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When compact sets are g-closed

Saheb. K. Jassim ***Haider* G. Ali***

The University of Al-Mustansiriya - College of Science - Department of Mathematics-Iraq

***E-Mail:** haiderali89@yahoo.com

المستخلص

كرس هذا البحث لتقديم مفاهيم جديده هي الفضاءات K(gc), gK(gc), L(gc), gL(gc) و- L(gc)محليا. حيث أن مبرهنات عديده ومتنوعه حول هذه المفاهيم قد برهنت. فضلا عن ذكر خصائصها وكذلك تحري العلاقات بين هذه المفاهيم والفضاءات LC.

Abstract

This paper is devoted to introduce new concepts which are called K(gc), gK(gc), L(gc), gL(gc) and locally L(gc)-spaces. Several various theorems about these concepts are proved. Further more properties are stated as well as the relationships between these concepts and LC-spaces are investigated.

Key words:
g-closed, KC-spaces and LC-spaces.

1-Introduction:

It is known that compact subset of a Hausdorff space is closed, this motivates the author [7] to introduce the concept of KC-space, these are the spaces in which every compact subset is closed. Lindelof spaces have always played a highly expressive role in topology. They were introduced by Alexandroff and Urysohn back in 1929. In 1979 the authors [5] introduce a new concept namely LC-spaces, these are the spaces whose lindelof sets are closed. The aim of this paper is to continue the study of KC-spaces (LC-spaces).

2-Preliminaries:

The basic definitions that needed in this work are recalled. In this work, spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated, a topological space is denoted by (X, τ) (or simply by X). For a subset A of X, the closure and the interior of A in X are denoted by cl(A) and Int(A) respectively. A space X is said to be K_2- space if cl(A) is compact, when- ever A is compact set in X[6]. Also a subset F of a space X is g-closed if cl(F)$\subset$U,
whenever U is open and containing F[4], X is said to be gT_1 if for every two distinct points x and y in X, there exist two g-open sets U and V such that $x \in U$ and $y \notin U$, also $x \notin V$ and $y \in V$ [3], and gT_2 if for every two distinct points x and y in X, there exist two disjoint g-open sets U and V containing x and y respectively [3]. A space X is said to be g-regular if whenever F is g-closed in X and $x \in X$ with $x \notin F$, then there are two disjoint g-open sets U and V containing x and F respectively [3]. A space X is said to be gT_3 if whenever it is gT_1 and g-regular [3] and X is said to be g-compact if for every g-open cover of X has a finite subcover[2]. A function f from a space X into a space Y is said to be g^{**}-continuous if $f^{-1}(U)$ is g-open, whenever U is g-open subset of a space Y. Also f is said to be g^{**}-closed if f(F) is g-closed, whenever F is g-closed [3].

3-Weak forms of KC-spaces:

The author in [7] introduce the concept KC-spaces; in the present paper we introduce a generalization of KC-spaces namely K(gc) and gK(gc), also we study the

1-Ass. Prof., Department of Mathematics, College of Science, Al-Mustansiriya University
2-Lectuerer, Department of Mathematics, College of Science, Al-Mustansiriya University, E-Mail: haiderali89@yahoo.com

Properties and facts about these concepts and the relationships between this concepts and KC-space.

Definition 3.1

A space X is said to be K(gc)-space if every compact set in X is g -closed. So every KC-space is K(gc), but the converse is not true in general.

Example 3.1:

Let $X \neq \phi$ and Γ be the indiscrete topology on X. Then (X, Γ) i s K(gc) but not KC-space. Since if B is a nonempty proper set in X. Clearly B is compact but not closed. Also it is g-closed, since the only open set which contains B is the whole space and cl(B) = X.

Definition 3.2

A space X is said to be gK(gc)-space if every g-compact set in X is g -closed. So every K(gc)-space is gK(gc), but the converse is not true in general.

Definition 3.3

A space X is said to be gK_2 if g-cl(A) is compact, whenever A is compact set in X.

Theorem 3.1:

Every K(gc)-space is gK_2.

Proof: Let K be compact set in K(gc)-space X, then it is g-closed, that is, $Cl_g(K) = K$, which implies to $Cl_g(K)$ is also compact.

