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Ulam-Type Stability in Non-Archimedean Banach Algebras

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ABSTRACT

This paper studies the Ulam-type stability of functional equations in non-Archimedean Banach algebras. Conditions are established under which approximate solutions of the considered functional equations are close to exact solutions. The obtained results generalize and extend several known stability results in non-Archimedean settings. These findings contribute to a deeper understanding of stability theory in the framework of non-Archimedean Banach algebras with the functional equation:

$$g(g(x) + g(h)) = g(x + h) + g(x - h) - g(x) - g(h)$$

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1. Introduction

Theory of (S.F.E.) has become an important topic in modern mathematical analysis. This field originated from a problem proposed by Stanisław ($\mathcal{U}.$) in 1940 concerning the stability of homomorphism between metric groups. In particular, ($\mathcal{U}.$) asked whether a mapping that approximately satisfies a (F.P.) must be close to an exact solution of that (E.). This (F.E.) initiated the development of what is now known as ($\mathcal{U}.$)-type stability theory [1,2]. The first significant answer to ($\mathcal{U}.$)'s problem was given by Donald ($\mathcal{H}.$) in 1941. ($\mathcal{H}.$) proved that if a mapping approximately satisfies the additive (F.E.) in Banach spaces, then there is a unique additive mapping near it [3,4]. Later, this result was ($\mathcal{G}.$) by Themistocles. Rassias who introduced a remarkable extension allowing the control of the Cauchy difference by an unbounded function. This generalization, known as the ($\mathcal{H}.$)-($\mathcal{U}.$)-Rassias stability, significantly expanded the theory and opened the door for many further investigations [5-7]. During the last decades, the stability theory of (F.E.) has been extended to various mathematical structures, including (N.A.S.). In such spaces, the norm satisfies the strong triangle inequality, which leads to analytical properties that are substantially different from those in classical Banach spaces. These spaces arise naturally in areas such as p-adic

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analysis and (N.A.) geometry, making them an important framework for studying (F.P.) and their stability properties [6, 8-10]. In particular, (N.A.) (B.A.) provide a rich structure for studying algebraic mappings such as homomorphisms and derivations. The stability of these mappings plays an important role in understanding whether approximate algebraic structures can be approximated by exact ones. Several authors have investigated the (S.) and superstability of ring homomorphism, derivations, and higher derivations in (N.A.) (B.A.) [4,11,12]. Various techniques have been developed to establish such stability results. Among them, the direct method and the (F.P.) method are the most widely used approaches. These methods have been successfully applied to investigate the stability of different (F.P.) and algebraic mappings in (N.A.) settings [1,9,10,13-18].

Motivated by these developments, the study of $(\mathcal{H})$ -type stability in (N.A.) (B.A.) continues to attract considerable attention. This topic combines ideas from (F.P.), (B.A.) theory, and (N.A.) analysis, providing a fruitful framework for exploring the relationship between approximate and exact algebraic mappings. we work in a (N.A.) (B.A.). This structure is appropriate for our study since it satisfies the multiplicative property between elements, which allows the use of algebraic homomorphism. We consider a sequence that forms a geometric progression in the (N.A.) (B.A.). By using the (N.A.) property of the norm, the terms of this sequence are convergent, and hence the limit tends to zero. Next, we apply a homomorphism that preserves the multiplicative structure of the algebra, that is, $\varphi(rh) = \varphi(r)\varphi(h)$. This property is essential for establishing the required relations in the (B.A.). Using these properties together with the convergence of the geometric sequence, we obtain the uniqueness of the mapping under the given conditions. Let

$$\varphi(\varphi(r) + \varphi(h)) = \varphi(r + h) + \varphi(r - h) - \varphi(r) - \varphi(h) \quad (1)$$

2. Preliminaries

Definition 2.1. [19] Let $\mathbb{K}$ be a (N.A.) valued field and let $\mathcal{Q}$ be an algebra over $\mathbb{K}$. Then is called (N.A.) (B.A.) if there exists a norm $\|\cdot\|: \mathcal{Q} \rightarrow [0, \infty)$ satisfying:

1. $\|r + h\| \leq \max\{\|r\|, \|h\|\}$;
2. $\|rh\| \leq \|r\|\|h\|$;
3. The space $\mathcal{Q}$ is complete with respect to the norm.

In this case, $\mathcal{Q}$ is called a (N.A.) (B.A.).

