), A(x_2)\} = 0$.

(ii) If $x_2 = 0$, then $((1-\lambda)A)(x_2) = \sup\limits_{z\in X} A(z)$ and

$(\lambda A + (1-\lambda)A)(x) = \sup\limits_{x_1=x} \min\{\lambda A(x_1), \sup\limits_{z\in X} A(z)\} \le \sup\limits_{x_1=x} A(x_1) = A(x)$.

(c) If $\lambda = 0$ and $1-\lambda \ne 0$, by the same way in (b).

(d) if $\lambda = 0$ and $1-\lambda = 0$, its impossible.

Definition 3.4.

Let X be a vector space over F. A fuzzy set A in X is called $S-fuzzy\,set$ if A is balanced and affine fuzzy set.

Theorem 3.5.

Let X be a vector space over F and $x_\circ \in X$, then the characteristic function of $\{x_\circ\}$ (in symbol $\chi_{\{x_\circ\}}$) is $S-fuzzy\,set$.

Proof

(1) To prove $\chi_{\{x_\circ\}}$ is an affine fuzzy set. Let $x, y \in X$, $\lambda \in F$ and suppose that $\chi_{\{x_\circ\}}(\lambda x + (1-\lambda)y) < \min\{\chi_{\{x_\circ\}}(x), \chi_{\{x_\circ\}}(y)\}$

Then $\chi_{\{x_\circ\}}(\lambda x + (1-\lambda)y) = 0$ and $\min\{\chi_{\{x_\circ\}}(x), \chi_{\{x_\circ\}}(y)\} = 1$. Thus, $\lambda x + (1-\lambda)y \ne x_\circ$ and $x = y = x_\circ$ implies that $x_\circ \ne x_\circ$ which is impossible. Hence,

$\chi_{\{x_\circ\}}(\lambda x+(1-\lambda)y)\geq\min\{\chi_{\{x_\circ\}}(x),\chi_{\{x_\circ\}}(y)\}$. From (Theorem 3.3), we get the result.

(2) To prove $\chi_{\{x_\circ\}}$ is a balanced fuzzy set. Let $\lambda\in F$ such that $|\lambda|\leq 1$, then from (1) we get for $x\in X$, $\chi_{\{x_\circ\}}(\lambda x)=\chi_{\{x_\circ\}}(\lambda x+(1-\lambda)0)\geq\min\{\chi_{\{x_\circ\}}(x),\chi_{\{x_\circ\}}(0)\}$. Now, the proof is completely if $\chi_{\{x_\circ\}}(x)<\chi_{\{x_\circ\}}(0)$. Suppose that $\chi_{\{x_\circ\}}(x)>\chi_{\{x_\circ\}}(0)$, then $\chi_{\{x_\circ\}}(x)=1$ and $\chi_{\{x_\circ\}}(0)=0$. That is mean $x=x_\circ$ for all $x\in X$ and $x_\circ\neq 0$. By other word $0\notin X$ which contradiction .

Theorem 3.5.

Let A be $S-$ *fuzzy set*. Define $B:X\to I$, $B(x)=\sup\{A(tx):t>1\}$, then B is $S-$ *fuzzy set*. Moreover, $B\subseteq A$ and $B(0)=A(0)$.

Proof

(1) To prove B is a balanced fuzzy set. Let $\lambda\in F$ such that $|\lambda|\leq 1$, $x\in X$.

$B(\lambda x)=\sup\{A(\lambda tx):t>1\}\geq\sup\{A(tx):t>1)=B(x)$.

(2) To prove B is an affine fuzzy set. Let $\lambda\in F$ and for any $x,y\in X$, we have

$$B(\lambda x+(1-\lambda)y)=\sup\{A(t(\lambda x+(1-\lambda)y)):t>1\}=\sup\{A(\lambda tx+(1-\lambda)ty):t>1\}$$

$$\geq\sup\{\min\{A(tx),A(ty)\}:t>1\}$$

$$\geq\min\{\sup A(tx),\sup A(ty)\}\text{ for } t>1.$$

$=\min\{B(x),B(y)\}$. Then B is $S-$ *fuzzy set*.

