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New Approach of Closure Operator By Using Mixture Path System

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ABSTRACT

The closure operator is the central idea of this paper, which depended on the mixture path system and used to introduce and investigate the new space called $M\chi$ -closed and $M\chi$ -closure . The essential foundation for the concepts of $M\chi$ -space is presented, and we investigate the ideas and discover certain features of mixing path systems that are connected to certain graphs.

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1. Introduction and Preliminaries:

Two main reasons exist for why graph theory is an important idea in many different academic domains and an helpful tool in many other mathematic subjects. Let's now examine the mathematical process that is utilized to construct graphs. selected according to cognitive capacity. Second, given in a graph structure, some topics are simpler to comprehend. change into a more rational individual. One tool used to depict the graph is topological concepts, such as turning a set of vertices or edges into topological space, in relation to the relationship between topology and graph theory. These earlier studies on mixed structure graphs are included below. Kh. Sh. Al'Dzhabri et al. in [1-5] proposed a groundbreaking idea about directed and undirected graphs created by a unique mixture set. They used a variety of mixture structures to study various aspects of this idea, including the closed link between topology and graph. Another articles which details new approach of mixture structure by using approximation spaces and domination sets see[7-10]. A graph is defined by an ordered pair $(\gamma(\omega), \mathcal{E}(\omega))$, where $\gamma(\omega)$ is a nonempty set whose elements are called points or vertices and $\mathcal{E}(\omega)$ is a set of unordered pairs of distinct elements of $\gamma(\omega)$. The elements of $\mathcal{E}(\omega)$ are called lines or edges of the graph[6]. A path of length $n-1$, denoted by p_n , is a sequence of distinct edges[11]. A closed path, a path with $\alpha_1 = \alpha_n$ is called a cycle. A cycle with n vertices is denoted by c_n [11].

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2. Mixture Path system and $\dot{M}\dot{x}$ -space

Definition 2.1: Let $\omega = (\gamma(\omega), E(\omega))$ be a graph and a vertex $\mathfrak{x} \in \gamma(\omega)$

(1) The shortest even path set of $\mathfrak{x}$ is denoted by $EP(\mathfrak{x})$ and defined by:

$$EP(\mathfrak{x}) = \{\nu \in \gamma(\omega) : d_p(\mathfrak{x}, \nu) \text{ is shortest even path} \}.$$

(2) The shortest odd path set of $\mathfrak{x}$ is denoted by $OP(\mathfrak{x})$ and defined by:

$$OP(\mathfrak{x}) = \{\nu \in \gamma(\omega) : d_p(\mathfrak{x}, \nu) \text{ is shortest odd path} \}.$$

Definition 2.2: Let $\omega = (\gamma(\omega), E(\omega))$ be a graph, then the even path system (resp. odd path system) of a vertex $\mathfrak{x} \in \gamma(\omega)$ is denoted by $EPS(\mathfrak{x})$ (resp. $OPS(\mathfrak{x})$) and defined by: $EPS(\mathfrak{x}) = \{EP(\mathfrak{x})\}$ (resp. $OPS(\mathfrak{x}) = \{OP(\mathfrak{x})\}$).

Example 2.3: Let $\omega = (\gamma(\omega), E(\omega))$ be a graph such that:

$$\gamma(\omega) = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_4, \mathfrak{x}_5, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\} \quad E(\omega) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}\}$$

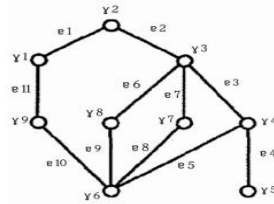


Figure 2.1 graph given in example

2.3

Then: $EP(\mathfrak{x}_1) = \{\mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6\}$, $EP(\mathfrak{x}_2) = \{\mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}$, $EP(\mathfrak{x}_3) = \{\mathfrak{x}_1, \mathfrak{x}_5, \mathfrak{x}_6\}$,

$EP(\mathfrak{x}_4) = \{\mathfrak{x}_2, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}$, $EP(\mathfrak{x}_5) = \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_6\}$, $EP(\mathfrak{x}_6) = \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5\}$, $EP(\mathfrak{x}_7) = \{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_8, \mathfrak{x}_9\}$, $EP(\mathfrak{x}_8) = \{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_9\}$, $EP(\mathfrak{x}_9) = \{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_8\}$

$OP(\mathfrak{x}_1) = \{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}$, $OP(\mathfrak{x}_2) = \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6\}$, $OP(\mathfrak{x}_3) = \{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}$, $OP(\mathfrak{x}_4) = \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6\}$, $OP(\mathfrak{x}_5) = \{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}$, $OP(\mathfrak{x}_6) = \{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}$, $OP(\mathfrak{x}_7) = \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6\}$, $OP(\mathfrak{x}_8) = \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6\}$, $OP(\mathfrak{x}_9) = \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6\}$

Theorem 2.6: let (C_n, H_{m_1}) and (P_n, H_{m_2}) be two $\dot{M}\dot{x}$ -space if n is even then $H_{m_1} = H_{m_2}$

Proof: since $M_p(C_n) = M_p(P_n)$ when n is even thus $H_{m_1} = H_{m_2}$

Theorem 2.7: let (ω_1, H_{m_1}) and (ω_2, H_{m_1}) be two $\dot{M}\dot{x}$ -space if $\omega_1 \cong \omega_2$ then $M_p(\omega_1) = M_p(\omega_2)$

Proof: $\omega_1 = (\gamma(\omega_1), E(\omega_1))$, $\omega_2 = (\gamma(\omega_2), E(\omega_2))$ be a two isomorphic graphs i.e $\omega_1 \cong \omega_2$ hence $|\gamma(\omega_1)| = |\gamma(\omega_2)|$ and $|E(\omega_1)| = |E(\omega_2)|$ and there exists bijective map $\vartheta : \gamma(\omega_1) \rightarrow \gamma(\omega_2)$ such that for any $d_p(w, v) \in EP(w)$ in $\gamma(\omega_1)$ iff $d_p(\vartheta(w) \cdot \vartheta(v)) \in EP(\vartheta(w))$ in $\gamma(\omega_2)$ incidence on w in ω_1 there exist the same path $d_p(w, v) \in \gamma(\omega_2)$ incidence on $\vartheta(w)$ in ω_2 and $d_p(w, v) \in OP(w)$ in $\gamma(\omega_1)$ iff $d_p(\vartheta(w) \cdot \vartheta(v)) \in OP(\vartheta(w))$ in $\gamma(\omega_2)$ then $EP(w)$ in $\gamma(\omega_1)$ equal to $EP(\vartheta(w))$ in $\gamma(\omega_2)$ also

$$OP(w) \text{ in } \gamma(\omega_1) \text{ equal to } OP(\vartheta(w)) \text{ in } \gamma(\omega_2) \text{ since } H_{m_1}(w) = \{EP(w), OP(w)\}$$