Example 2.2.[19] Let $\mathcal{Q}$ be a non-Archimedean Banach algebras and let $\partial \in \mathcal{Q}$ with $\|\partial\| < 1$. Define a mapping $\varphi: \mathcal{Q} \rightarrow \mathcal{Q}$ by

$$\varphi(r) = \partial r \quad \forall r \in \mathcal{Q}$$

Then, for all $r, h \in \mathcal{Q}$

$$\varphi(rh) - \varphi(r)\varphi(h) = \partial(1 - \partial)rh$$

Therefore,

$$\|\varphi(rh) - \varphi(r)\varphi(h)\| \leq \|\partial\| \|1 - \partial\| \|rh\|$$

which shows that φ is an approximate homomorphism. Moreover, the mapping $H(r) = 0$, is an exact homomorphism and

$$\|\varphi(r) - H(r)\| \leq \|\partial\| \|r\|$$

Which proves that φ is close to an exact homomorphism.

Definition 2.3. [20] Let $\mathcal{Q}, \mathcal{R}$ be (B.A.). A real linear mapping $\delta: \mathcal{Q} \rightarrow \mathcal{R}$ is said to be a (H.) if $\delta(rh) = \delta(r)\delta(h)$ for all $r, h \in \mathcal{Q}$.

Definition 2.4. [20] Let $\mathcal{Q}, \mathcal{R}$ be (B.A.). A real linear mapping $\delta: \mathcal{Q} \rightarrow \mathcal{R}$ is called to be a derivation if $\delta(rh) = \delta(r)h + r\delta(h)$ for all $r, h \in \mathcal{Q}$.

Definition 2.5. [1] Let $\mathcal{Q}$ be a set. A (F.) $t: \mathcal{Q} \times \mathcal{Q} \rightarrow [0, \infty]$ is said a $(\mathcal{G}, \mathcal{M})$ on $\mathcal{Q}$ if,

1. $t(r, h) = 0 \Leftrightarrow r = h$,
2. $t(r, h) = t(h, r)$ for all $r, h \in Q$,
3. $t(r, t) \leq t(r, h) + t(h, t)$ for all $r, h, t \in Q$.

Theorem 2.6. [1] Let (Q, t) be a $(\mathcal{C}.)$ $(\mathcal{G.M.})$ space, and let $J: Q \rightarrow \mathcal{R}$ be a strictly contractive mapping with the Lipschitz constant

$\ell < 1$. Then, for all $r \in Q$, either

$t(J^w r, J^{w+1} r) = +\infty$ for all $w \geq 0$ or there is a $w_0 \in \mathbb{N}$ such that

1. $t(J^w r, J^{w+1} r) < +\infty$ for all $w \geq w_0$,
2. The sequence $\{J^w r\}$ is convergent to a (F. P.) h^* of J ,
3. h^* is the unique (F. P.) of J in the set $\mathbb{Y} = \{h \in \mathbb{X}, t(J^{w_0} r, h) < \infty\}$,
4. $t(h, h^*) \leq \frac{1}{1-\ell} d(h, Jh)$ for all $h \in \mathbb{Y}$.

3. (S. F. E.) (1) in (N. A.) (B. A.)

We present several main results concerning the (S. F. E.) (1.1) in (N. A.) (B. A.). Let

$$D_g(r, h) = g(g(r) + g(h)) - g(r + h) - g(r - h) + g(r) + g(h) \quad (2)$$

Theorem 3.1. Let $Q, \mathcal{R}$ be two (N. A.) (B. A.). Let $\vartheta: Q \rightarrow [0, \infty)$ be a function such that

$$\sum_{w=1}^{\infty} \frac{\vartheta(2^w r, 2^w h)}{|2^w|} < \infty \text{ and } \lim_{w \rightarrow \infty} \frac{\vartheta(2^w r, 2^w h)}{|2^w|} = 0 \text{ for any } r, h \in Q.$$

Let $g: Q \rightarrow \mathcal{R}$ be a function satisfies the inequality

$$\|g(r, h)\|_{\mathcal{R}} \leq \vartheta(r, h) \quad (3)$$

and

$$\|g(rh) - g(r)g(h)\|_{\mathcal{R}} \leq \vartheta(r, h) \quad (4)$$

For any $r, h \in Q$. Then there is an (A.) mapping $\wp: Q \rightarrow \mathcal{R}$ satisfying (2) and satisfying (H.) property such that

$$\|g(r) - \wp(r)\| \leq \frac{1}{|2|} \lim_{w \rightarrow \infty} \max_{0 \leq \lambda < w} \left\{ \frac{\vartheta(2^\lambda r, 2^\lambda r)}{|2^\lambda|} \right\} \quad (5)$$

For any $r \in Q$. Then mapping $\wp(r)$ is defined by $\wp(r) = \lim_{w \rightarrow \infty} \frac{g(2^w r)}{2^w}$, for any $r \in Q$. $\wp$ is unique if

$$\frac{1}{|2|} \lim_{\lambda \rightarrow \infty} \lim_{w \rightarrow \infty} \max_{\lambda \leq j < w + \lambda} \left\{ \frac{\vartheta(2^j r, 2^j r)}{|2^j|} \right\} = 0 \quad (6)$$

