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Structure Model of Fuzzy Soft Homotopy Theory

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ABSTRACT

The paper aims to introduce a new concept of fuzzy soft homotopy (FSH) within the framework of fuzzy soft topological space (FSTS). The research provides a definition of the fuzzy soft set (FSS). Additionally, we define continuity in FSTSs and prove the main properties of this continuity. We also present the concept of fuzzy soft path homotopy as a flexible growth of classical homotopy, and it is proven that it forms a soft fuzzy equivalence relation. This work contributes to strengthening the topological foundations of fuzzy soft spaces and provides an analytical framework applicable to modeling complex systems characterized by high degrees of vagueness and uncertainty.

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1. Introduction

Building on Zadeh's fuzzy sets [1] and Molodtsov's soft sets [2], Maji et al. [3] introduced FSSs as a unified framework, with topological structures later established by Aygunoglu et al. [4]. Based on previous studies of fuzzy path homotopy by Sugapriya and Amudhambigai [5] and the results developed by Khalaf and Al Baydli [6], recent research has expanded into various properties of FSTSs. For instance, Sabri and Ahmed [7] as well as Maibed and Majeed [8] investigated compactness and continuity properties within FSSs, while Abd and Majeed [9] established several fundamental properties in this domain.

Despite these advancements and the results established in [10-12], there remains a distinct necessity to extend these concepts into the FSH framework. Accordingly, this study establishes a rigorous theoretical framework for FSH, aiming to characterize the structural properties of paths and formulate the necessary conditions for their classification.

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2. Preliminaries

In the sequel, X will be used to represent the original universe, and E is the compilation of parameters, $P(X)$ the power compilation of X , and I^X the compilation of all fuzzy subsets of X .

Definition 2.1: [1] The membership function $\mu_A: X \rightarrow [0,1]$ describes a fuzzy set A on the universe X , where $\mu_A(x)$ indicates how much the element $x \in X$ belongs to A .

Definition 2.2: [10, 11] Let $A \subseteq E$. Then the pair (γ_A, E) is a FSS over (X, E) , where $\gamma_A: E \rightarrow I^X$ is a mapping for which $\gamma_A(e) = \mu_{\gamma_A}^e$. For each parameter $e \in A$, the image $\mu_{\gamma_A}^e$ is different from the null fuzzy set whenever $e \in A$, while $\mu_{\gamma_A}^e = \bar{0}$ whenever $e \in A^c$, where the null fuzzy set $\bar{0}$ is given by $\bar{0}(x) = 0$ for every $x \in X$. The FSS (γ_A, E) is said to be null if $\gamma_A(e) = \bar{0}, \forall e \in E$, and it is said to be absolute if $\gamma_A(e) = \bar{1}, \forall e \in E$.

Definition 2.3: [12] Suppose that (γ_A, E) is a FSSs over (X, E) , and let T_{γ_A} represent the clan of every FSS of (γ_A, E) . Then T_{γ_A} is called a FSTS on (γ_A, E) if a few things axioms are met:

- $\Phi, (\gamma_A, E) \in T_{\gamma_A}$.
- If $(\gamma_{i_A}, E) \in T_{\gamma_A}$, then $\sqcup_i (\gamma_{i_A}, E) \in T_{\gamma_A}$.
- If $(\gamma_{1_A}, E), (\gamma_{2_A}, E) \in T_{\gamma_A}$, then $(\gamma_{1_A}, E) \cap (\gamma_{2_A}, E) \in T_{\gamma_A}$. The triplet $(\gamma_A, E, T_{\gamma_A})$ is called a FSTS over (γ_A, E) .

Definition 2.4: [13] Assume that $\mu_R \in F(X)$. The relationship is considered:

- Reflexive if $\mu_R(x, x) = 1$ for all $x \in X$.
- Symmetric if $\mu_R(x, y) = \mu_R(y, x)$ for all $x, y \in X$, i.e. $\mu_R = \mu_R^{-1}$.
- Transitive if $\mu_R \circ \mu_R \subset \mu_R$.

We define a relationship as a fuzzy equivalence on $\mu(x)$ if it are met 1, 2, and 3. The clan of all fuzzy equivalence relations on $\mu(x)$ is expressed by FER (X) .

3. Fuzzy soft mappings and continuity

This section presents the concepts of fuzzy soft (FS) mappings and their inverse images and provides the definition of FS continuity along with a proof of a theorem related to it.

Definition3.1: [14] Let $f: X \rightarrow Y$ and $\phi: E \rightarrow D$, be two functions, where parameter sets for the sharp sets X and Y are marked by E and D , respectively. A FS mapping from (X, E) to (Y, D) is the pair f_ϕ .

Definition3.2: [14] Let (γ_A, E) and (γ_B, D) be two FSSs over X and Y , in turn and let f_ϕ be a FS mapping from X to Y .