And $H_{m_2}(\vartheta(w)) = \{EP(\vartheta(w)), OP(\vartheta(w))\}$ so $H_{m_1} = H_{m_2}$ then by theorem (3.6) we have $M_p(\omega_1) = M_p(\omega_2)$

Definition 2.8: Let $\omega = (\gamma(\omega), \mathcal{E}(\omega))$ be a graph the mixture path system of a vertex $\mathfrak{x} \in \gamma(\omega)$ is denoted by $\dot{M}\dot{x}PS(\mathfrak{x})$ and defined by: $\dot{M}\dot{x}PS(\mathfrak{x}) = \{EPS(\mathfrak{x}), OPS(\mathfrak{x})\}$.

Definition 2.9: Let $\omega = (\gamma(\omega), \mathcal{E}(\omega))$ be a graph the mixture path of a vertex $\mathfrak{x} \in \gamma(\omega)$ is denoted by $\dot{M}\dot{x}P(\mathfrak{x})$ such that $\dot{M}\dot{x}P(\mathfrak{x}) \in \dot{M}\dot{x}PS(\mathfrak{x})$.

Example 2.10: According to example(2.3), the mixture path systems are given by:

$$\dot{M}\dot{x}PS(\mathfrak{x}_1) = \{\{\mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6\}, \{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}\}$$

$$\dot{M}\dot{x}PS(\mathfrak{x}_2) = \{\{\mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}, \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6\}\}$$

$$\dot{M}\dot{x}PS(\mathfrak{x}_3) = \{\{\mathfrak{x}_1, \mathfrak{x}_5, \mathfrak{x}_6\}, \{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}\}$$

$$\dot{M}\dot{x}PS(\mathfrak{x}_4) = \{\{\mathfrak{x}_2, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}, \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6\}\}$$

$$\dot{M}\dot{x}PS(\mathfrak{x}_5) = \{\{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_6\}, \{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}\}$$

$$\dot{M}\dot{x}PS(\mathfrak{x}_6) = \{\{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5\}, \{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}\}$$

$$\dot{M}\dot{x}PS(\mathfrak{x}_7) = \{\{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_8, \mathfrak{x}_9\}, \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6\}\}$$

$$\dot{M}\dot{x}PS(\mathfrak{x}_8) = \{\{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_9\}, \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6\}\}$$

$$\dot{M}\dot{x}PS(\mathfrak{x}_9) = \{\{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_8\}, \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6\}\}$$

Definition 2.11: Let $\omega = (\gamma(\omega), \mathcal{E}(\omega))$ be a graph and suppose that $H_m: \gamma(\omega) \rightarrow P(P(\gamma(\omega)))$ is a mapping which assigns for each $\mathfrak{x}$ in $\gamma(\omega)$ its mixture path system in $P(P(\gamma(\omega)))$. The pair (ω, H_m) is called the $\dot{M}\dot{x}$ -space.

Example 2.12: According to example (3.3), the mapping H_m is given by:

$$H_m(\mathfrak{x}_1) = \{\{\mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6\}, \{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}\}$$

$$H_m(\mathfrak{x}_2) = \{\{\mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}, \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6\}\}$$

$$H_m(\mathfrak{x}_3) = \{\{\mathfrak{x}_1, \mathfrak{x}_5, \mathfrak{x}_6\}, \{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}\}$$

$$H_m(\mathfrak{x}_4) = \{\{\mathfrak{x}_2, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}, \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6\}\}$$

$$H_m(\mathfrak{x}_5) = \{\{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_6\}, \{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}\}$$

$$H_m(\mathfrak{x}_6) = \{\{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5\}, \{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}\}$$

$$H_m(\mathfrak{x}_7) = \{\{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_8, \mathfrak{x}_9\}, \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6\}\}$$

$$H_m(\mathfrak{x}_8) = \{\{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_9\}, \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6\}\}$$

$$H_m(\mathfrak{x}_9) = \{\{\mathfrak{x}_2, \mathfrak{x}_4, \mathfrak{x}_7, \mathfrak{x}_8\}, \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6\}\}$$

3. Limit Mixture Path System

In this section, we introduced new concept called the $\dot{m}\dot{x}$ -derived of a subgraph H of a graph ω and is denoted by $[\gamma(H)]_{\dot{M}\dot{x}}$ which depended on mixture path. Some properties and examples are investigated. ¶

Definition 3.1: Let (ω, H_m) be a $\dot{M}\dot{x}$ – space. A vertex τ in $\gamma(\omega)$ is called a limit mixture vertex path of a graph $\check{Y} \subseteq \omega$ if every mixture path of τ contains at least one vertex of H different from τ . The set of all limit vertices of a $\check{Y} \subseteq \omega$ is called the $\dot{m}\dot{x}$ -derived of $\check{Y}$ and is denoted by $[\gamma(\check{Y})]_{\dot{M}\dot{x}}$ and

$$[\gamma(\check{Y})]_{\dot{M}\dot{x}} = \{\tau \in \gamma(\omega); \forall \dot{M}\dot{x}P(\tau), \dot{M}\dot{x}P(\tau) \cap (\gamma(\check{Y}) - \{\tau\}) \neq \emptyset\}.$$

Example 3.2: in example (3.2) $\check{Y} \subseteq \omega$ where $\gamma(\check{Y}) = \{\tau_3, \tau_4, \tau_5, \tau_6\}$ and $E(H) = \{e_3, e_4, e_5\}$ then we note that $[\gamma(\check{Y})]_{\dot{M}\dot{x}} = \{\tau_1, \tau_2, \tau_3, \tau_5, \tau_6, \tau_7, \tau_8, \tau_9\}$

