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**Abstract.**

Let  $R$  be a commutative ring with unity .A unitary  $R$ -module  $M$  is called a quasi-Dedekind module if  $Hom(M/N, M) = 0$  for all nonzero submodules  $N$  of  $M$  . In this paper we introduce and study the concept of small quasi-Dedekind module as a generalization of quasi-Dedekind module . Where an  $R$ -module  $M$  is called small quasi-Dedekind if, for each nonzero homomorphisms  $f$  from  $M$  to  $M$  , implies  $Kerf$  small in  $M$  (  $Kerf \ll M$  ). And an  $R$ -submodule  $N$  of an  $R$ -module  $M$  is called a small submodule of  $M$  ( $N \ll M$  , for short) if , for all  $K \leq M$  with  $N+K = M$  implies  $K = M$

**Key Words :**

quasi-Dedekind Modules ; small quasi-Dedekind Modules .

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## 1. Introduction

Let  $R$  be a commutative ring with unity and  $M$  be a unitary  $R$ -module. Mijbass A.S in [8] introduced and studied the concept of quasi-Dedekind, where an  $R$ -module  $M$  is called quasi-Dedekind if  $\text{Hom}(M/N, M) = 0$  for all nonzero submodules  $N$  of  $M$ . In this paper we introduce and study another generalization of the concept a quasi-Dedekind module namely "small quasi-Dedekind module". Also in this paper, we investigate the basic properties and characterizations about this concept. At the start of this paper we give some of the basic properties and characterizations of small quasi-Dedekind modules. Recall that an  $R$ -module  $P$  is projective if and only if, for any two  $R$ -modules  $A, B$  and for any epimorphism  $f: A \rightarrow B$  and for any homomorphism  $g: P \rightarrow B$ , there exists a homomorphism  $h: P \rightarrow A$  such that  $foh = g$  [6, p.117]. Among results we obtain in this paper, we prove that: Let  $M$  be an  $R$ -module such that  $M/U$  is projective for all  $U \ll M$ . If  $M$  is a small quasi-Dedekind  $R$ -module then  $M/N$  is a small quasi-Dedekind  $R$ -module for all  $N \leq M$ .

Recall that an  $R$ -module  $M$  is a quasi-Dedekind module if and only if for all  $f \in \text{End}_R(M)$ ,  $f \neq 0$  implies  $\text{Ker}f = 0$ , (see Th 1.5, P.26, 8).

Now we shall give a generalization to quasi-Dedekind module namely "small quasi-Dedekind module" as follows.

### Definition 1.1.

An  $R$ -module  $M$  is called a small quasi-Dedekind module if, for all  $f \in \text{End}_R(M)$ ,  $f \neq 0$  implies  $\text{Ker}f \ll M$  (i.e.  $\text{Ker}f$  is a small submodule in  $M$ ).

### Remarks and Examples 1.2.

1) It is clear that every quasi-Dedekind  $R$ -module is a small quasi-Dedekind  $R$ -module.

But the converse is not true in general, for example:  $Z_4$  as  $Z$ -module is small quasi-Dedekind

but it is not quasi-Dedekind, also it is not essentially quasi-Dedekind. Where an  $R$ -module  $M$  is called essentially quasi-Dedekind if  $\text{Hom}(M/N, M) = 0$  for all  $N \leq_e M$  [4, def.1.2.1].

2)  $Z_6$  as  $Z$ -module is not small quasi-Dedekind, since there exists,  $f: Z_6 \longrightarrow Z_6$

define by  $f(\bar{x}) = 3\bar{x}$ ,  $\bar{x} \in Z_6$ . So  $f \neq 0$ , but  $\text{Ker}f = \{\bar{x} \in Z_6 : f(\bar{x}) = \bar{0}\} = \{\bar{x} \in Z_6 : 3\bar{x} = \bar{0}\} = (\bar{2}) \not\ll Z_6$ . However,  $Z_6$  is an essentially quasi-Dedekind  $Z$ -module.

3)  $Z \oplus Z$  is not a small quasi-Dedekind  $Z$ -module, since there exists  $f: Z \oplus Z \longrightarrow Z \oplus Z$  such that  $f(x, y) = (x, 0)$ ;  $x, y \in Z$ . So  $f \neq 0$ , but  $\text{Ker}f = (0) \oplus Z \not\ll Z \oplus Z$ .