(3) To prove $B\subseteq A$ and $B(0)=A(0)$. Let $x\in X$, since A is a balanced fuzzy set, we get $B(x)=\sup_{t>1}A(tx)=\sup_{t>1}\frac{1}{t}A(x)\leq\sup_{t>1}A(x)=A(x)$. That is mean $B\subseteq A$.

$B(0)=\sup\{A(t0):t>1\}=A(0)$.

Definition 3.6. [3]

A fuzzy set A in a vector space X, absorbs a fuzzy set B if $A(0) > 0$ and for every $\theta < A(0)$, there exists $t > 0$ such that $C_\theta \cap tB \subseteq A$

Definition 3.7. [3]

A fuzzy set A in a fuzzy topological vector space X is called bounded if it is absorbs by every neighborhood of zero.

Definition 3.8.

A fuzzy topological vector space X is called locally affine if it has a base at zero consisting of affine fuzzy sets.

Theorem 3.9.[3]

Let A be a neighborhood of zero in a fuzzy topological vector space X. Then, there exist an open neighborhood B_1 of zero and a balanced neighborhood B_2 of zero such that $B_1(0) = B_2(0) = A(0)$, $B_1 \subseteq B_2 \subseteq A$ and $tB_1 \subseteq B_1$ for each scalar t with $|t| \leq 1$.

Remark

If a neighborhood A of zero is an affine, then we called an affine neighborhood of zero. Likewise is called S-neighborhood if it is balanced and affine fuzzy set.

Theorem 3.10.

Let A be an affine neighborhood of zero in a fuzzy topological vector space X. Then there exist S-neighborhood B of zero with $B \subseteq A$ and $B(0) = A(0)$.

Proof :

By (Theorem 3.9.), there exists an open neighborhood D_1 of zero and a balanced neighborhood D_2 of zero with $D_1 \subseteq D_2 \subseteq A$ and $D_1(0) = D_2(0) = A(0)$. Let $B = span(D_2)$, then B is $S-fuzzy\,set$ and $D_2 \subseteq B \subseteq A$. To get the result only prove that B is a neighborhood of zero and this fact is true, since $D_1 \subseteq B$ and $D_1(0) = B(0) > 0$.

Journal of Al-Qadisiyah for Computer Science and Mathematics
3nd. Sinentific Conference 19-20/ APRIL -2011
Vol 3 No.2 Year 2011

Theorem 3.11.

If X is a fuzzy topological vector space, $x_\circ \in X$ then $\chi_{\{x_\circ\}}$ is bounded set.

Proof:

Let $x \in X$ and A is a neighborhood of zero, then $A(0) > 0$. Let $\theta \in (0, A(0))$. By (6) of theorem 3.1., there exists a positive number δ such that $A(tx) > \theta$ for all scalars t with $|t| \le \delta$. Take $t = \delta$, we see that

$\min\{\theta, \chi_{\{x_\circ\}}(x)\} \le \theta < A(\delta x) = \frac{1}{\delta} A(x)$. Thus, $C_\theta \cap \delta\chi_{\{x_\circ\}} \subset A$. This complete the proof.

Definition 3.12.

A locally affine fuzzy topological vector space X is called finological if every $S-$ *fuzzy set* in X which absorbs bounded sets is a neighborhood of zero.

The next definition is related to a special kind of finological spaces that is the space which contains all $S-$ *fuzzy sets* which absorbs the characteristic function for $\{0\}$ (in symbol $\chi_{\{0\}}$ or $\chi_\circ$ for short).

Definition 3.13.

A locally affine fuzzy topological vector space X is called $S-space$ if every $S-$ *fuzzy set* in X which absorbs $\chi_\circ$ is a neighborhood of zero.

Theorem 3.14.

Any $S-space$ has a base at zero.

Proof :

Let (X, γ) be $S-space$ and let

$IB = \{ A \in I^X : A \text{ is } S- \text{ fuzzy set which absorbs } \chi_\circ \}$

<u>To prove</u> IB is a base at zero for $S-space$ X. By another word it suffices to show that IB satisfies the conditions (1)-(7) of (Theorem 3.1).