Proposition 3.3: Let (ω, H_m) be a $\dot{M}\dot{x}$ – space and if ω is null graph then for every a subgraph H of ω the $\dot{m}\dot{x}$ -derived of H is empty

Proof: $\omega = (\gamma(\omega), E(\omega))$ be a null graph then $H_m(\tau) = \{\emptyset, \emptyset\} \forall \tau \in \gamma(\omega)$ so $EP(\tau) = \emptyset$ and $OP(\tau) = \emptyset \forall \tau \in \gamma(\omega)$ thus every mixture path of τ it dose not contain any vertex of $\check{Y}$ where $\check{Y} \subseteq \omega$ ie. $[\gamma(\check{Y})]_{\dot{M}\dot{x}} = \emptyset$ for all $\check{Y} \subseteq \omega$

Theorem 3.4: Every singleton sub graph $\check{Y}$ of ω then the $\dot{m}\dot{x}$ -derived of $\check{Y}$ is empty set.

Proof: Let $\omega = (\gamma(\omega), E(\omega))$ be a graph and $\check{Y} \subseteq \omega$ such that $\gamma(\check{Y}) = \{\tau_i\}$. Now to prove that $[\gamma(\check{Y})]_{\dot{M}\dot{x}} = \emptyset$, we will prove that by contradiction. Suppose that $[\gamma(\check{Y})]_{\dot{M}\dot{x}} \neq \emptyset \Rightarrow \exists \tau \in \gamma(\omega)$ such that

$$\tau \in [\gamma(\check{Y})]_{\dot{M}\dot{x}} \Rightarrow \dot{M}\dot{x}P(\tau) \cap (\{\gamma(\check{Y})\} - \{\tau\}) \neq \emptyset, \forall \dot{M}\dot{x}P(\tau) \Rightarrow \tau \neq \gamma(\check{Y}) \quad \text{and} \quad \gamma(\check{Y}) \in \dot{M}\dot{x}P(\tau), \forall \dot{M}\dot{x}P(\tau) \\ \Rightarrow \gamma(\check{Y}) \in OP(\tau) \text{ and } \gamma(\check{Y}) \in EP(\tau) \text{ and this a contradiction then } \tau \notin [\gamma(\check{Y})]_{\dot{M}\dot{x}} \Rightarrow [\gamma(\check{Y})]_{\dot{M}\dot{x}} = \emptyset.$$

Theorem 3.5: If ω is complete graph. then for every sub graph H of ω the $\dot{m}\dot{x}$ -derived of H is empty set. i.e. $[\gamma(H)]_{\dot{M}\dot{x}} = \emptyset$.

Proof: Let ω be complete graph. and $\check{Y} \subseteq \omega$ To prove that $[\gamma(H)]_{\dot{M}\dot{x}} = \emptyset$, in complete graph $EP(\tau) = \emptyset \forall \tau \in \gamma(\omega)$ Hence $[\gamma(H)]_{\dot{M}\dot{x}} = \emptyset$.

Theorem 3.6: Let $\omega = K_{m,n}$ be a complete bipartite graph having bipartition (γ_1, γ_2) then every sub graph. $\check{Y}$ of ω we have If $\gamma(\check{Y}) \subseteq \gamma_1$ or $\gamma(\check{Y}) \subseteq \gamma_2$ then $[\gamma(\check{Y})]_{\dot{M}\dot{x}} = \emptyset$.

Proof: Let $\omega = K_{m,n}$ be a complete bipartite graph. and $\check{Y} \subseteq \omega$ such that $\gamma(\check{Y}) \subseteq \gamma_1$. To prove that $[\gamma(\check{Y})]_{\dot{M}\dot{x}} = \emptyset$. Assume that $[\gamma(\check{Y})]_{\dot{M}\dot{x}} \neq \emptyset$ Let $\tau \in \gamma(\omega)$ then $\tau \in [\gamma(\check{Y})]_{\dot{M}\dot{x}} \Rightarrow \dot{M}\dot{x}P(\tau) \cap (\{\gamma(\check{Y})\} - \{\tau\}) \neq \emptyset, \forall \dot{M}\dot{x}P(\tau)$ if $\tau \in \gamma_1$. $\Rightarrow EP(\tau) = \gamma_1$ and $OP(\tau) = \gamma_2$ thus contradiction if $\tau \in \gamma_2$. $\Rightarrow EP(\tau) = \gamma_2$ and $OP(\tau) = \gamma_1$ thus a contradiction $\Rightarrow [\gamma(\check{Y})]_{\dot{M}\dot{x}} = \emptyset$. And we can prove by the same way if $\gamma(\check{Y}) \subseteq \gamma_2$.

Theorem 3.7: Let (ω, H_m) be a $\dot{M}\dot{x}$ -space, and H is every subgraph of ω . If H has only two vertex then these two vertex are not limit mixture path of H .

Proof : Let $\omega = (\gamma(\omega), E(\omega))$ be a graph, and let $H \subseteq \omega$ suppose that $\gamma(H) = \{\tau_1, \tau_2\}$, to prove $\tau_1, \tau_2 \notin [\gamma(H)]_{\dot{M}\dot{x}}$ we will prove that by a contradiction. Now If $\tau_1 \in [\gamma(H)]_{\dot{M}\dot{x}}$ then, $\dot{M}\dot{x}P(\tau_1) \cap (\gamma(H) - \{\tau_1\}) \neq \emptyset, \forall \dot{M}\dot{x}P(\tau_1)$ since $\gamma(H) - \{\tau_1\} = \{\tau_2\}$ this means $\dot{M}\dot{x}P(\tau_1) \cap \{\tau_2\} \neq \emptyset$ it's means

$\tau_2 \in EP(\tau_1)$ and $\tau_2 \in OP(\tau_1)$ and this contradiction. Hence $\tau_1 \notin [\gamma(H)]_{\dot{M}\dot{x}}$. In the same way we can prove that $\tau_2 \notin [\gamma(H)]_{\dot{M}\dot{x}}$.

Corollary 3.8: Let (ω, H_m) be a $\dot{M}\dot{x}$ -space, if $|\gamma(\omega)| = 2$ then for every subgraph H of ω the $\dot{M}\dot{x}$ -derived of H is empty.