4) If  $M = 0$ , it is clear that  $M$  is a small quasi-Dedekind module.

5) Every integral domain  $R$  is a small quasi-Dedekind  $R$ -module, but the converse

is not true in general, for example:

$Z_4$  as  $Z_4$ -module is small quasi-Dedekind, but it is not an integral domain.

6) If  $M$  is a semisimple  $R$ -module, then it is not necessarily small quasi-Dedekind, (see Rem.and.Ex 1.2(2)).

7) Every semisimple small quasi-Dedekind  $R$ -module  $M$  is a quasi-Dedekind  $R$ -module.

**Proof :** Let  $f \in \text{End}_R(M)$ ,  $f \neq 0$ . Since  $M$  is small quasi-Dedekind, then  $\text{Ker}f \ll M$ . But  $M$  is semisimple, so  $\text{Ker}f = 0$ . Thus  $M$  is a quasi-Dedekind  $R$ -module.

The following theorem is a characterization of small quasi-Dedekind modules.

**Theorem 1.3.**

Let  $M$  be an  $R$ -module. Then  $M$  is small quasi-Dedekind if and only if  $\text{Hom}(M/N, M) = 0$  for all  $N \ll M$ .

**Proof :**

$\Rightarrow$ ) Suppose that there exists  $N \not\ll M$  such that  $Hom(M/N, M) \neq 0$ , then there exists  $\phi: M/N \rightarrow M$ ,  $\phi \neq 0$ . Hence  $\phi \circ \pi \in End_R(M)$ , where  $\pi$  is the canonical projection, and  $\phi \circ \pi \neq 0$  which implies  $Ker(\phi \circ \pi) \ll M$ , but  $N \subseteq Ker(\phi \circ \pi)$ , so  $N \ll M$  which is a contradiction.

$\Leftarrow$ ) Suppose that there exists  $f: M \rightarrow M$ ,  $f \neq 0$  such that  $Ker f \ll M$ , define  $g: M/Ker f \rightarrow M$  by  $g(m + Ker f) = f(m)$ , for all  $m \in M$ . So  $g$  is well-defined and  $g \neq 0$ . Hence  $Hom(M/Ker f, M) \neq 0$  which is a contradiction.

**Proposition 1.4.**

Let  $M$  be an  $R$ -module and let  $\bar{R} = R/J$ , where  $J$  is an ideal of  $R$  such that  $J \subseteq ann_R(M)$ . Then  $M$  is a small quasi-Dedekind  $R$ -module if and only if  $M$  is a small quasi-Dedekind  $\bar{R}$ -module.

**Proof :**

$\Rightarrow$ ) We have  $Hom_R(M/K, M) = Hom_{\bar{R}}(M/K, M)$ , for all  $K \leq M$ , by [6, p.51]. Thus, if  $M$  is a small quasi-Dedekind  $R$ -module, then  $Hom_R(M/K, M) = 0$  for all  $K \ll M$ , so  $Hom_{\bar{R}}(M/K, M) = 0$  for all  $K \ll M$ , thus  $M$  is a small quasi-Dedekind  $\bar{R}$ -module.

$\Leftarrow$ ) The proof of the converse is similarly.

**Proposition 1.5.**

Let  $M_1, M_2$  be  $R$ -modules such that  $M_1 \cong M_2$ .  $M_1$  is a small quasi-Dedekind  $R$ -module if and only if  $M_2$  is a small quasi-Dedekind  $R$ -module.

**Proof :**  $\Rightarrow$ ) Let  $f: M_2 \rightarrow M_2$ ,  $f \neq 0$ . To prove  $Ker f \ll M_2$ . Since  $M_1 \cong M_2$ , there exists an isomorphism  $g: M_1 \rightarrow M_2$ . Consider the following commutative diagram:

$$M_1 \xrightarrow{g} M_2 \xrightarrow{f} M_2 \xrightarrow{g^{-1}} M_1$$

Hence  $h = g^{-1} \circ f \circ g \in End_R(M_1)$ ,  $h \neq 0$ . So  $Ker h \ll M_1$  (since  $M_1$  is small quasi-Dedekind), then  $g(Ker h) \ll M_2$  by [6, lemma 5.1.3, p.108]. But we can show that  $g(Ker h) = Ker f$  as follows: let  $y \in g(Ker h)$ , so  $y = g(x)$ ,  $x \in Ker h$ . Hence  $h(x) = 0$ ; that is  $g^{-1} \circ f \circ g(x) = 0$ , then  $g^{-1} \circ f(y) = 0$ , so  $g^{-1}(f(y)) = 0$  and hence  $f(y) = 0$ , since  $g^{-1}$  is monomorphism, so that