Definition 3.4

A space X is said to be locally g-compact if for each point in X has a neighbourhood base which is consisting of g-compact sets. So every locally compact space is locally g-compact, but the converse is not true in general.

Lemma 3.1[1]:

A space X is gT_1 if and only if every singleton set is g-closed.

Theorem 3.2

Every K(gc)-space is gT_1.

Proof:

Suppose X is K(gc)-space and $x \in X$, since {x} is finite, then it is compact in X, which is K(gc)-space, then it is g-closed. So by lemma 3.1 X is gT_1.

Theorem 3.3

Every gT_3-space is gT_2.

Proof: Let x and **y** be two distinct points in X, so {x} is g-closed, since X is gT_1 and $y \notin \{x\}$, but X is g-regular, then there exist two disjoint g-open sets U and V such that $x \in \{x\} \subset U$ and $y \in V$. Therefore X is gT_2-space.

Definition 3.5:

A set M is said to be g-neighbourhood of a point $x \in X$ if there exists a g-open set U such that $x \in U \subset X$. Clearly every neighbourhood is g-neighbourhood but the converse may be not true.

Example 3.2:

Let $X \neq \phi$ and Γ be the indiscrete topology on X. Then in (X, Γ) the one point set {x} is g-neighbourhood but not neighbourhood.

Theorem 3.4

The following are equivalent for a space X:

1) X is g-regular
2) If U is g-open in X and $x \in X$ with $x \in U$, then there is a g-open set V containing x such that $g\text{-}cl(V) \subset U$.
3) Each $x \in X$ has ag-neighbourhood base consisting of g-closed sets.

Proof: $(1) \rightarrow (2)$ Suppose X is g-regular, U is g-open in X and $x \in U$, then X-U is a g
-closed set in X not containing x, so disjoint g-open sets V and W can be found with
$x \in V$ and $X\text{-}U \subset W$. Then X-W is a g-closed set contained in U and containing V, so $g\text{-}cl(V) \subset U$. $(2) \rightarrow (3)$ if (2) applies, then every g-open set U containing x contains a g-closed neighbourhood (namely g-cl(V)) of X, so the g-closed neighbourhoods of x form a neighbourhood base. $(3) \rightarrow (1)$ suppose (3) applies and A is a g-closed set in X not containing x. Then X-A is a g-neighbourhood of

x, so there is a g-closed neighbourhood B of x with $B \subset X$-A. Then g-Int(B) and X-B are disjoint g-open sets containing x and A respectively, where g-Int(B) the set of all g-interior points. Thus X is g-regular.

Theorem 3.5:

Every T_2-space is K(gc)space.

Theorem 3.6

If X is locally g-compact and K(gc)-space, then X is gT_2-space.

Proof: Given X is locally g-compact, then every $x \in X$ has a neighbourhood base consisting of g-compact sets, but X is K(gc), then these compact sets are g-closed and hence x has neighbourhood base consisting of g-closed sets, then by theorem 3.4, X is g-regular space and by theorem 3.2 X is gT_1, then it is gT_3-space, that is, X is gT_2.

Theorem 3.7:

Every g-compact set in gT_2-space is g-closed.

Proof: Let A be a g-compact set in a gT_2–space X. If $p \in X$-A, so for each $q \in A$, there are two disjoint g-open sets U and V containing q and p respectively. The collection $\{U(q):q \in A\}$ is a g-open cover of A which is g-compact, then there is finite subcover of A, that is, $A \subset \bigcup_{i=1}^{n} U(q_i)$. Put $V_1=\bigcap_{i=1}^{n} V_{qi}(p)$ and $U_1=\bigcup_{i=1}^{n} U(q_i)$. Then V_1 is a g-open set containing p. We claim that $U_1 \cap V_1=\phi$, so let $x \in U_1$, then $x \in U(q_i)$ for some i, so $x \notin V_{qi}(p)$, hence $x \notin V_1$. Thus $U_1 \cap V_1=\phi$. Also $A \subset U_1$, that is, $A \cap V_1= \phi$ which implies $V_1 \subset X$-A. Therefore A is g-closed.