Proof. Putting $r = h$ in (2) we have

$$\left\| \frac{g(2r)}{2} - g(r) \right\| \leq \frac{1}{|2|} \vartheta(r, r) \quad (7)$$

Replacing r by $2^w r$ and driving all side by 2^w we get

$$\left\| \frac{g(2^{w+1} r)}{2^{w+1}} - \frac{g(2^w r)}{2^w} \right\| \leq \frac{1}{|2^{w+1}|} \vartheta(r, r) \quad (8)$$

When $w \rightarrow \infty$ then the sequence $\left\{ \frac{g(2^w r)}{2^w} \right\}$ is Cauchy. Since Q is $(\mathcal{C}.)$ we conclude that $\left\{ \frac{g(2^w r)}{2^w} \right\}$ is convergent.

Set $\wp(r) = \lim_{w \rightarrow \infty} \frac{g(2^w r)}{2^w}$, we can show that

$$\begin{aligned} \frac{g(2^w r)}{2^w} - g(r) &= \sum_{\lambda=0}^{w-1} \left(\frac{g(2^{\lambda+1} r)}{2^{\lambda+1}} - \frac{g(2^\lambda r)}{2^\lambda} \right) \\ \left\| \frac{g(2^w r)}{2^w} - g(r) \right\| &\leq \max \left\{ \left\| \frac{g(2r)}{2} - g(r) \right\|, \left\| \frac{g(2^2 r)}{2^2} - \frac{g(2r)}{2} \right\|, \dots, \left\| \frac{g(2^w r)}{2^w} - \frac{g(2^{w-1} r)}{2^{w-1}} \right\| \right\} \\ &\leq \max \left\{ \frac{\vartheta(r, r)}{|2|}, \frac{\vartheta(2r, 2r)}{|2|^2}, \dots, \frac{\vartheta(2^{w-1} r, 2^{w-1} r)}{|2|^w} \right\} \\ &= \max_{0 \leq \lambda < w} \left\{ \frac{\vartheta(2^\lambda r, 2^\lambda r)}{|2|^\lambda} \right\} \\ \left\| \frac{g(2^w r)}{2^w} - g(r) \right\| &\leq \frac{1}{|2|} \max_{0 \leq \lambda < w} \left\{ \frac{\vartheta(2^\lambda r, 2^\lambda r)}{|2|^\lambda} \right\} \end{aligned}$$

When $w \rightarrow \infty$, we get

$$\|g(r) - g(r)\| \leq \frac{1}{|2|} \lim_{w \rightarrow \infty} \max_{0 \leq \lambda < w} \left\{ \frac{\vartheta(2^\lambda r, 2^\lambda r)}{|2|^\lambda} \right\}$$

Replacing r by $2^w r$, s by $2^w h$ and driving all side by 2^w in (2) we get

$$\frac{1}{|2^w|} \|D_g(r, h)\| \leq \frac{1}{|2^w|} \vartheta(2^w r, 2^w h)$$

Taking the limit as $w \rightarrow \infty$ we get

$$\|D_g(r, h)\| \leq \lim_{w \rightarrow \infty} \frac{\vartheta(2^w r, 2^w h)}{|2^w|} = 0$$

Then g satisfy the (F. P.) and g is an (A.) mapping.

$$\begin{aligned} \|g(rh) - g(r)g(h)\| &= \lim_{w \rightarrow \infty} \left\| \frac{g(2^w r 2^w h)}{2^{4w}} - \frac{g(2^w r)}{2^w} \frac{g(2^w h)}{2^w} \right\| \\ &= \lim_{w \rightarrow \infty} \frac{1}{|2^{4w}|} \|g(2^w r 2^w h) - g(2^w r)g(2^w h)\| \\ &\leq \lim_{w \rightarrow \infty} \frac{1}{|2^{4w}|} \vartheta(2^w r 2^w h) = 0 \end{aligned}$$

Then $g(rh) = g(r)g(h)$.

To prove g is unique, $\hat{g}: Q \rightarrow R$ satisfies the (F. E.) and (3):

$$\begin{aligned} \|g(r) - \hat{g}(r)\| &\leq \max\{\|g(r) - g(r)\|, \|g(r) - \hat{g}(r)\|\} \\ &\leq \max \left\{ \frac{1}{|2|} \lim_{w \rightarrow \infty} \max_{0 \leq \lambda < w} \left\{ \frac{\vartheta(2^\lambda r, 2^\lambda r)}{|2|^\lambda} \right\}, \frac{1}{|2|} \lim_{w \rightarrow \infty} \max_{0 \leq \lambda < w} \left\{ \frac{\vartheta(2^\lambda r, 2^\lambda r)}{|2|^\lambda} \right\} \right\} \\ &= \frac{1}{|2|} \lim_{w \rightarrow \infty} \max_{0 \leq \lambda < w} \left\{ \frac{\vartheta(2^\lambda r, 2^\lambda r)}{|2|^\lambda} \right\} \\ &\leq \frac{1}{|2|} \lim_{\lambda \rightarrow \infty} \lim_{w \rightarrow \infty} \max_{\lambda \leq j < w+\lambda} \left\{ \frac{\vartheta(2^j r, 2^j r)}{|2|^j|} \right\} = 0 \end{aligned}$$

Therefor $g = \hat{g}$. This complete the proof of uniqueness of g .