(1)

Let $A \in IB$ and let B be a bounded set. If $A(0) = 0$, then there is no $\theta < A(0)$ such that $C_\theta \cap tB \subseteq A$ for all $t > 0$. That is mean A is not absorbs B but this impossible with assume. Thus, $A(0) > 0$ for each $A \in IB$.

(2)

Let $0<\beta\le 1$ and suppose that $\alpha\in(0,\beta)$. Choose $\alpha_1\in(\alpha,\beta)$. Now, C_{α_1} is $S-fuzzy\ set$ and $C_{\alpha_1}\cap\chi_\circ\subseteq C_{\alpha_1}\subseteq C_\beta$. Thus, that $C_{\alpha_1}\in IB$ and $C_{\alpha_1}(0)=\alpha_1>\alpha$.

(3)

Let $A,B\in IB$ and $0<\alpha<\min\{A(0),B(0)\}$. Let $\alpha_1\in(\alpha,\min\{A(0),B(0)\})$, then $\alpha_1<A(0)$ and $\alpha_1<B(0)$ subsequently, there are $t_1,t_2>0$ such that $C_{\alpha_1}\cap t_1\chi_\circ\subseteq A$ and $C_{\alpha_1}\cap t_2\chi_\circ\subseteq B$. Choose $0<t<t_1,t_2$. Then $\left|t/t_1\right|<1$ and $\left|t/t_2\right|<1$. Since $\chi_{\{x_\circ\}}$ is a balanced fuzzy set then $t\chi_\circ\subseteq t_1\chi_\circ$ and $t\chi_\circ\subseteq t_2\chi_\circ$. Now, let $D=C_{\alpha_1}\cap t\chi_\circ$, we see that $D=C_{\alpha_1}\cap t\chi_\circ\subseteq(C_{\alpha_1}\cap t_1\chi_\circ)\cap(C_{\alpha_1}\cap t_2\chi_\circ)\subseteq A\cap B$. Likewise $D(0)=(C_{\alpha_1}\cap t\chi_\circ)(0)=\alpha_1>\alpha$.

(4)

Let $A\in IB$ and $0\ne t\in R$. Suppose that $0<\alpha<A(0)$. Choose $\alpha_1\in(\alpha,A(0))$. Since $A\in IB$, then there exist $t_1>0$ such that $C_{\alpha_1}\cap t_1\chi_\circ\subseteq A$. That is implies $t(C_{\alpha_1}\cap t_1\chi_\circ)\subseteq tA$. In other words $(C_{\alpha_1}\cap t_1\chi_\circ)(x/t)\le(tA)(x)$ for all $x\in X$. Let $B=C_{\alpha_1}\cap t.t_1\chi_\circ$. Now, $B(x)=(C_{\alpha_1}\cap t.t_1\chi_\circ)(x)=(C_{\alpha_1}\cap t_1\chi_\circ)(x/t)\le tA(x)$. Then $B\subseteq tA$ and $B(0)=\alpha_1>\alpha$.

(5)

Suppose that $A\in IB$ and $0<\theta<A(0)$. Choose $\theta_1\in(\theta,A(0)$. Since $A\in IB$, then there exist $t>0$ such that $C_{\theta_1}\cap t\chi_\circ\subseteq A$. Let $s=\dfrac{t}{2}$, $B=C_{\theta_1}\cap s\chi_\circ$. Now, to prove that B which was to be demonstrated.

$$\begin{aligned}(B+B)(x)&=\sup_{x_1+x_2=x}\min\{B(x_1),B(x_2)\}\\&=\sup_{x_1+x_2=x}\min\{(C_{\theta_1}\cap s\chi_\circ)(x_1),(C_{\theta_1}\cap s\chi_\circ)(x_2)\}\\&=\sup_{y\in X}\min\{C_{\theta_1},\min\{s\chi_\circ(y),s\chi_\circ(x-y)\}\}\\&=\sup_{y\in X}\min\{C_{\theta_1},\min\{\frac{t}{2}\chi_\circ(y),\frac{t}{2}\chi_\circ(x-y)\}\}\\&=\sup_{y\in X}\min\{C_{\theta_1},\min\{\chi_\circ(\frac{2y}{t}),\chi_\circ(\frac{2x-2y}{t})\}\}\\&\le\sup_{y\in X}\min\{C_{\theta_1},\chi_\circ((\frac{1}{2})(\frac{2y}{t})+(\frac{1}{2})(\frac{2x-2y}{t}))\},\ \chi_\circ \text{ is an affine fuzzy set.}\end{aligned}$$