- The image of γ_A under the FS mapping f_ϕ , expressed by $f_\phi(\gamma_A)$, is the FSS over Y described by:

$$f_\phi(\gamma_A)_k(y) = \begin{cases} \bigvee_{f(x)=y} \bigvee_{\phi(a)=k} (\gamma_A)_{(a)}(x), & \text{if } x \in f^{-1}(y) \\ 0, & \text{otherwise} \end{cases}$$

for each $k \in D, y \in B$

- The pre-image of (γ_B, D) in Y , expressed by $f_\phi^{-1}(\gamma_B)$, is the FSS over X described by $f_\phi^{-1}(\gamma_B)_e(x) = (\gamma_B)_{\phi(e)}(f(x))$, for each $e \in E, x \in X$. If f and ϕ are injective (surjective), then f_ϕ was claimed to be injective (surjective).
- Let f_ϕ be a FS mapping from X to Y and f_ϕ^* be a FS mapping from Y to Z . Then the composition of these mappings from X to Z is described as follows:

$$f_\phi \circ f_\phi^* = (f \circ f^*)_{\phi \circ \phi^*}, \text{ where } \phi: E \rightarrow D \text{ and } \phi^*: D \rightarrow C.$$

Definition 3.3: [14] A FS mapping $f_\phi: (\gamma_A, E, T_{\gamma_A}) \rightarrow (\gamma_B, D, T_{\gamma_B})$ is fuzzy soft continuous if $f_\phi^{-1}[(\gamma_G, D)] \in T_{\gamma_B}$, for all $(\gamma_G, D) \in T_{\gamma_A}$.

Theorem 3.4: (Fuzzy soft continuity Lemma) Let $(\gamma_{1A}, W), (\gamma_{2A}, W)$ be two FS closed subsets of a FSTS $(\gamma_A, W, T_{\gamma_A})$ for which $(\gamma_{1A}, W) \tilde{\cup} (\gamma_{2A}, W) = (\gamma_A, W)$. Then

$$f_\phi: (\gamma_A, W, T_{\gamma_A}) \rightarrow (\gamma_B, D, T_{\gamma_B})$$

is FS continuous if

$$f_\phi|_{(\gamma_{1A}, W)}: (\gamma_{1A}, W, T_{\gamma_A(\gamma_{1A}, W)}) \rightarrow (\gamma_B, D, T_{\gamma_B}),$$

and

$$f_\phi|_{(\gamma_{2A}, W)}: (\gamma_{2A}, W, T_{\gamma_A(\gamma_{2A}, W)}) \rightarrow (\gamma_B, D, T_{\gamma_B})$$

are FS continuous.

Proof: Since $(\gamma_{1A}, W), (\gamma_{2A}, W)$ be two FS closed subsets of $(\gamma_A, W, T_{\gamma_A})$ for which $(\gamma_{1A}, W) \tilde{\cup} (\gamma_{2A}, W) = (\gamma_A, W)$. It follows that $(\gamma_{1A}, W), (\gamma_{2A}, W) \in T_{\gamma_A}$.

Let (γ_G, D) be any member of T_{γ_B} . Since $(f_\phi)|_{(\gamma_{1A}, W)}: (\gamma_{1A}, W, T_{\gamma_A(\gamma_{1A}, W)}) \rightarrow (\gamma_B, D, T_{\gamma_B})$ is FS continuous, implies that $(f_\phi)^{-1}[(\gamma_G, D)] \tilde{\cap} (\gamma_{1A}, W) \in T_{\gamma_A(\gamma_{1A}, W)}$ and hence $\exists (\gamma_{1H}, W) \in T_{\gamma_A}$ for which:

$$(f_\phi)^{-1}[(\gamma_G, D)] \tilde{\cap} (\gamma_{1A}, W) = [(\gamma_{1H}, W) \tilde{\cap} (\gamma_{1A}, W)] \in T_{\gamma_A} \quad (1)$$

Similarly, $(f_\phi)|_{(\gamma_{2A}, W)}: (\gamma_{2A}, W, T_{\gamma_A(\gamma_{2A}, W)}) \rightarrow (\gamma_B, D, T_{\gamma_B})$ is FS continuous $\exists (\gamma_{2H}, W) \in T_{\gamma_A}$ for which:

$$(f_\phi)^{-1}[(\gamma_G, D)] \tilde{\cap} (\gamma_{2A}, W) = [(\gamma_{2H}, W) \tilde{\cap} (\gamma_{2A}, W)] \in T_{\gamma_A} \quad (2)$$