Corollary 3.1:

Every gT_2-space is gK(gc)-space.

Theorem 3.8:

The g^{**}-continuous image of g-compact set is g-compact.

Proof: Let f be g^{**}-continuous function from a space X into a space Y and suppose B is g-compact set in X. To show that B is also g-compact, let $\{U_\alpha\}_{\alpha \in \Lambda}$ be

g-open cover of f(B), that is, f(B)= $\bigcup_{\alpha\in\Lambda} U_\alpha$. So B$\subset f^{-1}$f(B)=
$f^{-1}(\bigcup_{\alpha\in\Lambda} U_\alpha)=\bigcup_{\alpha\in\Lambda} f^{-1}(U_\alpha)$, then { $f^{-1}(U_\alpha)$}is a g-open cover of B, which is g-compact, then B$\subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\alpha i})$. But f(B) $\subseteq$ f$\bigcup_{i=1}^{n} f^{-1}(U_{\alpha i})=\bigcup_{i=1}^{n}$ f $f^{-1}(U_{\alpha i})\subset \bigcup_{i=1}^{n} U_{\alpha i}$.
Therefore f(B) is g-compact set.

Theorem 3.9:

Every continuous function from compact into a K(gc)-space is g-closed function.
Proof: Let A be closed set in X, which is compact, then A is compact. But f is continuous, then f(A) is compact in Y, which is K(gc)-space, then f(A) is g-closed. Therefore f is g-closed.

Lemma 3.2[1]:

Every g-closed subset of g-compact space is g-compact.

Theorem 3.10:

Every g^{**}-continuous function from g-compact into K(gc)-space is g^{**}-closed function.
Proof: Let f be g^{**}-continuous function from g-compact X into K(gc)-space Y. Also let B be g-closed set in X. So by lemma 3.2 B is g-compact also by theorem 3.8 f(B) is g-compact, which implies it is compact in Y, which is K(gc), then f(B) is g-closed. Therefore f is g^{**}-closed.

Corollary 3.2:

Every g^{**}-continuous function from g-compact space into gK(gc) -space is g^{**}-closed.

Remark 3.2:

The continuous image of K(gc)-space is not necessarily K(gc).

Example 3.3:

Consider I_R: (R, Γ_u) $\rightarrow$(R, Γ), where I_R is the identity function, Γ_u and Γ are usual and cofinite topologies respectively. Clearly (R, Γ_u) is K(gc)-space.. Since every compact set in R is closed and bounded, this implies it g-closed. But I_R(R) = R and (R, Γ) K(gc)-space. Since if given [0, 1], which is compact and U=R-{5}, so U$\in\Gamma$, then [0, 1] $\subset$ U, but cl([0, 1])=R$\not\subset$U. So (R, Γ) is not K(gc).

Theorem 3.11:

Let f be g^{**}-continuous injective function from X into a gK(gc) –space Y, then X is also gK(gc).

Proof: Let W be any g-compact subset of X, then by theorem 3.7 f(W) is g-compact set in Y, which is gK(gc), then f(W) is g-closed also f is g^{**}-continuous, so f^{-1}(f(W))=W. Therefore X is gK(gc)-space.

Theorem 3.12:

The property of space being K(gc) is a hereditary property.

Proof: Let Y be a subspace of K(gc)-space X and A be any compact subset of Y, then A is compact in X, which is K(gc), then A is g-closed in X. But A= A$\cap$X, then A is g-closed in Y. Therefore Y is also K(gc).

Theorem 3.13:

Let f be a homeomorphism function from a space X into a space Y, if U is g-open set in X, then f(U) is also g-open.

Proof: Let F be any closed subset of f(U), so f^{-1}(F)$\subset$ f^{-1}f(U)=U, but U is g-closed, then f^{-1}(F)$\subset$Int(U), which implies F= f(f^{-1}(F))$\subset$f(Int(U))=Int(f(U)). Therefore f(U) is also g-open.

Corollary 3.3:

Let f be a homeomorphism function from a space X into a space Y, if U is g-closed set in X, then f(U) is also g-closed.