$y \in \text{Ker}f$ , hence  $g(\text{Ker}h) \subseteq \text{Ker}f$ . Now, Let  $y \in \text{Ker}f$ , then  $f(y) = 0$ , but  $y \in M_2$ , so there exists an  $x \in M_1$  such that  $y = g(x)$ , since  $g$  is onto. Thus  $f(g(x)) = 0$  and so  $g^{-1}(f(g(x))) = 0$ ; that is  $h(x) = 0$ . Hence  $x \in \text{Ker}h$ . This implies  $y = g(x) \in g(\text{Ker}h)$ , thus  $\text{Ker}f = g(\text{Ker}h) \ll M_2$ , hence  $\text{Ker}f \ll M_2$ .

$\Leftarrow$ ) The proof of the converse is similarly.

**Remark 1.6.**

Let  $N \leq M$ , and  $f \in \text{End}_R(M)$ ,  $f \neq 0$ . Note that if  $f(N) \ll f(M)$ , then it is not necessarily  $N \ll M$ . Consider the following example.

**Example 1.7.**

Let  $M = Z_6$  as  $Z$ -module, and let  $N = (\bar{2}) \leq Z_6$ . Let  $f: Z_6 \rightarrow Z_6$  define by  $f(\bar{x}) = 3\bar{x}$ ,  $\bar{x} \in Z_6$ . So  $f \neq 0$  and  $f(N) = f((\bar{2})) = \{0\} \ll \{0, \bar{3}\} = f(Z_6) = f(M)$ , but  $N = (\bar{2}) \not\ll Z_6 = M$ .

In the following proposition we give a condition under which the remark (1.6) is true in general.

**Proposition 1.8.**

Let  $M$  be a small quasi-Dedekind  $R$ -module and  $f \in \text{End}_R(M)$ ,  $f \neq 0$ ,  $N \leq M$ . If  $f(N) \ll f(M)$  then  $N \ll M$ .

**Proof :** Let  $B \leq M$  and  $N + B = M$  then  $f(N) + f(B) = f(M)$ . But  $f(N) \ll f(M)$  implies  $f(B) = f(M)$ . Now, we can show that  $\text{Ker}f + B = M$ . Let  $m \in M$ , hence  $f(m) \in f(M) = f(B)$ . So that there exists  $b \in B$  such that  $f(m) = f(b)$ , hence  $m - b \in \text{Ker}f$ . It follows that  $m = (m - b) + b$ , thus  $M \subseteq \text{Ker}f + B$ . Thus  $\text{Ker}f + B = M$ , but  $M$  is a small quasi-Dedekind  $R$ -module, so  $\text{Ker}f \ll M$  which implies that  $B = M$ . Therefore  $N \ll M$ .

**Corollary 1.9.**

Let  $M$  be a small quasi-Dedekind  $R$ -module and  $f \in \text{End}_R(M)$ ,  $f$  is surjective. Then  $N \ll M$  if and only if  $f(N) \ll M$ .



Then  $g(N) = f(N) \neq 0$ ; that is  $g \neq 0$ . So that  $\text{Ker}g \ll M$ , since  $M$  is small quasi-Dedekind. But  $\text{Ker}f \subseteq \text{Ker}g$ , hence  $\text{Ker}f \ll M$ . On the other hand  $\text{Ker}f \leq N$ , so by hypothesis  $\text{Ker}f \ll N$ . Thus  $N$  is a small quasi-Dedekind  $R$ -module.  $\square$

We are now in a position to recall the definition of coclosed submodule which was introduced by Golan [5]. Recall that an  $R$ -submodule  $N$  of  $M$  is coclosed in  $M$ , if whenever  $N/K \ll M/K$  then  $N = K$  for all submodules  $K$  of  $M$  contained in  $N$ .