Proof: The proof it's clear from theorem(3.7)

Proposition 4.9: Let (ω, H_m) be a $\dot{M}\dot{x}$ -space, for every H subgraph of $\mathbb{G}$ if $\tau \in [\gamma(H)]_{\dot{M}\dot{x}}$ then $\tau \in [\gamma(H - \tau)]_{\dot{M}\dot{x}}$.

Proof: Let $\omega = (\gamma(\omega), E(\omega))$ be a graph and let H is a subgraph of ω , suppos that $\tau \in [\gamma(H)]_{\dot{M}\dot{x}}$ then $\dot{M}\dot{x}P(\tau) \cap (\gamma(H) - \{\tau\}) \neq \emptyset$, $\forall \dot{M}\dot{x}P(\tau)$ then $\dot{M}\dot{x}P(\tau) \cap (\gamma(H) \cap \{\tau\}^c) \neq \emptyset \Rightarrow \dot{M}\dot{x}P(\tau) \cap (\gamma(H - \tau) \cap \{\tau\}^c) \neq \emptyset \Rightarrow \dot{M}\dot{x}P(\tau) \cap (\gamma(H - \tau) - \{\tau\}) \neq \emptyset$ thus $\tau \in [\gamma(H - \tau)]_{\dot{M}\dot{x}}$.

Proposition 3.10: Let (ω, H_m) be a $\dot{M}\dot{x}$ -space for all $H, K \subseteq \omega$ then

$$(i) [\gamma(H)]_{\dot{M}\dot{x}} \cup [\gamma(K)]_{\dot{M}\dot{x}} \subseteq [\gamma(H \cup K)]_{\dot{M}\dot{x}}$$

$$(ii) [\gamma(H \cap K)]_{\dot{M}\dot{x}} \subseteq [\gamma(H)]_{\dot{M}\dot{x}} \cap [\gamma(K)]_{\dot{M}\dot{x}}$$

Proof:

(i) Let $\omega = (\gamma(\omega), E(\omega))$ be a graph and $H, K \subseteq \omega$. to prove that $[\gamma(H)]_{\dot{M}\dot{x}} \cup [\gamma(K)]_{\dot{M}\dot{x}} \subseteq [\gamma(H \cup K)]_{\dot{M}\dot{x}}$. Since $\gamma(H) \subseteq \gamma(H \cup K)$ by proposition (3.9) we get $[\gamma(H)]_{\dot{M}\dot{x}} \subseteq [\gamma(H \cup K)]_{\dot{M}\dot{x}}$ --- (1). And $\gamma(K) \subseteq \gamma(H \cup K)$ by proposition(3.9) we get $[\gamma(K)]_{\dot{M}\dot{x}} \subseteq [\gamma(H \cup K)]_{\dot{M}\dot{x}}$ --- (2). From(1) and (2) we get $[\gamma(H)]_{\dot{M}\dot{x}} \cup [\gamma(K)]_{\dot{M}\dot{x}} \subseteq [\gamma(H \cup K)]_{\dot{M}\dot{x}}$.

(ii) Let $\omega = (\gamma(\omega), E(\omega))$ be a graph and $H, K \subseteq \omega$. to prove $[\gamma(H \cap K)]_{\dot{M}\dot{x}} \subseteq [\gamma(H)]_{\dot{M}\dot{x}} \cap [\gamma(K)]_{\dot{M}\dot{x}}$.

Since $\gamma(H \cap K) \subseteq \gamma(H)$ by proposition (3.9)

we get $[\gamma(H \cap K)]_{\dot{M}\dot{x}} \subseteq [\gamma(H)]_{\dot{M}\dot{x}}$ --- (1). and $\gamma(H \cap K) \subseteq \gamma(K)$ proposition (3.8) we get $[\gamma(H \cap K)]_{\dot{M}\dot{x}} \subseteq [\gamma(K)]_{\dot{M}\dot{x}}$ --- (2). From (1) and(2) we get $[\gamma(H \cap K)]_{\dot{M}\dot{x}} \subseteq [\gamma(H)]_{\dot{M}\dot{x}} \cap [\gamma(K)]_{\dot{M}\dot{x}}$. $[\gamma(H)]_{\dot{M}\dot{x}} \cap [\gamma(K)]_{\dot{M}\dot{x}} \subseteq [\gamma(H \cap K)]_{\dot{M}\dot{x}}$.

4. $\dot{M}\dot{x}$ -closed and $\dot{M}\dot{x}$ -closure operators

In this section we intróduce the notions of $\dot{M}\dot{x}$ -closed and $\dot{M}\dot{x}$ -clósure operators and we sttudy sóme of their próperties.

Definition 4.1: Given an $\dot{M}\dot{x}$ -space (ω, H_m) , a subgraph $\check{H} \subseteq \omega$ which cóntains all its limit mixture path is called $\dot{M}\dot{x}$ -closed. The family $\mathcal{F}_{H_m}$ of all $\dot{M}\dot{x}$ -closed of an $\dot{M}\dot{x}$ -space is defined by:

$$\mathcal{F}_{H_m} = \{\gamma(\check{H}) \subseteq \gamma(\omega); [\gamma(\check{H})]_{\dot{M}\dot{x}} \subseteq \gamma(\check{H})\}.$$

Theorem 4.2: Let (ω, H_m) be an $\dot{M}\dot{x}$ -space and let $\mathcal{F}_{H_m}$ be a family of all $\dot{M}\dot{x}$ -closed subgraphs of ω , then :

1. $\emptyset$ and $\omega \in \mathcal{F}_{H_m}$.

2. For any subfamily $\{H_i\}_{i \in I} \subseteq \mathcal{F}_{H_m}$ then $\bigcap_{i \in I} H_i \in \mathcal{F}_{H_m}$

Proof: 1. Its clear since $\emptyset$ and ω are $\dot{M}\dot{x}$ -closed.