Theorem 3.2. Let $\mathcal{Q}, \mathcal{R}$ be two (N. A.)(B. A.). Let $\vartheta: \mathcal{Q} \rightarrow [0, \infty)$ be a function such that

$$\sum_{w=1}^{\infty} \frac{\vartheta(2^w \mathbf{r}, 2^w \mathbf{h})}{|2^w|} < \infty \text{ and } \lim_{w \rightarrow \infty} \frac{\vartheta(2^w \mathbf{r}, 2^w \mathbf{h})}{|2^w|} = 0 \text{ for any } \mathbf{r}, \mathbf{h} \in \mathcal{Q}.$$

Let $\varrho: \mathcal{Q} \rightarrow \mathcal{R}$ be a function satisfies the inequality

$$\|D_{\varrho}(\mathbf{r}, \mathbf{h})\|_{\mathcal{R}} \leq \vartheta(\mathbf{r}, \mathbf{h}) \quad (9)$$

and

$$\|\varrho(\mathbf{r}\mathbf{h}) - \mathbf{h}\varrho(\mathbf{r})\mathbf{r}\varrho(\mathbf{h})\|_{\mathcal{R}} \leq \vartheta(\mathbf{r}, \mathbf{h}) \quad (10)$$

For any $\mathbf{r}, \mathbf{h} \in \mathcal{Q}$. Then there is an (A.) mapping $\wp: \mathcal{Q} \rightarrow \mathcal{R}$ satisfying (3.6) and satisfying derivation property such that

$$\|\varrho(\mathbf{r}) - \wp(\mathbf{r})\| \leq \frac{1}{|2|} \lim_{w \rightarrow \infty} \max_{0 \leq \lambda < w} \left\{ \frac{\vartheta(2^\lambda \mathbf{r}, 2^\lambda \mathbf{r})}{|2^\lambda|} \right\} \quad (11)$$

For any $\mathbf{r} \in \mathcal{Q}$. Then mapping $\wp(\mathbf{r})$ is defined by $\wp(\mathbf{r}) = \lim_{w \rightarrow \infty} \frac{\varrho(2^w \mathbf{r})}{2^w}$, for any $\mathbf{r} \in \mathcal{Q}$.

$\wp$ is unique if

$$\frac{1}{|2|} \lim_{\lambda \rightarrow \infty} \lim_{w \rightarrow \infty} \max_{\lambda \leq \mathbb{J} < w + \lambda} \left\{ \frac{\vartheta(2^{\mathbb{J}} \mathbf{r}, 2^{\mathbb{J}} \mathbf{r})}{|2^{\mathbb{J}}|} \right\} = 0$$

Proof. We proceed as in the proof of theorem (2), except for the additional step described below:

$$\begin{aligned} \|\wp(\mathbf{r}\mathbf{h}) - \mathbf{r}\wp(\mathbf{h}) - \mathbf{h}\wp(\mathbf{r})\|_{\mathcal{Q}} &= \lim_{w \rightarrow \infty} \left\| \frac{\varrho(2^w \mathbf{r} 2^w \mathbf{h})}{2^{2w}} - \frac{2^w \mathbf{r} \varrho(2^w \mathbf{h})}{2^{2w}} - \frac{2^w \mathbf{h} \varrho(2^w \mathbf{r})}{2^{2w}} \right\| \\ &= \lim_{w \rightarrow \infty} \frac{1}{|2^{2w}|} \|\varrho(2^w \mathbf{r} 2^w \mathbf{h}) - 2^w \mathbf{r} \varrho(2^w \mathbf{h}) - 2^w \mathbf{h} \varrho(2^w \mathbf{r})\| \\ &\leq \lim_{w \rightarrow \infty} \frac{1}{|2^{2w}|} \vartheta(2^w \mathbf{r} 2^w \mathbf{h}) = 0 \end{aligned} \quad (12)$$

then

$$\wp(\mathbf{r}\mathbf{h}) = \mathbf{r}\wp(\mathbf{h}) + \mathbf{h}\wp(\mathbf{r})$$