Thus, from (1) and (2), we get:

$$\begin{aligned} & [(f_\phi)^{-1}[(\gamma_G, D)] \tilde{\cap} (\gamma_{1A}, W)] \tilde{\cup} [(f_\phi)^{-1}[(\gamma_G, D)] \tilde{\cap} (\gamma_{2A}, W)] \in T_{\gamma_A} \\ & \Rightarrow [(f_\phi)^{-1}[(\gamma_G, D)] \tilde{\cap} (\gamma_{1A}, W) \tilde{\cup} (\gamma_{2A}, W)] = [(f_\phi)^{-1}[(\gamma_G, D)] \tilde{\cap} (\gamma_A, W)] \in T_{\gamma_A} \\ & \Rightarrow (f_\phi)^{-1}(\gamma_G, D) \in T_{\gamma_A} \end{aligned}$$

Therefore, $(f_\phi)^{-1}(\gamma_G, D) \in T_{\gamma_A}, (f_\phi)^{-1}(\gamma_G, D) \in T_{\gamma_B}$ and hence $f_\phi: (\gamma_A, W, T_{\gamma_A}) \rightarrow (\gamma_B, D, T_{\gamma_B})$ is FS continuous.

4. Fuzzy soft homotopy theory

In this section introduces the concept of FS path homotopy in FSTS, establishing it as a fuzzy soft equivalence relation.

Definition 4.1: Let $(\gamma_A, E, T_{\gamma_A})$ and $(\gamma_B, D, T_{\gamma_B})$ be FSTSs. Then two FS continuous mappings $f_\phi, g_\phi: (\gamma_A, E, T_{\gamma_A}) \rightarrow (\gamma_B, D, T_{\gamma_B})$ are said to be FSH, written as $f_\phi \sim g_\phi$, for which there exists a FS continuous map $h_\sigma: (\gamma_A \times \gamma_I, I \times I, T_{\gamma_A} \times T_{\gamma_I}) \rightarrow (\gamma_B, D, T_{\gamma_B})$ for which $h_\sigma[\mu_{\gamma_A}^e \times (\bar{0}, E)] = f_\phi(\mu_{\gamma_A}^e, E)$ and $h_\sigma[\mu_{\gamma_A}^e \times (\bar{1}, E)] = g_\phi(\mu_{\gamma_A}^e, E)$, also h_σ is called a FSH between the maps f_ϕ and g_ϕ , i.e., $h_\sigma(\tilde{s}, \bar{0}) = f_\phi(\tilde{s}), h_\sigma(\tilde{s}, \bar{1}) = g_\phi(\tilde{s})$ for all $\tilde{s} \in (\gamma_I, E)$.

$$\begin{array}{ccc}
 & & (\gamma_A, E, T_{\gamma_A}) \\
 & & \downarrow f_\phi g_\varphi \\
 (\gamma_A \times \gamma_I, I \times I, T_{\gamma_A} \times T_{\gamma_I}) & \xrightarrow{h_\sigma} & (\gamma_B, D, T_{\gamma_B})
 \end{array}$$

Fig. 1 - FSH.

Proposition 4.2: If $(\gamma_A, E, T_{\gamma_A})$ be a FSTS, let (γ_A, E) , (γ_B, D) and (γ_C, C) be a FSSs and consider $f_\phi, g_\varphi: (\gamma_A, E, T_{\gamma_A}) \rightarrow (\gamma_B, D, T_{\gamma_B})$ be any two FS continuous maps, then:

- Let $h_\psi: (\gamma_A, E, T_{\gamma_A}) \rightarrow (\gamma_B, D, T_{\gamma_B})$ be a FS continuous map. If $f_\phi \sim g_\varphi$ and $g_\varphi \sim h_\psi$, then $f_\phi \sim h_\psi$.
- If $f_\phi \sim g_\varphi$ and $h_\psi: (\gamma_B, D, T_{\gamma_B}) \rightarrow (\gamma_C, C, T_{\gamma_C})$, then $h_\psi \circ f_\phi \simeq h_\psi \circ g_\varphi$.
- Let $f_\phi, g_\varphi: (\gamma_B, D, T_{\gamma_B}) \rightarrow (\gamma_C, C, T_{\gamma_C})$ be to continuous maps for which $f_\phi \simeq g_\varphi$ and $h_\psi: (\gamma_A, E, T_{\gamma_A}) \rightarrow (\gamma_B, D, T_{\gamma_B})$ then $f_\phi \circ h_\psi \simeq g_\varphi \circ h_\psi$.