2. Let (ω, H_m) be an $\dot{M}\dot{x}$ -space such that $K \subseteq \omega$ and $\gamma(K) = \cap_i \gamma(\check{Y}_i); i \in I$, the intersection of the $\dot{M}\dot{x}$ -closed $\check{Y}_i \subseteq \omega, i \in I$. Hence $\gamma(K) \subseteq \gamma(\check{Y}_i)$ for all $i \in I$ which implies $[\gamma(K)]_{\dot{M}\dot{x}} \subseteq [\gamma(\check{Y}_i)]_{\dot{M}\dot{x}}$ for all $i \in I$. But $[\gamma(\check{Y}_i)]_{\dot{M}\dot{x}} \subseteq \gamma(\check{Y}_i)$ for all $i \in I$ since $\check{Y}_i$ is $\dot{M}\dot{x}$ -closed and so $[\gamma(K)]_{\dot{M}\dot{x}} \subseteq \gamma(\check{Y}_i)$ for all $i \in I$ thus, $[\gamma(K)]_{\dot{M}\dot{x}} \subseteq \cap_i (\gamma(\check{Y}_i)) = \gamma(K)$, hence K is $\dot{M}\dot{x}$ -closed.

If $\cap_i (\gamma(\check{Y}_i)) = \emptyset$ by (1) above we get $\cap_i (\gamma(\check{Y}_i))$ is $\dot{M}\dot{x}$ -closed.

If $\cap_i (\gamma(\check{Y}_i)) = \omega$ by (1) above we get $\cap_i (\gamma(\check{Y}_i))$ is $\dot{M}\dot{x}$ -closed.

Remark 4.3:. The union of two $\dot{M}\dot{x}$ -closed contained in an $\dot{M}\dot{x}$ -space do not need to be $\dot{M}\dot{x}$ -closed as shown in the following example.

Example 4.4: Let $\omega = (\gamma(\omega), E(\omega))$ be an graph shown in figure (2.1) such that $\gamma(\omega) = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$,

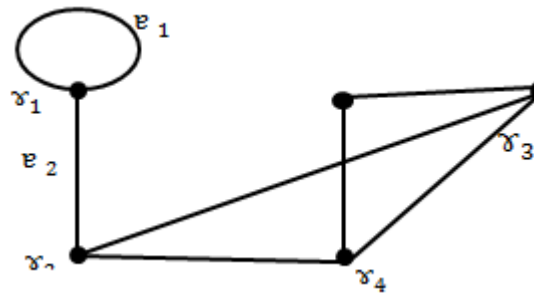


Figure 4.1: given in Example (4.4).

$$\dot{M}\dot{x}PS(\gamma_1) = \{\{\gamma_4, \gamma_5\}, \{\gamma_1, \gamma_2, \gamma_3\}\}$$

$$\dot{M}\dot{x}PS(\gamma_2) = \{\{\gamma_3\}, \{\gamma_1, \gamma_4, \gamma_5\}\}$$

$$\dot{M}\dot{x}PS(\gamma_3) = \{\{\gamma_2\}, \{\gamma_5, \gamma_1, \gamma_4\}\}$$

$$\dot{M}\dot{x}PS(\gamma_4) = \{\{\gamma_1\}, \{\gamma_3, \gamma_2, \gamma_5\}\}$$

$$\dot{M}\dot{x}PS(\gamma_5) = \{\emptyset, \{\gamma_1, \gamma_3, \gamma_2, \gamma_4\}\}$$

Accordingly, the family $\mathcal{F}_{H_m}$ of all $\dot{M}\dot{x}$ -closed graphs of this $\dot{M}\dot{x}$ -space is given by:

$$\mathcal{F}_{H_m} = \{\gamma(\omega), \emptyset, \{\gamma_1\}, \{\gamma_2\}, \{\gamma_3\}, \{\gamma_4\}, \{\gamma_5\}, \{\gamma_1, \gamma_4\}, \{\gamma_3, \gamma_2\}, \{\gamma_5, \gamma_4\}, \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}\}$$

Obviously, the graphs $H = (\gamma(\check{H}), E(H))$ such that $\gamma(H) = \{\gamma_1, \gamma_4\}$, and $K = (\gamma(K), E(K))$ such that $\gamma(K) = \{\gamma_5\}$, are $\dot{M}\dot{x}$ -closed, but their union $H \cup K = (\gamma(H \cup K), E(H \cup K))$ such that $\gamma(H \cup K) = \{\gamma_1, \gamma_4, \gamma_5\}$ is not $\dot{M}\dot{x}$ -closed, because $[\gamma(H \cup K)]_{\dot{M}\dot{x}} = \{\gamma_1, \gamma_4, \gamma_5\}$ and this leads to $[\gamma(H \cup K)]_{\dot{M}\dot{x}} \not\subseteq \gamma(H \cup K)$

Theorem 4.5: Let (ω, H_m) be an $\dot{M}\dot{x}$ -space then

- (i) Every singleton sub graph H of ω is $\dot{M}\dot{x}$ -closed.
- (ii) If ω is complete then every sub graph H of ω is $\dot{M}\dot{x}$ -closed.

Proof:

(i) Let $\check{Y} \subseteq \omega$ be singleton sub i.e. $\check{Y} = \{\tau\}, \tau \in \check{Y}(\omega)$ by (theorem(3.4), we get $[\check{Y}(\check{Y})]_{\check{M}\check{X}}^{\cdot} = \phi$. And hence $[\check{Y}(\check{Y})]_{\check{M}\check{X}}^{\cdot} \subseteq \check{Y}(\check{Y})$. Therefore $\check{Y}$ is $\check{M}\check{X}$ -closed.

(ii) Let ω be a complete and let $\check{Y} \subseteq \omega$ be any subgraph by (theorem(3.4), we get $[\check{Y}(\check{Y})]_{\check{M}\check{X}}^{\cdot} = \phi$. And hence $[\check{Y}(\check{Y})]_{\check{M}\check{X}}^{\cdot} \subseteq \check{Y}(\check{Y})$. Therefore H is $\check{M}\check{X}$ -closed.

Corollary 4.6: Let (ω, H_m) be an $\check{M}\check{X}$ -space If ω is complete graph then $\mathcal{F}_{H_m} = P(\check{Y}(\check{Y}))$.

Proof: The proof its obviously from theorem (4.5)

Proposition 4.7: Let $\omega = K_{m,n}$ be a complete bipartite graph having bipartition (Y_1, Y_2) then every subgraph $\check{Y}$ of ω if $\check{Y} \subseteq Y_1$ or $\check{Y} \subseteq Y_2$ is $\check{M}\check{X}$ -closed.