This complete the proof. ■

Corollary 3.3. If $[3] < 1$. Suppose that a mapping $\varrho: \mathcal{Q} \rightarrow \mathcal{R}$ satisfies

$$\|D_{\varrho}(\mathbf{r}, \mathbf{h})\|_{\mathcal{R}} \leq \begin{cases} \|\mathbf{r}\|^{\delta} + \|\mathbf{h}\|^{\delta}, & \delta > 1; \\ \|\mathbf{r}\|^{\delta} \|\mathbf{h}\|^{\delta}, & \delta \geq 1 \\ \frac{\|\mathbf{r}\|}{\|\mathbf{h}\|^{\delta}}, & \delta < 0; \\ \frac{\|\mathbf{r}\| \|\mathbf{h}\|}{\|\mathbf{r}\|^{\delta} + \|\mathbf{h}\|^{\delta}}, & \delta < 1; \\ \frac{1}{\|\mathbf{r}\|^{\delta} + \|\mathbf{h}\|^{\delta}}, & \delta > 1 \end{cases} \quad (13)$$

and

$$\|g(rh) - g(r)g(h)\|_Q \leq \begin{cases} \|r\|^\delta + \|h\|^\delta, & \delta > 1; \\ \|r\|^\delta \|h\|^\delta, & \delta \geq 1 \\ \frac{\|r\|}{\|h\|^\delta}, & \delta < 0; \\ \frac{\|r\| \|h\|}{\|r\|^\delta + \|h\|^\delta}, & \delta < 1; \\ \frac{1}{\|r\|^\delta + \|h\|^\delta}, & \delta > 1 \end{cases}$$

For any $r, h \in Q$.

Then there is a unique (H.) mapping $\wp: Q \rightarrow \mathcal{R}$ such that

$$\|g(r) - \wp(r)\| \leq \frac{1}{|2|} \lim_{w \rightarrow \infty} \max_{0 \leq \lambda < w} \begin{cases} 2|2^\lambda|^{\delta-1} \|r\|^\delta \\ |2^\lambda|^{(2\delta-1)} \|r\|^{2\delta} \\ |2^\lambda|^{-\delta} \|r\|^{1-\delta} \\ |2^\lambda|^{1-\delta} \frac{\|r\|^2}{2\|r\|^\delta} \\ |2^\lambda|^\delta \frac{2\|r\|^\delta}{|2^\lambda|} \end{cases} \quad (14)$$

For any $r \in Q$.

Proof. The proof is straightforward from the assumptions. ■

Theorem 3.4. Let $Q, \mathcal{R}$ be two (N. A.)(B. A.). Let $\vartheta: Q \rightarrow [0, \infty)$ be a function such that

$$\sum_{w=1}^{\infty} |2^w| \vartheta\left(\frac{r}{2^w}, \frac{h}{2^w}\right) < \infty \text{ and } \lim_{w \rightarrow \infty} \left[|2^w| \vartheta\left(\frac{r}{2^w}, \frac{h}{2^w}\right) \right] = 0$$

Let $g: Q \rightarrow \mathcal{R}$ be a function satisfies the inequality

$$\|D_g(r, h)\|_{\mathcal{R}} \leq \vartheta(r, h) \quad (15)$$

and

$$\|g(rh) - g(r)g(h)\|_{\mathcal{R}} \leq \vartheta(r, h) \quad (16)$$

For any $r, h \in Q$. Then there is an (A.) mapping $\wp: Q \rightarrow \mathcal{R}$ satisfying (15) and satisfy (H.) property such that

$$\|g(r) - \wp(r)\| \leq \lim_{w \rightarrow \infty} \max_{0 \leq \lambda < w} \left\{ |2^\lambda| \vartheta\left(\frac{r}{2^{\lambda+1}}, \frac{r}{2^{\lambda+1}}\right) \right\} \quad (17)$$

For any $r \in Q$. Then mapping $\wp(r)$ is defined by $\wp(r) = \lim_{w \rightarrow \infty} 2^w g\left(\frac{r}{2^w}\right)$ for any $r \in Q$. $\wp$ is unique if

$$\lim_{\lambda \rightarrow \infty} \lim_{w \rightarrow \infty} \max_{\lambda \leq j < w+\lambda} |2^j| \vartheta\left(\frac{x}{2^j}, \frac{y}{2^j}\right) = 0 \quad (18)$$

Proof. This theorem can be demonstrated using the same way as in the Theorem 3.1. ■

Theorem 3.5. Let $Q, \mathcal{R}$ be two (N. A.)(B. A.). Let $\vartheta: Q \rightarrow [0, \infty)$ be a function such that

$$\sum_{w=1}^{\infty} |2^w| \vartheta\left(\frac{r}{2^w}, \frac{h}{2^w}\right) < \infty \text{ and } \lim_{w \rightarrow \infty} \left[|2^w| \vartheta\left(\frac{r}{2^w}, \frac{h}{2^w}\right) \right] = 0$$