$$= \sup_{y\in X} \min\{C_{\theta_1}, t\chi_\circ(y+x-y))\} = \min\{C_{\theta_1}, t\chi_\circ(x)\} \le A(x).$$

Also, $B(0) = C_{\theta_1} \cap t\chi_\circ(0) = \min\{\theta_1, 1\} = \theta_1 > \theta$.

(6)

Let $x_\circ \in X$, $A \in IB$ and $0 < \theta < A(0)$. Choose $\theta_1 \in (\theta, A(0))$. Since the fuzzy set $\chi_{\{x_\circ\}}$ is bounded, then there is $t > 0$ such that $C_{\theta_1} \cap t\chi_{\{x_\circ\}} \subseteq A$. Now, if $|s| \le t$, then $A(sx_\circ) \ge A(tx_\circ) \ge \min\{\theta_1, \chi_{\{x_\circ\}}(x_\circ) = 1\} = \theta_1 > \theta$.

(7)

Let $A \in IB$. Define $B : X \to I$, $B(x) = \sup\{A(tx) : t > 1\}$. Form (Theorem 3.5) B is $S-fuzzy\,set$. Moreover, $B \subseteq A$ and $B(0) = A(0)$. Also, B absorbs $\chi_\circ$. In fact, let $0 < \theta < B(0)$. Since A absorbs $\chi_\circ$, then there exists $t > 0$ such that $C_\theta \cap t\chi_\circ \subseteq A$. Now, $(C_\theta \cap t\chi_\circ)(2x) \le A(2x) \le B(x)$. And so $C_\theta \cap (\frac{1}{2}t\chi_\circ) \subseteq B$ which proves that B absorbs $\chi_\circ$. By other word $B \in IB$. If $0 < \theta < B(x)$, let $\theta_1 \in (\theta, B(x))$, there exists $t > 1$ such that $A(tx) > \theta_1$. Let $1 < s < t$. Then $B(sx) \ge A(tx) > \theta_1$ for $x \in X$.

Take $\eta = \frac{1}{s}$, $\mu = 1 - \eta$. Using the fact that B is an affine fuzzy set and for $y \in X$, we get,

$$(-x+B)(y) = B(x+y) = B(\eta(sx) + \mu(\mu^{-1}y)) \ge \min\{B(sx), B(\mu^{-1}y)\} \ge \min\{\theta_1, (\mu B)(y)\}$$

let $B_2 = C_{\theta_1} \cap \mu B$. Now, $B_2 \subseteq -x + B$ and $B_2(0) = \theta_1 > \theta$. We just prove $B_2 \in IB$. It is clear that B_2 is $S-fuzzy\,set$ and $C_\theta \cap \frac{1}{2}t\mu^{-1}\chi_\circ \subset C_{\theta_1} \cap \mu B = B_2$ which means B_2 absorbs $\chi_\circ$. Hereupon, Which complete the proof.

Acknowledgment. The author would like to express their thanks to the referee Asst. Prof. Dr. Noori. F. AL-Mayahi for his detailed examinations of this paper.

References

[1] R. Lowen , Fuzzy Topological Spaces and Fuzzy Compactness, *J. Math. Anal. Appl.*, **56** (1976) 621-633.
[2] A. K. Katsaras and D. B. Liu, Fuzzy Vector Spaces and Fuzzy Topological Vector Spaces, *J. Math. Anal. Appl*. **58** (1977) 135-146.
[3] A. K. Katsaras, Fuzzy Topological Vector Spaces I, *Fuzzy Sets and Systems*,**6** (1981) 85-95.