Proof : By (theorem 3.6) we get $[\check{Y}(\check{Y})]_{\check{M}\check{X}}^{\cdot} = \phi$. And hence $[\check{Y}(\check{Y})]_{\check{M}\check{X}}^{\cdot} \subseteq \check{Y}(\check{Y})$. Therefore H is $\check{M}\check{X}$ -closed.

Proposition 4.8: Let (ω, H_m) be an $\check{M}\check{X}$ -space if $H \subseteq \omega$ is $\check{M}\check{X}$ -closed then every sub graph contained in $\check{Y}$ and containing $[\check{Y}(\check{Y})]_{\check{M}\check{X}}^{\cdot}$ is $\check{M}\check{X}$ -closed.

Proof : Let (ω, H_m) be an $\check{M}\check{X}$ -space and $H, K \subseteq \omega$ such that $\check{Y}$ is $\check{M}\check{X}$ -closed and $[\check{Y}(\check{Y})]_{\check{M}\check{X}}^{\cdot} \subseteq \check{Y}(K) \subseteq \check{Y}(H)$. Since $\check{Y}(K) \subseteq \check{Y}(H)$ then $[\check{Y}(K)]_{\check{M}\check{X}}^{\cdot} \subseteq [\check{Y}(\check{Y})]_{\check{M}\check{X}}^{\cdot}$ and so $[\check{Y}(K)]_{\check{M}\check{X}}^{\cdot} \subseteq \check{Y}(K)$, thus K is $\check{M}\check{X}$ -closed.

Corollary 4.9: Let (ω, H_m) be an $\check{M}\check{X}$ -space, if $H \subseteq \omega$ is $\check{M}\check{X}$ -closed then $[\check{Y}(\check{Y})]_{\check{M}\check{X}}^{\cdot}$ it's also $\check{M}\check{X}$ -closed.

Proof : Let $H \subseteq \omega$ and $\check{M}\check{X}$ -closed $\Rightarrow [\check{Y}(\check{Y})]_{\check{M}\check{X}}^{\cdot} \subseteq \check{Y}(\check{Y})$ by proposition (3.10) we get $[[\check{Y}(\check{Y})]_{\check{M}\check{X}}^{\cdot}]_{\check{M}\check{X}}^{\cdot} \subseteq [\check{Y}(\check{Y})]_{\check{M}\check{X}}^{\cdot} \Rightarrow [\check{Y}(\check{Y})]_{\check{M}\check{X}}^{\cdot}$ is $\check{M}\check{X}$ -closed.

Remark 4.10: The converse of the corollary (3.9) above does not hold in general. This will be shown by the following example.

Example 4.11: Let $\omega = P_5$, such that $\check{Y}(\omega) = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$ let us take H a sub graph of ω such that $\check{Y}(H) = \{\tau_1, \tau_2, \tau_3, \tau_4\}$ then $[\check{Y}(H)]_{\check{M}\check{X}}^{\cdot} = \check{Y}(\omega)$, we note that $\check{Y}(\omega)$ is $\check{M}\check{X}$ -closed but H is not $\check{M}\check{X}$ -closed

Proposition 4.12: Let (ω, H_m) be an $\check{M}\check{X}$ -space, and $H \subseteq \omega$ is $\check{M}\check{X}$ -closed if $K \subseteq H$ then $[\check{Y}(K)]_{\check{M}\check{X}}^{\cdot} \subseteq \check{Y}(H)$.

Proof: Let (ω, H_m) be an $\check{M}\check{X}$ -space and $\check{Y}(H) \subseteq \check{Y}(\omega)$ is $\check{M}\check{X}$ -closed, $K \subseteq H$ by proposition (3.10) then $[\check{Y}(K)]_{\check{M}\check{X}}^{\cdot} \subseteq [\check{Y}(\check{Y})]_{\check{M}\check{X}}^{\cdot}$, since $\check{Y}$ is $\check{M}\check{X}$ -closed then $[\check{Y}(\check{Y})]_{\check{M}\check{X}}^{\cdot} \subseteq \check{Y}(H)$ and thus $[\check{Y}(K)]_{\check{M}\check{X}}^{\cdot} \subseteq \check{Y}(H)$.

Remark 4.13: The converse of proposition (4.12) above does not hold in general. This will be shown by the following example.

Example 4.14: In example(2.4), let us take H, K are sub graphs of ω such that $\check{Y}(\check{Y}) = \{\tau_1, \tau_2, \tau_3, \tau_4\}$ is $\check{M}\check{X}$ -closed, and $\check{Y}(K) = \{\tau_1, \tau_2, \tau_3, \tau_5\}$ we note that $[\check{Y}(K)]_{\check{M}\check{X}}^{\cdot} = \{\tau_1, \tau_2, \tau_3, \tau_4\} \subseteq H$, but $K \not\subseteq H$.

Definition 4.15: Let $\check{Y}$ be an subgraph of an $\check{M}\check{X}$ -space (ω, H_m) . The intersection of all $\check{M}\check{X}$ -closed containing $\check{Y}$ is called $\check{M}\check{X}$ -closure of $\check{Y}$ and is denoted by $Cl_{\check{M}\check{X}}(\check{Y}(H))$, i.e. : $Cl_{\check{M}\check{X}}(\check{Y}(H)) = \cap \{\check{Y}(K) \in \mathcal{F}_{H_m} ; \check{Y}(H) \subseteq \check{Y}(K)\}$.

Remark 4.16: From theorem(4.2 (2)), $Cl_{\check{M}\check{x}}(\gamma(H))$ is a-closed for all $\check{H} \subseteq \omega$ Moreover, it is the smallest $\check{M}\check{x}$ -closed containing $\check{H}$.