Proof:

- Since $f_\phi \sim g_\varphi$ and $f_\phi \circ h_\psi$ then there exist two FS continuous maps $h_{\sigma_1}^1, h_{\sigma_2}^2: (\gamma_A \times \gamma_I, I \times I, T_{\gamma_A} \times T_{\gamma_I}) \rightarrow (\gamma_B, D, T_{\gamma_B})$ for all $\tilde{t} \in \gamma_I$ for which $h_{\sigma_1}^1(\tilde{x}, \bar{0}) = f_\phi(\tilde{x})$, $h_{\sigma_1}^1(\tilde{x}, \bar{1}) = g_\varphi(\tilde{x})$, $h_{\sigma_2}^2(\tilde{x}, \bar{0}) = g_\varphi(\tilde{x})$ and $h_{\sigma_2}^2(\tilde{x}, \bar{1}) = h_\psi(\tilde{x})$. Define the function $h_\sigma: (\gamma_A \times \gamma_I, I \times I, T_{\gamma_A} \times T_{\gamma_I}) \rightarrow (\gamma_B, D, T_{\gamma_B})$ by:

$$h_\sigma(\tilde{x}, \tilde{t}) = \begin{cases} h_{\sigma_1}^1(\tilde{x}, \bar{2}\tilde{t}) & \text{if } 0 \leq \tilde{t} \leq \frac{1}{2} \\ h_{\sigma_1}^1(\tilde{x}, \bar{2}\tilde{t} - 1) & \text{if } \frac{1}{2} \leq \tilde{t} \leq 1 \end{cases}$$

This is a FS continuous map and has a property for which $h_\sigma(\tilde{x}, \bar{0}) = h_{\sigma_1}^1(\tilde{x}, \bar{0}) = f_\phi(\tilde{x})$ while $h_\sigma(\tilde{x}, \bar{1}) = h_{\sigma_2}^2(\tilde{x}, \bar{1}) = h_\psi(\tilde{x})$ for all $\tilde{t} \in \gamma_I$, thus $f_\phi \circ h_\psi$.

- Let (γ_A, E) , (γ_B, D) and (γ_C, C) be a FSSs and let $f_\phi, g_\varphi: (\gamma_A, E, T_{\gamma_A}) \rightarrow (\gamma_B, D, T_{\gamma_B})$ and $h_\psi: (\gamma_B, D, T_{\gamma_B}) \rightarrow (\gamma_C, C, T_{\gamma_C})$ to prove $h_\psi \circ f_\phi \sim h_\psi \circ g_\varphi$. Since $f_\phi \sim g_\varphi$ then there exists a FS continuous map $h_{\sigma_1}^1: (\gamma_A \times \gamma_I, I \times I, T_{\gamma_A} \times T_{\gamma_I}) \rightarrow (\gamma_B, D, T_{\gamma_B})$ for which $h_{\sigma_1}^1(\tilde{x}, \bar{0}) = f_\phi(\tilde{x})$ and $h_{\sigma_1}^1(\tilde{x}, \bar{1}) = g_\varphi(\tilde{x})$, $\forall \tilde{x} \in \gamma_I$. Define $h_{\sigma_1}^1: (\gamma_A \times \gamma_I, I \times I, T_{\gamma_A} \times T_{\gamma_I}) \rightarrow (\gamma_C, C, T_{\gamma_C})$ by $h_{\sigma_2}^2(\tilde{x}, \tilde{t}) = h_\psi \circ h_{\sigma_1}^1(\tilde{x}, \tilde{t})$ be a FS continuous mapping for which $h_{\sigma_2}^2(\tilde{x}, \bar{0}) = h_\psi \circ f_\phi$ and $h_{\sigma_2}^2(\tilde{x}, \bar{1}) = h_\psi \circ g_\varphi$ for all $\tilde{t} \in \gamma_I$, then $h_\psi \circ f_\phi \sim h_\psi \circ g_\varphi$.
- Since $f_\phi \simeq g_\varphi$, then there is a FSH continuous map $h_{\sigma_1}^1: (\gamma_B \times \gamma_I, I \times I, T_{\gamma_B} \times T_{\gamma_I}) \rightarrow (\gamma_C, C, T_{\gamma_C})$ for which $h_{\sigma_1}^1(\tilde{x}, \bar{0}) = f_\phi(\tilde{x})$ and $h_{\sigma_1}^1(\tilde{x}, \bar{1}) = g_\varphi(\tilde{x})$ to prove $f_\phi \circ h_\psi \simeq g_\varphi \circ h_\psi$.