Let $g: Q \rightarrow \mathcal{R}$ be a function satisfies the inequality

$$\|D_{\vartheta}(\mathbf{r}, \hbar)\|_{\mathcal{R}} \leq \vartheta(\mathbf{r}, \hbar) \quad (19)$$

and

$$\|\vartheta(\mathbf{r}\hbar) - \vartheta(\mathbf{r})\vartheta(\hbar)\|_{\mathcal{R}} \leq \vartheta(\mathbf{r}, \hbar) \quad (20)$$

For any $\mathbf{r}, \hbar \in \mathcal{Q}$. Then there is an (A.) derivation $\wp: \mathcal{Q} \rightarrow \mathcal{R}$ satisfying (19) such that

$$\|\vartheta(\mathbf{r}) - \wp(\mathbf{r})\| \leq \lim_{w \rightarrow \infty} \max_{0 \leq \lambda < w} \left\{ |2^\lambda| \vartheta \left(\frac{\mathbf{r}}{2^{\lambda+1}}, \frac{\mathbf{r}}{2^{\lambda+1}} \right) \right\} \quad (21)$$

For any $\mathbf{r} \in \mathcal{Q}$. Then mapping $\wp(\mathbf{r})$ is defined by $\wp(\mathbf{r}) = \lim_{w \rightarrow \infty} 2^w \vartheta \left(\frac{\mathbf{r}}{2^w} \right)$, for any $\mathbf{r} \in \mathcal{Q}$. $\wp$ is unique if

$$\lim_{\lambda \rightarrow \infty} \lim_{w \rightarrow \infty} \max_{\lambda \leq j < w} |2^j| \vartheta \left(\frac{\mathbf{r}}{2^j}, \frac{\mathbf{s}}{2^j} \right) = 0 \quad (22)$$

Proof. We proceed as in the proof of Theorem 3.4, except for the additional step described below:

$$\begin{aligned} \|\wp(\mathbf{r}\hbar) - \mathbf{r}\wp(\hbar) - \hbar\wp(\mathbf{r})\|_{\mathcal{S}} &= \lim_{w \rightarrow \infty} |2^{2w}| \left\| \vartheta \left(\frac{\mathbf{r}\hbar}{2^{2w}} \right) - \frac{\mathbf{r}}{2^w} \vartheta \left(\frac{\hbar}{2^w} \right) - \frac{\hbar}{2^w} \vartheta \left(\frac{\mathbf{r}}{2^w} \right) \right\| \\ &\leq \lim_{w \rightarrow \infty} |2^w| \vartheta \left(\frac{\mathbf{r}}{2^w}, \frac{\hbar}{2^w} \right) = 0 \end{aligned}$$

Then we have

$$\wp(\mathbf{r}\hbar) = \mathbf{r}\wp(\hbar) + \hbar\wp(\mathbf{r})$$

This complete the proof. ■

Corollary 3.6. If $[3] < 1$. Suppose that a mapping $\vartheta: \mathcal{Q} \rightarrow \mathcal{R}$ satisfies

$$\|D_v(\mathbf{r}, \hbar)\|_{\mathcal{R}} \leq \begin{cases} \|\mathbf{r}\|^{\vartheta} + \|\hbar\|^{\vartheta}, & \vartheta < 1; \\ \|\mathbf{r}\|^{\vartheta} \|\hbar\|^{\vartheta}, & \vartheta < \frac{1}{2}; \\ \frac{\|\mathbf{r}\|}{\|\hbar\|^{\vartheta}}, & \vartheta > 0; \\ \frac{\|\mathbf{r}\| \|\hbar\|}{\|\mathbf{r}\|^{\vartheta} + \|\hbar\|^{\vartheta}}, & \vartheta < \frac{1}{2}; \\ \frac{1}{\|\mathbf{r}\|^{\vartheta} + \|\hbar\|^{\vartheta}}, & \vartheta > 0 \end{cases} \quad (23)$$

and

$$\|\vartheta(\mathbf{r}\hbar) - \vartheta(\mathbf{r})\vartheta(\hbar)\|_{\mathcal{R}} \leq \begin{cases} \|\mathbf{r}\|^{\vartheta} + \|\hbar\|^{\vartheta}, & \vartheta < 1; \\ \|\mathbf{r}\|^{\vartheta} \|\hbar\|^{\vartheta}, & \vartheta < \frac{1}{2}; \\ \frac{\|\mathbf{r}\|}{\|\hbar\|^{\vartheta}}, & \vartheta > 0; \\ \frac{\|\mathbf{r}\| \|\hbar\|}{\|\mathbf{r}\|^{\vartheta} + \|\hbar\|^{\vartheta}}, & \vartheta < \frac{1}{2}; \\ \frac{1}{\|\mathbf{r}\|^{\vartheta} + \|\hbar\|^{\vartheta}}, & \vartheta > 0 \end{cases}$$

For any $r, h \in Q$.