Example 4.17: let $\omega = (\gamma(\omega), E(\omega))$ be an graph shown in the following figure (4.2) and $H \subseteq \omega$, such that $\gamma(H) = \{\gamma_1, \gamma_3, \gamma_4\}$,

Of this $\check{M}\check{x}$ -space is given by:

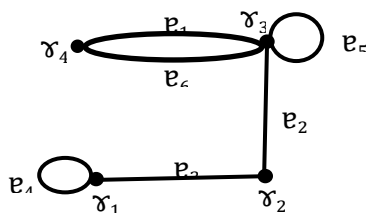


Figure 4.2: given in Example (4.17).

$$\check{M}\check{x}PS(\gamma_1) = \{\{\gamma_3\}, \{\gamma_1, \gamma_2, \gamma_4\}\}$$

$$\check{M}\check{x}PS(\gamma_2) = \{\{\gamma_4\}, \{\gamma_1, \gamma_3\}\}$$

$$\check{M}\check{x}PS(\gamma_3) = \{\{\gamma_1\}, \{\gamma_3, \gamma_2, \gamma_4\}\}$$

$$\check{M}\check{x}PS(\gamma_4) = \{\{\gamma_2\}, \{\gamma_1, \gamma_3\}\}$$

$$\mathcal{F}_{H_m} = \{\gamma(\omega), \phi, \{\gamma_1\}, \{\gamma_2\}, \{\gamma_3\}, \{\gamma_4\}, \{\gamma_1, \gamma_3\}, \{\gamma_4, \gamma_2\}\}$$

$$\text{So } Cl_{\check{M}\check{x}}(\gamma(H)) = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$$

Theorem 4.18: Let (ω, H_m) be an $\check{M}\check{x}$ -space and let $H \subseteq \omega$ then:

$$(1) \gamma(H) \subseteq Cl_{\check{M}\check{x}}(\gamma(H)).$$

$$(2) Cl_{\check{M}\check{x}}(\gamma(H)) = \gamma(H) \Leftrightarrow \gamma(H) \text{ is } \check{M}\check{x} \text{-closed.}$$

Proof:

$$(1) \text{ By Definition (4.15) we get } \gamma(H) \subseteq Cl_{\check{M}\check{x}}(\gamma(H)).$$

(2) Let (ω, H_m) be an $\check{M}\check{x}$ -space and let $H \subseteq \omega$ Suppos that $Cl_{\check{M}\check{x}}(\gamma(H)) = \gamma(H)$ to prove $\check{H}$ is $\check{M}\check{x}$ -closed from definition (4.15) such that $Cl_{\check{M}\check{x}}(\gamma(H)) = \cap \{\gamma(k) \in \mathcal{F}_{H_m} ; \gamma(H) \subseteq \gamma(k)\}$ and from theoem (4.2 , (2)) hence

$\gamma(H)$ is $\check{M}\check{x}$ - closed . Now suppose that $\gamma(H)$ is $\check{M}\check{x}$ - closed to prove that $Cl_{\check{M}\check{x}}(\gamma(H)) = \gamma(H)$ fome the remark(4.16) we can get the results.

Proposition 4.19 : Let (ω, H_m) be an $\check{M}\check{x}$ -space and $H \subseteq \omega$ then $\gamma(H) \cup [\gamma(H)]_{\check{M}\check{x}} \subseteq Cl_{\check{M}\check{x}}(\gamma(H))$.

Proof: Let (ω, H_m) be an $\check{M}\check{X}$ -space and $H \subseteq \omega$. Since $\gamma(H) \subseteq Cl_{\check{M}\check{X}}(\gamma(H))$ then $[\gamma(H)]_{\check{M}\check{X}} \subseteq [Cl_{\check{M}\check{X}}(\gamma(H))]_{\check{M}\check{X}}$ but $[Cl_{\check{M}\check{X}}(\gamma(H))]_{\check{M}\check{X}} \subseteq Cl_{\check{M}\check{X}}(\gamma(H))$ because $Cl_{\check{M}\check{X}}(\gamma(H))$ is $\check{M}\check{X}$ -closed and so $[\gamma(H)]_{\check{M}\check{X}} \subseteq Cl_{\check{M}\check{X}}(\gamma(H))$. Accordingly $\gamma(H) \cup [\gamma(H)]_{\check{M}\check{X}} \subseteq Cl_{\check{M}\check{X}}(\gamma(H))$.

Remark 4.20: If (ω, H_m) is an $\check{M}\check{X}$ -space and $H \subseteq \omega$, then the relation $\gamma(H) \cup [\gamma(H)]_{\check{M}\check{X}} = Cl_{\check{M}\check{X}}(\gamma(H))$ is not necessarily true in general. The next example is employed as a counter example to show the above remark. $[\gamma(H)]_{\check{M}\check{X}} \subseteq [Cl_{\check{M}\check{X}}(\gamma(H))]_{\check{M}\check{X}}$.

Example 4.21: In Example (4.17), Let $H \subseteq \omega$, $H = (\gamma(H), \mathcal{E}(H))$ such that $\gamma(H) = \{\tau_1, \tau_3\}$, $\mathcal{E}(H) = \{e_3, e_2\}$ then $[\gamma(H)]_{\check{M}\check{X}} = \{\tau_3\}$ and $Cl_{\check{M}\check{X}}(\gamma(H)) = \{\tau_3\}$. Obviously, $\gamma(H) \cup [\gamma(H)]_{\check{M}\check{X}} \neq Cl_{\check{M}\check{X}}(\gamma(H))$.

Proposition 4.22: Let (ω, H_m) be an $\check{M}\check{X}$ -space and $H \subseteq \omega$ if $\check{H}$ is $\check{M}\check{X}$ -closed then $\gamma(H) \cup [\gamma(H)]_{\check{M}\check{X}} = Cl_{\check{M}\check{X}}(\gamma(H))$.

Proof: Let (ω, H_m) be an $\check{M}\check{X}$ -space and $H \subseteq \omega$ since $\check{H}$ is $\check{M}\check{X}$ -closed from definition (4.1) $[\gamma(H)]_{\check{M}\check{X}} \subseteq \gamma(H)$ thus $\gamma(H) \cup [\gamma(H)]_{\check{M}\check{X}} = \gamma(H)$ and from theorem (4.15 (2)). Hence $\gamma(H) \cup [\gamma(H)]_{\check{M}\check{X}} = Cl_{\check{M}\check{X}}(\gamma(H))$.

Proposition 4.23: If (ω, H_m) be an $\check{M}\check{X}$ -space then:

$$(1) Cl_{\check{M}\check{X}}(\phi) = \phi.$$

$$(2) Cl_{\check{M}\check{X}}(\omega) = \omega.$$

Proof: Let (ω, H_m) be an $\check{M}\check{X}$ -space

(1) By Remark (4.2) we get ϕ is $\check{M}\check{X}$ -closed and by theorem (4.18 (2)) we get $Cl_{\check{M}\check{X}}(\phi) = \phi$.