Define $h_{\sigma_2}^2: (\gamma_A \times \gamma_I, I \times I, T_{\gamma_A} \times T_{\gamma_I}) \rightarrow (\gamma_C, C, T_{\gamma_C})$ by $h_{\sigma_2}^2(\tilde{x}, \tilde{t}) = h_{\sigma_1}^1(h_\psi(x), \tilde{t})$ this is satisfy:

$$h_{\sigma_2}^2(\tilde{x}, \bar{0}) = h_{\sigma_1}^1(h_\psi(\tilde{x}), \bar{0}) = f_\phi(h_\psi(\tilde{x}))$$

and

$$h_{\sigma_2}^2(\tilde{x}, \bar{1}) = h_{\sigma_1}^1(h_\psi(x), \bar{1}) = g_\varphi(h_\psi(\tilde{x}))$$

so,

$$f_\phi \circ h_\psi \simeq g_\varphi \circ h_\psi.$$

Definition 4.3: Let $(\gamma_A, E, T_{\gamma_A})$ a FSTS and $\tilde{x}_0, \tilde{x}_1$ be the initial and end points with respect on (γ_A, E) , then:

- A FS path from $\tilde{x}_0$ to $\tilde{x}_1$ is a FS continuous mapping $f_\phi: (\gamma_I, I, T_{\gamma_I}) \rightarrow (\gamma_A, E, T_{\gamma_A})$ for which $f_\phi(\bar{0}) = \tilde{x}_0$ and $f_\phi(\bar{1}) = \tilde{x}_1$.

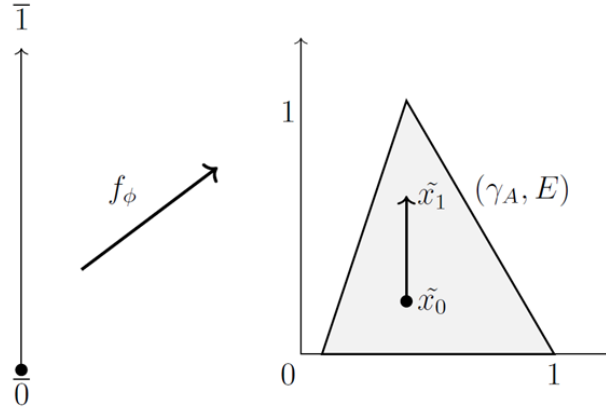


Fig. 2 – FS path.

- A FS loop at $\tilde{x}_0$ to $\tilde{x}_1$ is a FS continuous mapping $f_\phi: (\gamma_I, I, T_{\gamma_I}) \rightarrow (\gamma_A, E, T_{\gamma_A})$ for which $f_\phi(\bar{0}) = \tilde{x}_0 = f_\phi(\bar{1})$.

Definition 4.4: Let the FSTS $(\gamma_A, E, T_{\gamma_A})$ and let $f_\phi, g_\varphi: (\gamma_I, I, T_{\gamma_I}) \rightarrow (\gamma_A, E, T_{\gamma_A})$ be two FS paths from $\tilde{x}_0$ to $\tilde{x}_1$. Then f_ϕ, g_φ is referred to as a FSH path, written as $f_\phi \sim_p g_\varphi$ if there is a FS continuous mapping $h_\sigma: (\gamma_I \times \gamma_I, I \times I, T_{\gamma_I} \times T_{\gamma_I}) \rightarrow (\gamma_A, E, T_{\gamma_A})$ for which:

- $h_\sigma(\tilde{s}, \bar{0}) = f_\phi(\tilde{s}), h_\sigma(\tilde{s}, \bar{1}) = g_\varphi(\tilde{s})$, for all $\tilde{s} \in (\gamma_I, E)$.
- $h_\sigma(\bar{0}, \tilde{t}) = f_\phi(\bar{0}) = \tilde{x}_0, h_\sigma(\bar{1}, \tilde{t}) = g_\varphi(\bar{1}) = \tilde{x}_1$, for all $\tilde{t} \in (\gamma_I, E)$.

Also, h_σ is called a FS path homotopy between the FS paths f_ϕ and g_φ .

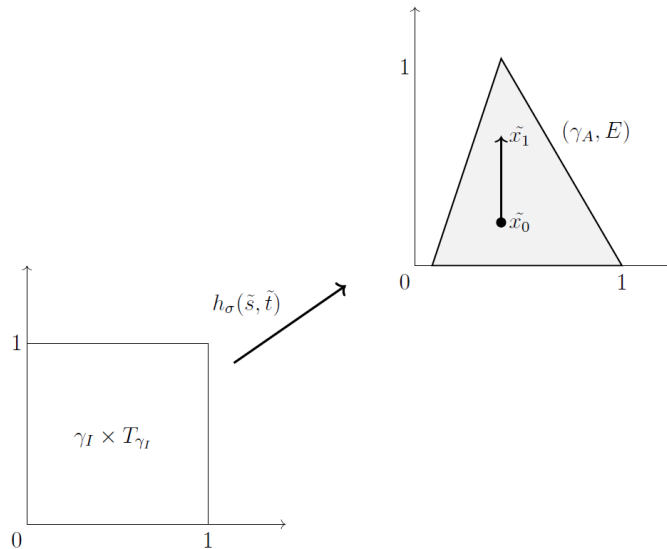


Fig. 3 – FSH path.

Theorem 4.5: The relation FS path homotopy is a FS equivalence relation.