And let  $U$  be a submodule of  $M$ , a submodule  $V$  of  $M$  is called a supplement (or addition complement) of  $U$  in  $M$  if  $V$  is a minimal element in the set of all submodules  $L$  of  $M$  with  $U + L = M$ .  $V$  is called a supplement submodule of  $M$  if,  $V$  is a supplement of some submodule of  $M$ , [7].

**Corollary 1.12.**

Let  $M$  be a small quasi-Dedekind and quasi-injective  $R$ -module, let  $N \leq M$ . If  $N$  is a supplement (or coclosed) submodule, then  $N$  is a small quasi-Dedekind  $R$ -module.

**Proof :** By [1, Prop 1.2.6],  $N$  is supplement then  $N$  is coclosed, and hence for all  $U \leq N$ ,  $U \ll M$  implies  $U \ll N$ . So the result follows by ( Prop 1.11).

An  $R$ -module  $M$  is called a quasi-injective  $R$ -module if for each monomorphism  $f: N \longrightarrow M$ ,  $N \leq M$  and any homomorphism  $g: N \longrightarrow M$ , there exists a homomorphism  $h: M \longrightarrow M$  such that  $hof = g$ . A quasi-injective  $R$ -module  $\overline{M}$  is called a quasi-injective hull (a quasi-injective envelope) of an  $R$ -module  $M$  if there is a monomorphism  $f: M \longrightarrow \overline{M}$  such that  $\text{Im} f \leq_e \overline{M}$ .

**Corollary 1.13.**

Let  $M$  be an  $R$ -module such that  $\overline{M}$  is a small quasi-Dedekind  $R$ -module, and for all  $U \leq M$ ,  $U \ll \overline{M}$  implies  $U \ll M$ . Then  $M$  is a small quasi-Dedekind  $R$ -module.

**proof :** Since  $\bar{M}$  is a small quasi-Dedekind and quasi-injective  $R$ -module , so by (Prop 1.11) ,  $M$  is a small quasi-Dedekind  $R$ -module .

**Proposition 1.14.**

Let  $M$  be a small quasi-Dedekind  $R$ -module .Then for all  $N \ll M$   
 $ann_R(N) = ann_R(M)$  .

**Proof :** Since  $M$  is a small quasi-Dedekind  $R$ -module, so by (Th 1.3),  
 $Hom(M/N, M) = 0$  for all  $N \ll M$  which implies  $N$  is a quasi-invertible  
submodule for all  $N \ll M$  . Thus by (8, Prop1.4, P.7) , for all  $N \ll M$   
 $ann_R(N) = ann_R(M)$  .

**Remark (1.15)**

Let  $N \leq M$  . If  $M/N$  is a small quasi-Dedekind  $R$ -module , then it is not necessarily that  $M$  is a small quasi-Dedekind  $R$ -module, for example : If  $M = Z_6$  as  $Z$ -module , and let  $N = (\bar{2}) \leq M = Z_6$  , then  $Z_6/(\bar{2}) \cong Z_2$  which is a small quasi-Dedekind  $Z$ -module . But  $M = Z_6$  as  $Z$ -module is not small quasi-Dedekind .

**Remark 1.16.**

If  $M$  is a small quasi-Dedekind  $R$ -module ,  $N \leq M$  . Then it is not necessarily that  $M/N$  is a small quasi-Dedekind  $R$ -module . Consider the following example .

**Example 1.17.**

The  $Z$ -module  $M=Z$  is small quasi-Dedekind .Let  $N = 6Z \leq Z$  , then  
 $M/N = Z/6Z \cong Z_6$  is not a small quasi-Dedekind  $Z$ -module .

The following result shows that under certain condition , the module  $M/N$  is small quasi-Dedekind .