(2) By Remark (4.2) we get ω is $\check{M}\check{X}$ -closed and by Theorem (4.18 (2)) we get $Cl_{\check{M}\check{X}}(\omega) = \omega$.

Proposition 4.24: If (ω, H_m) is an $\check{M}\check{X}$ -space and let $\check{H} \subseteq \omega$ then:

$$(1) Cl_{\check{M}\check{X}}(Cl_{\check{M}\check{X}}(\check{H})) = Cl_{\check{M}\check{X}}(\check{H}).$$

$$(2) \text{ If } \check{k} \subseteq \check{H} \text{ then } Cl_{\check{M}\check{X}}(\check{k}) \subseteq Cl_{\check{M}\check{X}}(\check{H}).$$

$$(3) Cl_{\check{M}\check{X}}(\check{H} \cap \check{k}) \subseteq Cl_{\check{M}\check{X}}(\check{H}) \cap Cl_{\check{M}\check{X}}(\check{k}).$$

$$(4) Cl_{\check{M}\check{X}}(\check{H}) \cup Cl_{\check{M}\check{X}}(\check{k}) \subseteq Cl_{\check{M}\check{X}}(\check{H} \cup \check{k}).$$

Proof:

Let (ω, H_m) be an $\check{M}\check{X}$ -space and let $\check{H} \subseteq \omega$.

(1) By Definition (4.15) and theorem (4.2) and (4.18 (2)) we get $Cl_{\check{M}\check{X}}(Cl_{\check{M}\check{X}}(\check{H})) = Cl_{\check{M}\check{X}}(\check{H})$.

(2) Let $k \subseteq \check{H} \subseteq \omega$ and let $\gamma \notin Cl_{M\check{x}}(\check{H}) \Rightarrow \gamma \notin \cap \{ \gamma(\mathcal{A}) \in \mathcal{F}_{H_m} ; \gamma(\check{H}) \subseteq \gamma(\mathcal{A}) \} \Rightarrow \gamma \notin \cap \{ \gamma(\mathcal{A}) \in \mathcal{F}_{H_m M\check{x}} ; \gamma(k) \subseteq \gamma(\check{H}) \subseteq \gamma(\mathcal{A}) \} \Rightarrow \gamma \notin \cap \{ \gamma(\mathcal{A}) \in \mathcal{F}_{H_m} ; \gamma(k) \subseteq \gamma(\mathcal{A}) \} \Rightarrow \gamma \notin Cl_{M\check{x}}(k)$. Thus $Cl_{M\check{x}}(k) \subseteq Cl_{M\check{x}}(H)$.

(3) Let $k, \check{H} \subseteq \omega$ since $H \cap k \subseteq \check{H}$ by (2) above we get $Cl_{M\check{x}}(\check{H} \cap k) \subseteq Cl_{M\check{x}}(\check{H})$ — — (1). Since $\check{H} \cap k \subseteq k$ by (2) above we get $Cl_{M\check{x}}(\check{H} \cap k) \subseteq Cl_{M\check{x}}(k)$ — — (2). From (1) and (2) we get $Cl_{M\check{x}}(\check{H} \cap k) \subseteq Cl_{M\check{x}}(\check{H}) \cap Cl_{M\check{x}}(k)$.

(4) Let $k, \check{H} \subseteq \omega$ since $\check{H} \subseteq \check{H} \cup k$ by (2) above we get $Cl_{M\check{x}}(\check{H}) \subseteq Cl_{M\check{x}}(\check{H} \cup k)$ — — (1). Since $k \subseteq \check{H} \cup k$ by (2) above we get $Cl_{M\check{x}}(k) \subseteq Cl_{M\check{x}}(\check{H} \cup k)$ — — (2). From (1) and (2) we get $Cl_{M\check{x}}(\check{H}) \cup Cl_{M\check{x}}(k) \subseteq Cl_{M\check{x}}(\check{H} \cup k)$

Remark 4.25: In above proposition we not that:

(1) $Cl_{M\check{x}}(\check{H}) \cap Cl_{M\check{x}}(k) \not\subseteq Cl_{M\check{x}}(\check{H} \cap k)$.

(2) $Cl_{M\check{x}}(\check{H} \cup k) \not\subseteq Cl_{M\check{x}}(\check{H}) \cup Cl_{M\check{x}}(k)$.

The following examples shows the remark above.

Example 4.26:(1)

In example (4.17) let $H = (\gamma(H), E(H))$ such that $\gamma(H) = \{\gamma_1, \gamma_2, \gamma_3\}$, and $k = (\gamma(k), E(k))$ such that $\gamma(k) = \{\gamma_1, \gamma_3, \gamma_4\}$, then we note that $Cl_{M\check{x}}(H) = \omega$, $Cl_{M\check{x}}(k) = \omega$, $Cl_{M\check{x}}(H \cap k) = \{\gamma_3\}$, $Cl_{M\check{x}}(H) \cap Cl_{M\check{x}}(k) = \omega$ and hence $Cl_{M\check{x}}(H) \cap Cl_{M\check{x}}(k) \not\subseteq Cl_{M\check{x}}(H \cap k)$

(2) In example (4.17) let $H = (\gamma(H), E(H))$ such that $\gamma(H) = \{\gamma_1\}$, and $k = (\gamma(k), E(k))$ such that $\gamma(k) = \{\gamma_3\}$, then we note that $Cl_{M\check{x}}(H) = \emptyset$

$Cl_{M\check{x}}(k) = \emptyset$, $Cl_{M\check{x}}(H \cup k) = \{\gamma_3\}$, $Cl_{M\check{x}}(H) \cup Cl_{M\check{x}}(k) = \emptyset$ and hence $Cl_{M\check{x}}(H \cup k) \not\subseteq Cl_{M\check{x}}(H) \cup Cl_{M\check{x}}(k)$.

Conclusion: In this work, we established an unusual approach to closure operator structure, known as the mixture route system, and associated it to certain graphs in order to obtain additional concepts and results. We also provide some rationales about specific connections.

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