Proof: In this theory, we have to prove that the relation is reflexive, symmetric, and transitive, so that the FS symmetry path is a FS equivalence relation.

i. Relative

Let $f_\phi: (\gamma_I, I, T_{\gamma_I}) \rightarrow (\gamma_A, E, T_{\gamma_A})$ be any FS path from $\tilde{x}_0$ to $\tilde{x}_1$ then, by the definition of the FS path, we have:

$$f_\phi(\bar{0}) = \tilde{x}_0, f_\phi(\bar{1}) = \tilde{x}_1$$

We now define a FS mapping $h_\sigma: (\gamma_I \times \gamma_I, I \times I, T_{\gamma_I} \times T_{\gamma_I}) \rightarrow (\gamma_A, E, T_{\gamma_A})$ by:

$$h_\sigma[(\tilde{s}, \tilde{t})] = f_\phi(\tilde{s})$$

Then, h_σ is a FS continuous mapping. Also $h_\sigma[(\tilde{s}, \bar{0})] = f_\phi(\tilde{s})$ and $h_\sigma[(\tilde{s}, \bar{1})] = f_\phi(\tilde{s}), \forall \tilde{s} \in (\gamma_I, E)$ and $h_\sigma[(\bar{0}, \tilde{t})] = f_\phi(\bar{0}) = \tilde{x}_0$ and $h_\sigma[(\bar{1}, \tilde{t})] = f_\phi(\bar{1}) = \tilde{x}_1, \forall \tilde{t} \in (\gamma_I, E)$.

Thus, h_σ is a FS continuous and it meets all of the conditions of Definition 4.4, so h_σ is a FS path homotopic to itself.

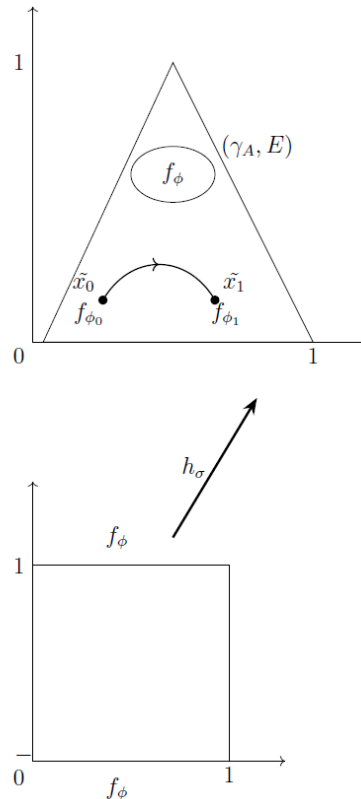


Fig. 4 – FS equivalence relation– reflexive.

ii. Symmetric

Suppose that f_ϕ, g_ϕ are two FS paths from $\tilde{x}_0$ to $\tilde{x}_1$ for which $f_\phi \sim_p g_\phi$. Let $h_\sigma: (\gamma_I \times \gamma_I, I \times I, T_{\gamma_I} \times T_{\gamma_I}) \rightarrow (\gamma_A, E, T_{\gamma_A})$ be the corresponding FS path homotopy. Then h_σ satisfies:

$$(a) \quad h_\sigma[(\tilde{s}, \bar{0})] = f_\phi(\tilde{s}), h_\sigma[(\tilde{s}, \bar{1})] = g_\phi(\tilde{s}), \forall \tilde{s} \in (\gamma_I, E), \forall \tilde{s} \in (\gamma_I, E).$$

$$(b) \quad h_\sigma[(\bar{0}, \tilde{t})] = \tilde{x}_0, h_\sigma[(\bar{1}, \tilde{t})] = \tilde{x}_1, \forall \tilde{t} \in (\gamma_I, E).$$

Now define

$$h'_{\sigma'}: (\gamma_I \times \gamma_I, I \times I, T_{\gamma_I} \times T_{\gamma_I}) \rightarrow (\gamma_A, E, T_{\gamma_A})$$

By

$$h'_{\sigma'}[(\tilde{s}, \tilde{t})] = h_{\sigma}[(\tilde{s}, \bar{1} - \tilde{t})].$$

Then $h'_{\sigma'}$ is FS continuous and:

$$h'_{\sigma'}[(\tilde{s}, \bar{0})] = h_{\sigma}[(\tilde{s}, \bar{1})] = g_{\varphi}(\tilde{s});$$

$$h'_{\sigma'}[(\tilde{s}, \bar{1})] = h_{\sigma}[(\tilde{s}, \bar{0})] = f_{\phi}(\tilde{s}) \quad \forall \tilde{s} \in (\gamma_I, E).$$

Also $h'_{\sigma'}[(\bar{0}, \tilde{t})] = h_{\sigma}[(\bar{0}, \bar{1} - \tilde{t})] = f_{\phi}(\bar{0}) = g_{\varphi}(\bar{0}) = \tilde{x}_0$ and $h'_{\sigma'}[(\bar{1}, \tilde{t})] = h_{\sigma}[(\bar{1}, \bar{1} - \tilde{t})] = f_{\phi}(\bar{1}) = g_{\varphi}(\bar{1}) = \tilde{x}_1$, $\forall \tilde{t} \in (\gamma_I, E)$. Thus, $h'_{\sigma'}$ is a fuzzy soft path homotopy between the paths g_{φ} and f_{ϕ} and hence $g_{\varphi} \sim_p f_{\phi}$.

