

**Page 123-137 A CLASS OF UNIVALENT FUNCTIONS WITH NEGATIVE
 COEFFICIENTS DEFINED BY HADAMARD PRODUCT
 WITH RAFID - OPERATOR**

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Abstract:

In this paper, we study a class $WR(\lambda, \beta, \alpha, \mu, \theta)$, which consists of analytic and univalent functions with negative coefficients in the open unit disk $U = \{z \in C : |z| < 1\}$ defined by Hadamard product (or convolution) with Rafid – Operator, we obtain coefficient bounds for this class. Also and some results for this class are obtained.

Key words:

Univalent Function, Hadamard Product, Rafid Operator, Integral Representation, Close – to – Convex, (n, τ) – Neighborhoods .

Mathematics Subject Classification : 30C45.

1- Introduction :

Let R denote the class of functions of the form :

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, (a_n \geq 0, n \in IN = \{1,2,3,\dots\}) \tag{1}$$

which are analytic and univalent in the unit disk $U = \{z \in C : |z| < 1\}$.

If $f \in R$ is given by (1) and $g \in R$ given by

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n, b_n \geq 0$$

then the Hadamard product (or convolution) (f_*g) of f and g is defined by

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) \quad (2)$$

Lemma 1: The *Rafid* –Operator of $f \in R$ for $0 \leq \mu < 1, 0 \leq \theta \leq 1$ is denoted by R_{μ}^{θ} and defined as following :

$$\begin{aligned} R_{\mu}^{\theta}(f(z)) &= \frac{1}{(1-\mu)^{1+\theta} \Gamma(\theta+1)} \int_0^{\infty} t^{\theta-1} e^{-\left(\frac{t}{1-\mu}\right)} f(zt) dt \\ &= z - \sum_{n=2}^{\infty} K(n, \mu, \theta) a_n z^n, \end{aligned} \quad (3)$$

where $K(n, \mu, \theta) = \frac{(1-\mu)^{n-1} \Gamma(\theta+n)}{\Gamma(\theta+1)}$.

Proof :
$$\begin{aligned} R_{\mu}^{\theta}(f(z)) &= \frac{1}{(1-\mu)^{1+\theta} \Gamma(\theta+1)} \int_0^{\infty} t^{\theta-1} e^{-\left(\frac{t}{1-\mu}\right)} f(zt) dt \\ &= \frac{1}{(1-\mu)^{1+\theta} \Gamma(\theta+1)} \int_0^{\infty} t^{\theta-1} e^{-\left(\frac{t}{1-\mu}\right)} \left[zt - \sum_{n=2}^{\infty} a_n (zt)^n \right] dt \\ &= \frac{1}{(1-\mu)^{1+\theta} \Gamma(\theta+1)} \left[z \int_0^{\infty} t^{\theta} e^{-\left(\frac{t}{1-\mu}\right)} dt - \sum_{n=2}^{\infty} a_n z^n \int_0^{\infty} t^{\theta-1+n} e^{-\left(\frac{t}{1-\mu}\right)} dt \right] \end{aligned}$$

Let $x = \frac{t}{1-\mu}$, then if $t=0$, we get $x=0$

$t = \infty$, we get $x = \infty$

and $t = (1-\mu)x$, then $dt = (1-\mu)dx$.

Thus

$$\begin{aligned} R_{\mu}^{\theta}(f(z)) &= \frac{1}{(1-\mu)^{1+\theta} \Gamma(\theta+1)} \left[z \int_0^{\infty} (1-\mu)^{1+\theta} e^{-x} x^{\theta} dx - \sum_{n=2}^{\infty} a_n z^n \int_0^{\infty} (1-\mu)^{\theta+n} e^{-x} x^{\theta-1+n} dx \right] \\ &= \frac{1}{(1-\mu)^{1+\theta} \Gamma(\theta+1)} \left[z(1-\mu)^{1+\theta} \Gamma(\theta+1) - \sum_{n=2}^{\infty} a_n z^n (1-\mu)^{\theta+n} \Gamma(\theta+n) \right] \\ &= z - \sum_{n=2}^{\infty} \frac{(1-\mu)^{n-1} \Gamma(\theta+n)}{\Gamma(\theta+1)} a_n z^n \\ &= z - \sum_{n=2}^{\infty} K(n, \mu, \theta) a_n z^n. \end{aligned}$$

Definition 1:

A function $f \in R, z \in U$ is said to be in the class $WR(\lambda, \beta, \alpha, \mu, \theta)$ if and only if satisfies the inequality :

$$\operatorname{Re} \left\{ \frac{z(R_{\mu}^{\theta}((f * g)(z)))' + \lambda z^2 (R_{\mu}^{\theta}((f * g)(z)))''}{(1 - \lambda)R_{\mu}^{\theta}((f * g)(z)) + \lambda z (R_{\mu}^{\theta}((f * g)(z)))'} \right\} \geq \beta \left\{ \frac{z(R_{\mu}^{\theta}((f * g)(z)))' + \lambda z^2 (R_{\mu}^{\theta}((f * g)(z)))''}{(1 - \lambda)R_{\mu}^{\theta}((f * g)(z)) + \lambda z (R_{\mu}^{\theta}((f * g)(z)))'} - 1 \right\} + \alpha, \quad (4)$$

where $0 \leq \alpha < 1, 0 \leq \lambda \leq 1, \beta \geq 0, z \in U, 0 \leq \mu < 1, 0 \leq \theta \leq 1$ and

$$g(z) \in R \text{ given by } g(z) = z - \sum_{n=2}^{\infty} b_n z^n, b_n \geq 0.$$

Lemma 2[1] : Let $w = u + iv$. Then $\operatorname{Re} w \geq \sigma$ if and only if $|w - (1 + \sigma)| \leq |w + (1 - \sigma)|$

Lemma 3[1] : Let $w = u + iv$ and σ, γ are real numbers. Then $\operatorname{Re} w > \sigma|w - 1| + \gamma$ if and only if $\operatorname{Re} \{w(1 + \sigma e^{i\phi}) - \sigma e^{i\phi}\} > \gamma$.

Here ,we aim to study the coefficient bounds, Hadamard product of the class $WR(\lambda, \beta, \alpha, \mu, \theta)$, radius of close-to-convexity, integral representation for $R_{\mu}^{\theta}((f * g)(z))$, inclusive properties of the class $WR(\lambda, \beta, \alpha, \mu, \theta)$, (n, τ) - neighborhoods on $WR^{\ell}(\lambda, \beta, \alpha, \mu, \theta)$.

2. Coefficient Bounds:

Here we obtain a necessary and sufficient condition and extreme points for the functions f in the class $WR(\lambda, \beta, \alpha, \mu, \theta)$.

Theorem 1:

The function f defined by (1) is in the class $WR(\lambda, \beta, \alpha, \mu, \theta)$ if and only if

$$\sum_{n=2}^{\infty} (1 - \lambda + n\lambda)[n(1 + \beta) - (\beta + \alpha)]K(n, \mu, \theta)a_n b_n \leq 1 - \alpha, \quad (5)$$

where $0 \leq \alpha < 1, \beta \geq 0, 0 \leq \lambda \leq 1, 0 \leq \mu < 1$ and $0 \leq \theta \leq 1$.

Proof: Let (5) holds true . Then we must to show that $f \in WR(\lambda, \beta, \alpha, \mu, \theta)$.By Definition 1 , we have

$$\operatorname{Re} \left\{ \frac{z(R_{\mu}^{\theta}((f_*g)(z)))' + \lambda z^2 (R_{\mu}^{\theta}((f_*g)(z)))''}{(1-\lambda)R_{\mu}^{\theta}((f_*g)(z)) + \lambda z(R_{\mu}^{\theta}((f_*g)(z)))'} \right\} \geq \beta \left| \frac{z(R_{\mu}^{\theta}((f_*g)(z)))' + \lambda z^2 (R_{\mu}^{\theta}((f_*g)(z)))''}{(1-\lambda)R_{\mu}^{\theta}((f_*g)(z)) + \lambda z(R_{\mu}^{\theta}((f_*g)(z)))'} - 1 \right| + \alpha.$$

Then by Lemma 3 , we have

$$\operatorname{Re} \left\{ \frac{z(R_{\mu}^{\theta}((f_*g)(z)))' + \lambda z^2 (R_{\mu}^{\theta}((f_*g)(z)))''}{(1-\lambda)R_{\mu}^{\theta}((f_*g)(z)) + \lambda z(R_{\mu}^{\theta}((f_*g)(z)))'} (1 + \beta e^{i\phi}) - \beta e^{i\phi} \right\} \geq \alpha, \\ -\pi < \phi \leq \pi$$

or equivalently

$$\operatorname{Re} \left\{ \frac{(z(R_{\mu}^{\theta}((f_*g)(z)))' + \lambda z^2 (R_{\mu}^{\theta}((f_*g)(z)))'' (1 + \beta e^{i\phi}))}{(1-\lambda)(R_{\mu}^{\theta}((f_*g)(z))) + \lambda z(R_{\mu}^{\theta}((f_*g)(z)))'} \right. \\ \left. - \frac{\beta e^{i\phi} ((1-\lambda)R_{\mu}^{\theta}((f_*g)(z)) + \lambda z^2 (R_{\mu}^{\theta}((f_*g)(z)))')}{(1-\lambda)R_{\mu}^{\theta}((f_*g)(z)) + \lambda z(R_{\mu}^{\theta}((f_*g)(z)))'} \right\} \geq \alpha. \quad (6)$$

$$\text{Let } F(z) = [z(R_{\mu}^{\theta}((f_*g)(z)))' + \lambda z^2 (R_{\mu}^{\theta}((f_*g)(z)))''] (1 + \beta e^{i\phi}) \\ - \beta e^{i\phi} [(1-\lambda)(R_{\mu}^{\theta}((f_*g)(z))) + \lambda z(R_{\mu}^{\theta}((f_*g)(z)))'],$$

$$\text{and } E(z) = (1-\lambda)R_{\mu}^{\theta}((f_*g)(z)) + \lambda z(R_{\mu}^{\theta}((f_*g)(z)))'.$$