Corollary 3.4:

Let f be a homeomorphism function from a space X into a space Y, if M is g-compact set in X, then f(M) is also g-compact.

Theorem 3.14:

The property of space being K(gc) is a topological property.

Proof: Let f be a homeomorphism function from a K(gc)-space X into a space Y and B be compact set in Y, then f^{-1}(B) is compact in X, which is K(gc), then f^{-1}(B) is g-closed and by corollary 3.3 f(f^{-1}(B))=B is g-closed set in Y.

Corollary 3.5:

The property of space being gK(gc) is a topological property.

4. Further type of LC-spaces:

In 1979 the authors [5] introduce a new concept namely LC-spaces, these are the spaces in which every lindelof sets are closed. In the present paper we introduce a new concept namely L(gc)-spaces which is a weak form of LC-spaces.

Definition 4.1

A space X is said to be L(gc)-space if every lindelof set is g-closed. So every LC-space is L(gc) but the converse is not true in general.

Example 4.1:

Let R with the indiscrete topology Γ. Clearly (R, Γ) is L(gc), since for every Lindelof set difference from R and ϕ is g-closed but not closed.

Theorem 4.1

Every L(gc)-space is gT_1.

Theorem 4.2

Every locally g-compact L(gc) is gT_2.

Proof: Let X be a locally g-compact and L(gc)-space, then X is K(gc). So by theorem 3.6 X is gT_2-space.

Theorem 4.3

The property of space being L(gc) is a hereditary property.

Proof: The proof is similar to theorem 3.12.

Theorem 4.4:

If X is L(gc) and $T_{\frac{1}{2}}$ -space, then every compact set in X is finite.

Proof: Let A be compact set in X. If A is finite, then the proof is finished, if A is infinite, then either A is countable or uncountable. Suppose A is countable and U is any set in A, then U is countable, so U is lindelof in A, which implies it is lindelof in X, which is L(gc), then U is g-closed in X. But X is $T_{\frac{1}{2}}$, and then U is closed in X. But $U \cap A = U$, then U is closed in A, that is, A is discrete but A is compact, then A is finite, which is a contradiction. If A is uncountable, then there exists a subset K of A is countable and so K is lindelof in A, so it is lindelof in X, which is L(gc) and $T_{\frac{1}{2}}$ -space, then K is closed. Put K= $\{a_1, a_2 \ldots\}$. Let $U_1 = K^c$, now $a_1 \in U_2 = A - \{a_1, a_2, \ldots\}$ and $a_2 \in A - \{a_3, a_4 \ldots\} \ldots$, then $\{U_i\}_{i=1}^{\infty}$ is an open cover of A, which has no finite subcover, which is a contradiction. Then A is finite.

Definition 4.3:

A space X is said to be g-lindelof if for every g-open cover of X has a countable subcovre. Clearly every g-lindelof-space is lindelof but the converse may be not true.

Example 4.2:

Let R with the indiscrete topology Γ. Clearly every subset of R is lindelof, since the only open cover of any set is just R. But (R, Γ) is not g-lindelof, since if given Q^c=R-Q , then it is not g-lindelof, since $\{\{x\}\}: x \in Q^c\}$ is a cover of Q^c consisting of g-open sets, which can not be reduce to a countable subcover.

Theorem 4.5:

The g^{**}-continuous image of g-lindelof set is also g-lindelof.

Proof: Let f be g^{**}-continuous function from a space X into a space Y and let K be g-lindelof set in X. To show that f(K) is also g-lindelof, let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a g-open cover of f(K), that is, f(K)$\subseteq \bigcup_{\alpha\in\Lambda}\{U_\alpha\}$, then K$\subset f^{-1}$f(K)$\subseteq f^{-1}\bigcup_{\alpha\in\Lambda}\{U_\alpha\}$= $\bigcup_{\alpha\in\Lambda}\{f^{-1}U_\alpha\}$, which is also g-open cover of K, but K is g-lindelof, then it is has a countable subcover, that is, K$\subseteq \bigcup_{i\equiv1}^{\infty}\{f^{-1}U_{\alpha i}\}$, which implies to f(K)$\subseteq \bigcup_{i\equiv1}^{\infty}\{U_{\alpha i}\}$. Therefore f(K) is g-lindelof.