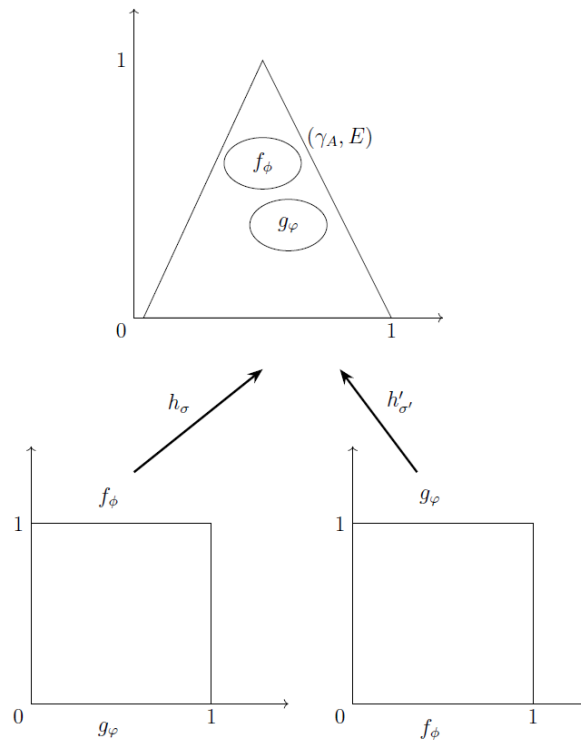


Fig. 5 – FS equivalence relation– symmetric.

ii. Transitive

Assume that $f_{\phi}, g_{\varphi}, k_{\psi}$ be three FS paths from $\tilde{x}_0$ to $\tilde{x}_1$ for which $f_{\phi} \sim_p g_{\varphi}$ and $g_{\varphi} \sim_p k_{\psi}$.

Also, let $h^1_{\sigma_1}$ and $h^2_{\sigma_2}$ be the corresponding FS path homotopies. Then $h^1_{\sigma_1}$ and $h^2_{\sigma_2}$ are FS continuous and both will satisfy the following conditions:

$$h^1_{\sigma_1}[(\tilde{s}, \bar{0})] = f_{\phi}(\tilde{s}), \quad h^1_{\sigma_1}[(\tilde{s}, \bar{1})] = g_{\varphi}(\tilde{s}), \quad \forall \tilde{s} \in (\gamma_I, E),$$

and

$$h^1_{\sigma_1}[(\bar{0}, \tilde{t})] = \tilde{x}_0, \quad h^1_{\sigma_1}[(\bar{1}, \tilde{t})] = \tilde{x}_1, \quad \forall \tilde{t} \in (\gamma_I, E),$$

also,

$$h^2_{\sigma_2}[(\tilde{s}, \bar{0})] = g_{\varphi}(\tilde{s}), \quad h^2_{\sigma_2}[(\tilde{s}, \bar{1})] = k_{\psi}(\tilde{s}), \quad \forall \tilde{s} \in (\gamma_I, E),$$

and

$$h_{\sigma_2}^2[(\bar{0}, \bar{t})] = \tilde{x}_0, \quad h_{\sigma_2}^2[(\bar{1}, \bar{t})] = \tilde{x}_1, \quad \forall \bar{t} \in (\gamma_I, E)$$

Now, define a FS mapping $h_\sigma: (\gamma_I \times \gamma_I, I \times I, T_{\gamma_I} \times T_{\gamma_I}) \rightarrow (\gamma_A, E, T_{\gamma_A})$ by:

$$h_\sigma[(\bar{s}, \bar{t})] = \begin{cases} h_{\sigma_1}^1[(\bar{s}, 2\bar{t})], & \bar{t} \in \left([0, \frac{1}{2}], E\right) \\ h_{\sigma_2}^2[(\bar{s}, 2\bar{t} - \bar{1})], & \bar{t} \in \left([\frac{1}{2}, 1], E\right) \\ g_\varphi(\bar{s}) & \text{otherwise} \end{cases}$$

Since $h_{\sigma_1}^1$ and $h_{\sigma_2}^2$ are FS continuous and agree on their intersection, then h_σ is FS continuous.