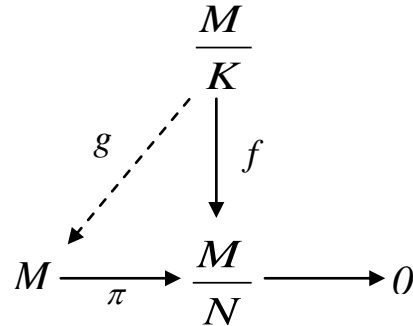
**Proposition 1.18.**

Let  $M$  be an  $R$ -module such that  $M/U$  is projective for all  $U \ll M$ . If  $M$  is a small quasi-Dedekind  $R$ -module , then  $M/N$  is a small quasi-Dedekind  $R$ -module for all  $N \leq M$  .

**Proof :** Let  $K/N \not\ll M/N$  , so by [1, Prop 1.1.2 , p.10] ,  $K \not\ll M$  .

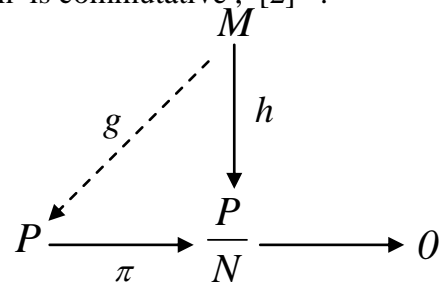


Suppose that  $\text{Hom}(\frac{M/N}{K/N}, \frac{M}{N}) \neq 0$ , but  $\text{Hom}(\frac{M/N}{K/N}, \frac{M}{N}) \cong \text{Hom}(\frac{M}{K}, \frac{M}{N})$ , so there exists  $f: M/K \longrightarrow M/N, f \neq 0$ . Since  $M/K$  is projective, then there exists  $g: M/K \longrightarrow M$  such that  $\pi \circ g = f$ , where  $\pi$  is the canonical projection.



Hence  $\pi \circ g(M/K) = f(M/K) \neq 0$ , so  $g \neq 0$ , but  $g \in \text{Hom}(M/K, M), K \ll M$ . Thus  $\text{Hom}(M/K, M) \neq 0, K \ll M$ ; that is  $M$  is not small quasi-Dedekind, which is a contradiction. Thus  $M/N$  is a small quasi-Dedekind  $R$ -module.

Let  $M$  and  $P$  be modules, then  $M$  is called  $P$ -projective in case for each  $N \leq P$  and every homomorphism  $h: M \longrightarrow P/N$ , there exists a homomorphism  $g: M \longrightarrow P$  such that  $\pi \circ g = h$ . (where  $\pi$  is the natural epimorphism); that is the following diagram is commutative, [2].

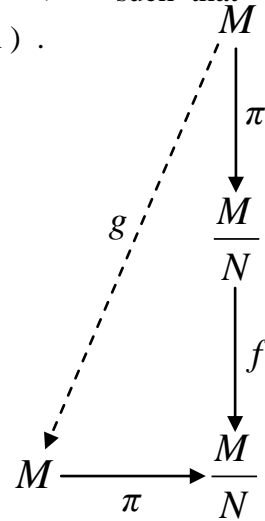


An  $R$ -module  $M$  is called quasi-projective if,  $M$  is  $M$ -projective; that is for each  $N \leq M$  and every homomorphism  $h: M \longrightarrow M/N$ , there exists a homomorphism  $g: M \longrightarrow M$  such that  $\pi \circ g = h$ . (where  $\pi$  is the natural epimorphism), [9].

**Theorem 1.19.**

Let  $M$  be a quasi-projective  $R$ -module, let  $N \leq M$  such that  $g^{-1}(N) \ll M$  for each  $g \in \text{End}_R(M)$ , then  $M/N$  is a small quasi-Dedekind  $R$ -module.

**Proof :** Let  $f: M/N \rightarrow M/N$  such that  $f \neq 0$ . Since  $M$  is quasi-projective, there exists a homomorphism  $g: M \rightarrow M$  such that  $\pi \circ g = f \circ \pi$  (where  $\pi$  is the canonical projection).



Let  $\text{Ker} f = L/N = \{x + N : f(x + N) = N\} = \{x + N : f \circ \pi(x) = N\} = \{x + N : \pi \circ g(x) = N\} = \{x + N : g(x) + N = N\} = \{x + N : g(x) \in N\} = \{x + N : x \in g^{-1}(N)\}$ . Thus  $\text{Ker} f = g^{-1}(N)/N$ , but  $g^{-1}(N) \ll M$ , so by [6, Lemma 5.1.3, p.108],  $g^{-1}(N)/N \ll M/N$ ; that is  $\text{Ker} f \ll M/N$ .