Then there exists a unique (H.) mapping $\wp: Q \rightarrow \mathcal{R}$ such that

$$\|\wp(r) - \wp(r)\| \leq \frac{1}{|2|} \lim_{w \rightarrow \infty} \max_{0 \leq \lambda < w} \begin{cases} \frac{|2^\lambda|^{1-\delta}}{|2|^\delta} (2\|r\|^\delta) \\ \frac{|2^\lambda|^{1-\delta}}{|2|^\delta} (\|r\|^{2\delta}) \\ |2^\lambda|^{2\delta-1} \frac{\|r\|}{\|r\|^\delta} \\ |2^\lambda|^{\delta-2} \frac{\|r\|^{2\delta}}{2\|r\|} \\ \frac{|2^\lambda||2|^{\delta+2}}{2\|r\|^\delta} \end{cases}$$

For any $r \in Q$.

Proof. The proof is straightforward from the assumptions. ■

Remark 3.7. The results obtained in the context of (H.) is also applicable to derivations.

4. (S. F. E.) (1) in (N. A.) (B. A.) via the (F. P.) method

We investigate the (S. F. E.) (1) in (N. A.) (B. A.) by applying the (F. P.) method. Let

$$D_\wp(r, h) = \wp(\wp(r) + \wp(h)) - \wp(r + h) - \wp(r - h) + \wp(r) + \wp(h)$$

Theorem 4.1. Suppose that $\vartheta: Q \times Q \rightarrow [0, \infty)$ is a (F.) such that there is a constant $\ell < 1$ with

$$(\vartheta 2r, 2h) \leq \ell \vartheta(r, h) \quad (24)$$

For all $r, h \in Q$. Consider a mapping $\wp: Q \rightarrow \mathcal{M}$ for which

$$(\vartheta 2r, 2h) \leq \ell \vartheta(r, h) \quad (25)$$

For any $r, h \in Q$. There is a unique (A.) mapping $C: Q \rightarrow \mathcal{M}$ satisfying

$$\|\wp(r) - C(r)\| \leq \frac{1}{(1-\ell)|2|} \vartheta(r, r) \quad (26)$$

For any $r \in Q$

Proof. putting $r = h$ in (25), we get

$$\left\| \frac{\wp(2r)}{2} - \wp(r) \right\| \leq \frac{1}{|2|} \vartheta(r, r)$$

For any $r \in Q$. We operate within the class

$$\mathcal{M} = \{d: Q \rightarrow \mathcal{R}\}$$

In addition, we define $(G. \mathcal{M}.)$ by:

$$d(\wp, d) = \inf \{\sigma \in \mathcal{R}^+: \|\wp(r) - d(r)\|_{\mathcal{R}} \leq \sigma \vartheta(r, r), \forall r \in Q\}$$

It is immediate that $(\mathcal{M}, d)$ ($\mathcal{C. \mathcal{M}.)}$ space. A linear mapping is defined as follows $\mathbb{J}: \mathcal{M} \rightarrow \mathcal{M}$ such that

$$\mathbb{J}(\mathcal{P}(\mathbf{r})) = \frac{\mathcal{P}(2\mathbf{r})}{2}, \forall \mathbf{r} \in \mathcal{Q}$$

Then

$$\|\mathbb{J}\mathcal{G}(\mathbf{r}) - \mathcal{G}(\mathbf{r})\| \leq \frac{1}{|2|} \vartheta(\mathbf{r}, \mathbf{r})$$

$$d(\mathbb{J}\mathcal{G}, \mathcal{G}) \leq \frac{1}{|2|}$$

Let $\mathcal{G}, \mathcal{H} \in \mathcal{M}$

$$d(\mathcal{G}, \mathcal{H}) = \sigma$$

$$\|\mathcal{G}(\mathbf{r}) - \mathcal{H}(\mathbf{r})\| \leq \sigma \vartheta(\mathbf{r}, \mathbf{r})$$

$$\begin{aligned} \|\mathbb{J}\mathcal{G} - \mathbb{J}\mathcal{H}\| &= \left\| \frac{\mathcal{G}(2\mathbf{r})}{2} - \frac{\mathcal{H}(2\mathbf{r})}{2} \right\| \\ &= \frac{1}{|2|} \|\mathcal{G}(2\mathbf{r}) - \mathcal{H}(2\mathbf{r})\| \\ &\leq \frac{1}{|2|} \sigma \vartheta(2\mathbf{r}, 2\mathbf{r}) \\ &\leq \frac{1}{|2|} \sigma \ell |2| \vartheta(\mathbf{r}, \mathbf{r}) \\ &\leq \sigma \ell \vartheta(\mathbf{r}, \mathbf{r}) \end{aligned}$$

Then

$$d(\mathbb{J}\mathcal{G}, \mathbb{J}\mathcal{H}) \leq \sigma \ell$$

$$d(\mathbb{J}\mathcal{G}, \mathbb{J}\mathcal{H}) \leq \ell d(\mathcal{G}, \mathcal{H})$$

Then $\mathbb{J}$ is contraction.