By Lemma 2 , (6) is equivalent to

$$|F(z) + (1-\alpha)E(z)| \geq |F(z) - (1+\alpha)E(z)| \quad \text{for } 0 \leq \alpha < 1.$$

$$\text{But } |F(z) + (1-\alpha)E(z)|$$

$$= \left| \left[z - \sum_{n=2}^{\infty} K(n, \mu, \theta) a_n b_n z^n - \lambda \sum_{n=2}^{\infty} n(n-1) K(n, \mu, \theta) a_n b_n z^n \right] (1 + \beta e^{i\phi}) \right|$$

$$\begin{aligned}
 & -\beta e^{i\phi} \left[(1-\lambda) \left(z - \sum_{n=2}^{\infty} K(n, \mu, \theta) a_n b_n z^n \right) + \lambda z - \lambda \sum_{n=2}^{\infty} n K(n, \mu, \theta) a_n b_n z^n \right] \\
 & + (1-\alpha) \left[z - \sum_{n=2}^{\infty} (1-\lambda + n\lambda) K(n, \mu, \theta) a_n b_n z^n \right] \\
 & = \left| (2-\alpha) z - \sum_{n=2}^{\infty} [(n + \lambda n(n-1)) + (1-\alpha)(1-\lambda + n\lambda)] K(n, \mu, \theta) a_n b_n z^n \right. \\
 & \left. - \beta e^{i\phi} \sum_{n=2}^{\infty} [n + \lambda n(n-1) - (1-\lambda + n\lambda)] K(n, \mu, \theta) a_n b_n z^n \right| \\
 & \geq (2-\alpha) |z| - \sum_{n=2}^{\infty} [(n + \lambda n(n-1)) + (1-\alpha)(1-\lambda + n\lambda)] K(n, \mu, \theta) a_n b_n |z|^n \\
 & \quad - \beta \sum_{n=2}^{\infty} [n + \lambda n(n-2) - 1 + \lambda] K(n, \mu, \theta) a_n b_n |z|^n.
 \end{aligned}$$

$$\begin{aligned}
 \text{Also } |F(z) - (1+\alpha)E(z)| & = \left| \left[z - \sum_{n=2}^{\infty} n K(n, \mu, \theta) a_n b_n z^n \right. \right. \\
 & \left. \left. - \lambda \sum_{n=2}^{\infty} n(n-1) K(n, \mu, \theta) a_n b_n z^n \right] (1 + \beta e^{i\phi}) \right. \\
 & \left. - \beta e^{i\phi} \left[z - (1-\lambda) \sum_{n=2}^{\infty} K(n, \mu, \theta) a_n b_n z^n - \lambda \sum_{n=2}^{\infty} n K(n, \mu, \theta) a_n b_n z^n \right] \right. \\
 & \left. - (1+\alpha) \left[z - \sum_{n=2}^{\infty} (1-\lambda + n\lambda) K(n, \mu, \theta) a_n b_n z^n \right] \right| \\
 & = \left| -\alpha z - \sum_{n=2}^{\infty} [(n + \lambda n(n-1)) - (1+\alpha)(1-\lambda + n\lambda)] K(n, \mu, \theta) a_n b_n z^n \right. \\
 & \left. - \beta e^{i\phi} \sum_{n=2}^{\infty} [n + n\lambda(n-1) - (1-\lambda + n\lambda)] K(n, \mu, \theta) a_n b_n z^n \right| \\
 & \leq \alpha |z| + \sum_{n=2}^{\infty} [(n + n\lambda(n-1)) - (1+\alpha)(1-\lambda + n\lambda)] K(n, \mu, \theta) a_n b_n |z|^n \\
 & \quad + \beta \sum_{n=2}^{\infty} [n + n\lambda(n-1) - (1-\lambda + n\lambda)] K(n, \mu, \theta) a_n b_n |z|^n
 \end{aligned}$$

$$\begin{aligned}
 \text{and so } |F(z) + (1-\alpha)E(z)| - |F(z) - (1+\alpha)E(z)| & \geq 2(1-\alpha)|z| \\
 & - \sum_{n=2}^{\infty} [(2n + 2n\lambda(n-1)) - 2\alpha(1-\lambda + n\lambda) - \beta(2n + 2n\lambda(n-1) - \\
 & - 2(1-\lambda + n\lambda))] K(n, \mu, \theta) a_n b_n |z|^n \geq 0
 \end{aligned}$$

or

$$\sum_{n=2}^{\infty} [n(1+\beta) + n\lambda(n-1)(1+\beta) - (1-\lambda+n\lambda)(\alpha+\beta)] K(n, \mu, \theta) a_n b_n \leq 1-\alpha.$$