$$h_\sigma[(\bar{s}, \bar{0})] = h_{\sigma_1}^1[(\bar{s}, \bar{0})] = f_\phi(\bar{s}),$$

and

$$h_\sigma[(\bar{s}, \bar{1})] = h_{\sigma_2}^2[(\bar{s}, \bar{1})] = k_\psi(\bar{s}), \quad \forall \bar{s} \in (\gamma_I, E) \quad (3)$$

also,

$$h_\sigma[(\bar{0}, \bar{t})] = \begin{cases} h_{\sigma_1}^1[(\bar{0}, 2\bar{t})], & \bar{t} \in \left([0, \frac{1}{2}], E\right) \\ h_{\sigma_2}^2[(\bar{0}, 2\bar{t} - \bar{1})], & \bar{t} \in \left([\frac{1}{2}, 1], E\right) \\ g_\varphi(\bar{0}) & \text{otherwise} \end{cases} \quad (4)$$

$$= \begin{cases} \tilde{x}_0, & \bar{t} \in \left([0, \frac{1}{2}], E\right) \\ \tilde{x}_0, & \bar{t} \in \left([\frac{1}{2}, 1], E\right) \\ \tilde{x}_0, & \text{otherwise} \end{cases} = \tilde{x}_0, \forall \bar{t} \in (\gamma_I, E)$$

$$h_\sigma[(\bar{1}, \bar{t})] = \begin{cases} h_{\sigma_1}^1[(\bar{1}, 2\bar{t})], & \bar{t} \in \left([0, \frac{1}{2}], E\right) \\ h_{\sigma_2}^2[(\bar{1}, 2\bar{t} - \bar{1})], & \bar{t} \in \left([\frac{1}{2}, 1], E\right) \\ g_\varphi(\bar{1}) & \text{otherwise} \end{cases} \quad (5)$$

$$= \begin{cases} \tilde{x}_1, & \bar{t} \in \left([0, \frac{1}{2}], E\right) \\ \tilde{x}_1, & \bar{t} \in \left([\frac{1}{2}, 1], E\right) \\ \tilde{x}_1, & \text{otherwise} \end{cases} = \tilde{x}_1, \forall \bar{t} \in (\gamma_I, E)$$

Thus, from Equations (3-5), we have h_σ which is a FS path homotopy between the FS paths f_ϕ and k_ψ , so, $f_\phi \sim_p k_\psi$. Therefore, the relation of FSH path is a FS equivalence relation.

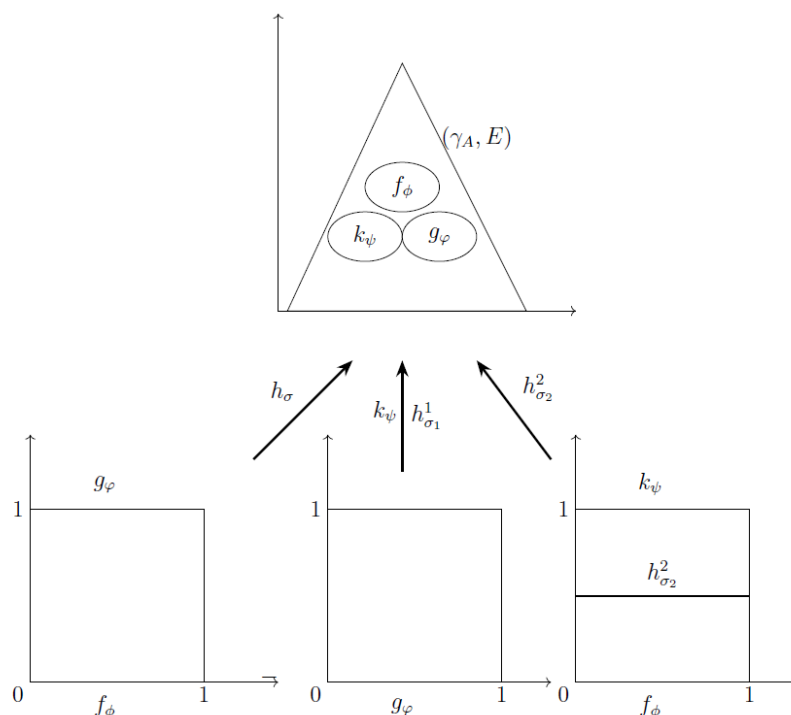


Fig. 6 – Fuzzy soft equivalence relation – transitive.

5. Conclusion

We present a novel idea of FSH in this study, focusing on constructing a theoretical framework for FS homotopical spaces. By exploring the relationship between FS path homotopy and demonstrating that it is a FS equivalence relationship, we are able to classify paths into homotopical classes. These results make a significant contribution to the development of a topological understanding of this type of space.

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