By applying Theorem 2.6, then we have

1. $\mathbb{J}$ is unique (F. P.) of $\mathcal{K}$

$$\mathbb{J}(\mathcal{K}(\mathbf{r})) = \mathcal{K}(\mathbf{r})$$

$$\frac{\mathcal{K}(2\mathbf{r})}{2} = \mathcal{K}(\mathbf{r})$$

$$\mathcal{K}(2\mathbf{r}) = 2\mathcal{K}(\mathbf{r})$$

For any $\mathbf{r} \in \mathcal{Q}$, then $\mathcal{K}$ is unique (F. P.) of $\mathbb{J}$ in the set

$$\Omega = \{\mathcal{F} \in \mathcal{M} : (g, d) < \infty\}$$

There is $\sigma \in (0, \infty)$ such that $d(\mathbb{J}, \mathcal{G}) < \sigma$

$$\|\mathcal{K}(\mathbf{r}) - \mathcal{G}(\mathbf{r})\| \leq \frac{\sigma}{|2|} \vartheta(\mathbf{r}, \mathbf{r})$$

2. $d(\mathbb{J}\mathcal{G}^{\mathfrak{w}}, \mathcal{K}) \rightarrow 0$ as $\mathfrak{w} \rightarrow \infty$, this gives that, for any $\mathbf{r} \in \mathcal{Q}$

$$\lim_{\mathfrak{w} \rightarrow \infty} \frac{\mathcal{G}(2^{\mathfrak{w}}\mathbf{r})}{2^{\mathfrak{w}}} = \mathcal{K}(\mathbf{r})$$

3. $d(\mathcal{g}, \mathcal{K}) \leq \frac{1}{1-\ell} d(\mathcal{g}, \mathbb{J}\mathcal{g})$, then

$$d(\mathcal{g}, \mathcal{K}) \leq \frac{1}{|2|(1-\ell)}$$

$$\|\mathcal{g}(\mathbf{r}) - \mathcal{K}(\mathbf{r})\| \leq \frac{1}{(1-\ell)|2|} \vartheta(\mathbf{r}, \mathbf{r})$$

This complete the proof. ■

Corollary 4.2. If $0 \leq \ell < 1$ and $\delta \in \mathbb{R}$. Suppose that a mapping $\mathcal{g}: \mathcal{Q} \rightarrow \mathcal{R}$, satisfies

$$\|D_{\mathcal{Q}}(\mathbf{r}, \hbar)\|_{\mathcal{R}} \leq \begin{cases} \frac{\|\mathbf{r}\|}{\|\hbar\|^{\delta}} \\ \frac{\|\mathbf{r}\|\|\hbar\|}{\|\mathbf{r}\|^{\delta} + \|\hbar\|^{\delta}} \\ \frac{1}{\|\mathbf{r}\|^{\delta} + \|\hbar\|^{\delta}} \end{cases} \quad (27)$$

for any $\mathbf{r}, \hbar \in \mathcal{Q}$, and

$$\|\mathcal{g}(\mathbf{r}\hbar) - \mathcal{g}(\mathbf{r})\mathcal{g}(\hbar)\|_{\mathcal{R}} \leq \begin{cases} \frac{\|\mathbf{r}\|}{\|\hbar\|^{\delta}} \\ \frac{\|\mathbf{r}\|\|\hbar\|}{\|\mathbf{r}\|^{\delta} + \|\hbar\|^{\delta}} \\ \frac{1}{\|\mathbf{r}\|^{\delta} + \|\hbar\|^{\delta}} \end{cases} \quad (28)$$

for any $\mathbf{r}, \hbar \in \mathcal{Q}$.

Then there is a unique (H.) mapping $\mathcal{C}: \mathcal{Q} \rightarrow \mathcal{R}$ such that

$$\|\mathcal{g}(\mathbf{r}) - \mathcal{C}(\mathbf{r})\| \leq \frac{1}{(1-\ell)|2|} \begin{cases} \|\mathbf{r}\|^{1-\delta} \\ \|\mathbf{r}\|^{2-\delta} \\ \|\mathbf{r}\|^{-\delta} \end{cases} \quad (29)$$

For any $\mathbf{r} \in \mathcal{Q}$.

Proof. The proof is straightforward from the assumptions. ■

Remark 4.3. The results obtained in the context of (H.) is also applicable to derivations.

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