Theorem 4.6:

The property of space being g-lidelof is a topological property.

Proof: Let f be a homeomorphism function from a g-lindelof space X into a space Y. Suppose $\{U_\alpha\}_{\alpha\in\Lambda}$ be g-open cover of Y, that is, Y= $\bigcup_{\alpha\in\Lambda}\{U_\alpha\}$, then X= f^{-1}(Y) = $f^{-1}\bigcup_{\alpha\in\Lambda}\{U_\alpha\}$. So by theorem 3.13 $\{f^{-1}U_\alpha\}$ is g-open cover of X, which is g-lindelof, then X=$\bigcup_{i\equiv1}^{\infty}\{f^{-1}U_{\alpha i}\}$, which implies to Y=f(X)=f($\bigcup_{i\equiv1}^{\infty}\{f^{-1}U_{\alpha i}\}$)=$\bigcup_{i\equiv1}^{\infty}$ f$\{f^{-1}U_{\alpha i}\}$=$\bigcup_{i\equiv1}^{\infty}\{U_{\alpha i}\}$. Therefore Y is also g-lindelof.

Definition 4.3:

A space X is said to be gL(gc)-space if every g-lindelof set in X is g-closed. So every LC-space is gL(gc) and every L(gc)-space is gL(gc) but the converses are not true in general.

Theorem 4.7:

Let f be a homeomorphism function from a space X into a space Y if X is gL(gc)-space, then Y is also is gL(gc).

Proof: Let B be a g-lindelof set in Y, then f^{-1}(B) is g-lindelof in X, which is gL(gc)-space, then it is g-closed, but f is a homeomorphism. So by theorem 3.13 f(f^{-1}(B))=B is g-closed in Y. Therefore Y is also gL(gc).

Definition 4.4:

A space X is said to be locally L(gc)-space if every point in X has L(gc)-neighbourhood. So every L(gc)-space is locally L(gc).

Lemma 4.1[3]:

If (Y, Γ_Y) is a g-closed subspace of a space (X, Γ_X), then if B is g-closed in Y, then it is g-closed in X.

Theorem 4.8:

A space X is an L(gc)-space if and only if each point has closed neighbourhoood which is an L(gc)-subspace.

Proof: If X is L(gc)-space, then for each $x \in X$, X itself is a closed neighbourhood of x, which is L(gc). Conversely, Let L be a lindelof set in X and a point $x \in X$ such that $x \notin L$. Choose a closed neighbourhood W_x of x, which is L(gc)-subspace, then $W_x \cap L$ is closed in L, which is lindelof, then $W_x \cap L$ is lidelof in W_x, but W_x is L(gc)-subspace, then $W_x \cap L$ is g-closed in W_x, which is closed so it is g-closed. So by lemma 4.1 $W_x \cap L$ is g-closed in X. Then W_x-($W_x \cap L$) =W_x-L is a g-open set containing x and disjoint with L. Therefore L is g-closed set in X.

References

[1] H. J. Ali and A. J. Mohammed, On g, g^*, g^{**}-compact functions, Journal of Al-Nahrain University Vol. 13(2), June, 2010, pp.205-209.
[2] K. Balachandran, P. Sandram and H. Maki, On generalized continuous maps in topological spaces, Mem. Fac. Sci. Kochi Uni. Ser. A Math. 12(1991), pp.5-13.
[3] J. M.. Esmail, On g-closed functions, M.Sc. Thesis, college of education Al-Mustansiriyah Univeresity, 2002.
[4] N. Liven, Generalized closed sets in topology, Rend. Circ. Math. Paleremo(2) 19 (1970), 89-96.
[5] T. K. Mukherji and M. Sarkar, On a class of almost discrete spaces, Mat. Vesnik 3(16) (31) (1979), 459-474.
[6] N. V. Velicko, H-closed topological spaces, Mat. sb. (N.S) 70(112) (1966), 98-112.
[7] Wilansky A., Between T_1 and T_2, Amer. Math. Monthly 74(1967) 261-266.