**Corollary 1.20.**

Let  $M$  be a quasi-projective  $R$ -module such that for each  $N \leq M$ ,  $N \ll h(M)$  for all  $h \in \text{End}_R(M)$ . Then  $M$  is a small quasi-Dedekind  $R$ -module if and only if  $M/N$  is a small quasi-Dedekind  $R$ -module.

**Proof :**  $\Leftarrow$ ) It is clear by taking  $N = (0)$ .

$\Rightarrow$ ) By (prop 1.10),  $N \ll h(M)$  implies  $h^{-1}(N) \ll M$ . Hence the result follows by the previous theorem.  $\square$

Recall that an  $R$ -submodule  $N$  of an  $R$ -module  $M$  is invariant if  $f(N) \subseteq N$  for each  $f \in \text{End}_R(M)$ . Some authors called an invariant submodule, fully invariant submodule, by [3].

**Theorem 1.21.**

Let  $M$  be an  $R$ -module. Then  $M$  is small quasi-Dedekind if and only if there exists  $N \ll M$ ,  $N$  is fully invariant such that for each  $f \in \text{End}_R(M)$ ,  $f \neq 0$ ,  $f(M) \not\subseteq N$  and  $M/N$  is small quasi-Dedekind.

**Proof :**  $\Rightarrow$ ) Choose  $N = (0)$  implies  $N \ll M$  and  $N$  is fully invariant and for all  $f \in \text{End}_R(M)$ ,  $f \neq 0$ , hence  $f(M) \not\subseteq (0) = N$  and  $M/N = M/(0) \cong M$  is small quasi-Dedekind.

$\Leftarrow$ ) If  $N = 0$ , then  $M$  is small quasi-Dedekind. Suppose that  $N \neq (0)$ ,  $N \ll M$ . Let  $f \in \text{End}_R(M)$ ,  $f \neq 0$ . To prove  $\text{Ker} f \ll M$ . Define  $g : M/N \rightarrow M/N$  by  $g(m + N) = f(m) + N$  for all  $m \in M$ .  $g$  is well-defined, since if  $m_1 + N = m_2 + N$  where  $m_1, m_2 \in M$ , then  $m_1 - m_2 \in N$  and  $f(m_1 - m_2) \in f(N) \subseteq N$ , since  $N$  is fully invariant. This implies  $f(m_1) - f(m_2) \in N$ ; that is  $f(m_1) + N = f(m_2) + N$ , thus  $g(m_1 + N) = g(m_2 + N)$ .  $g \neq 0$ , because if  $g = 0$  then  $g(M/N) = N = 0_{M/N}$ . Hence  $f(M) + N = N$ , it follows that  $f(M) \subseteq N$  which is a contradiction with the hypothesis. Thus  $\text{Ker} g \ll M/N$ , since  $M/N$  is a small quasi-Dedekind  $R$ -module. Let  $\text{Ker} g = L/N \ll M/N$ , but  $N \ll M$ , so by [1, Prop 1.1.2, p.10],  $L \ll M$ . On the other hand it is easy to see that  $\text{Ker} f \subseteq L$ , so  $\text{Ker} f \ll M$ , thus  $M$  is a small quasi-Dedekind  $R$ -module.

An  $R$ -module  $M$  is called multiplication if for each submodule  $N$  of  $M$ ,  $N = IM$  for some ideal  $I$  of  $R$ . Equivalently,  $M$  is a multiplication  $R$ -module if, for each submodule  $N$  of  $M$ ,  $N = [N : M]M$ , where  $[N : M] = \{r \in R : rM \subseteq N\}$ .

**Corollary 1.22.**

Let  $M$  be a multiplication  $R$ -module. Then  $M$  is small quasi-Dedekind if and only if there exists  $N \ll M$  such that for all  $f \in \text{End}_R(M)$ ,  $f \neq 0$ ,  $f(M) \not\subseteq N$  and  $M/N$  is small quasi-Dedekind.

**Proof :** Since  $M$  is a multiplication  $R$ -module, every proper submodule of  $M$  is fully invariant. Thus the result is obtained by (Th 1.21).

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