This is equivalent to

$$\sum_{n=2}^{\infty} (1-\lambda+n\lambda)[n(1+\beta) - (\beta+\alpha)] K(n, \mu, \theta) a_n b_n \leq 1-\alpha.$$

Conversely, suppose that (5) holds. Then we must show

$$\operatorname{Re} \left\{ \frac{[z(R_{\mu}^{\phi}((f * g)(z)))' + \lambda z^2 (R_{\mu}^{\theta}((f * g)(z)))''](1 + \beta e^{i\phi})}{(1-\lambda)R_{\mu}^{\theta}((f * g)(z)) + \lambda z (R_{\mu}^{\theta}((f * g)(z)))'} - \frac{\beta e^{i\phi} [(1-\lambda)R_{\mu}^{\phi}((f * g)(z)) + \lambda z (R_{\mu}^{\phi}((f * g)(z)))']}{(1-\lambda)R_{\mu}^{\theta}((f * g)(z)) + \lambda z (R_{\mu}^{\theta}((f * g)(z)))'} \right\} \geq \alpha.$$

Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, the above inequality reduces to

$$\operatorname{Re} \left\{ \frac{(1-\alpha) - \sum_{n=2}^{\infty} [n(1+\beta e^{i\phi})(1-\lambda+\lambda n) - (\alpha+\beta e^{i\phi})(1-\lambda+n\lambda)] K(n, \mu, \theta) a_n b_n r^{n-1}}{1 - \sum_{n=2}^{\infty} (1-\lambda+n\lambda) K(n, \mu, \theta) a_n b_n r^{n-1}} \right\} \geq 0.$$

Since $\operatorname{Re}(-e^{i\phi}) \geq -|e^{i\phi}| = -1$, the above inequality reduces to

$$\operatorname{Re} \left\{ \frac{(1-\alpha) - \sum_{n=2}^{\infty} [n(1+\beta)(1-\lambda+\lambda n) - (\alpha+\beta)(1-\lambda+n\lambda)] K(n, \mu, \theta) a_n b_n r^{n-1}}{1 - \sum_{n=2}^{\infty} (1-\lambda+n\lambda) K(n, \mu, \theta) a_n b_n r^{n-1}} \right\} \geq 0.$$

Letting $r \rightarrow 1^-$, we get desired conclusion.

There are many authors who have studied the various interesting properties of the classes, Y. Komatu [4], W. G. Atshan and S. R. Kulkarni [2],

S. Kanas and A. Wisniowska [3], Rosy et. al.[5].

In the next discussion, we concentrate upon getting the radius of close-to-convexity.

Theorem 2:

Let the function f defined by (1) be in the class $WR(\lambda, \beta, \alpha, \mu, \theta)$. Then f is close – to–convex of order δ ($0 \leq \delta < 1$) in $|z| < r(\lambda, \beta, \alpha, \mu, \theta, \delta)$, where

$$r(\lambda, \beta, \alpha, \mu, \theta, \delta) = \inf_n \left\{ \frac{(1-\delta)(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]k(n, \mu, \theta)b_n}{n(1-\alpha)} \right\}^{\frac{1}{n-1}}, n \geq 2. (7)$$

Proof: We must show that $|f'(z) - 1| \leq 1 - \delta$ for $|z| < r(\lambda, \beta, \alpha, \mu, \theta, \delta)$,

we have

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} na_n |z|^{n-1}.$$

Thus $|f'(z) - 1| \leq 1 - \delta$ if $\sum_{n=2}^{\infty} \left(\frac{n}{1-\delta} \right) a_n |z|^{n-1} \leq 1. (8)$

According to Theorem 1, we have

$$\sum_{n=2}^{\infty} \frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]K(n, \mu, \theta)}{(1-\alpha)} a_n b_n \leq 1. (9)$$

Hence (8) will be true if

$$\frac{n|z|^{n-1}}{1-\delta} \leq \frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]K(n, \mu, \theta)b_n}{(1-\alpha)}$$

equivalently if

$$|z| \leq \left\{ \frac{(1-\delta)(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]K(n, \mu, \theta)b_n}{n(1-\alpha)} \right\}^{\frac{1}{n-1}}, n \geq 2. (10)$$

The theorem follows from (10).

The following theorem provides integral representation for $R_{\mu}^{\theta}((f * g)(z)), z \in U$.

Theorem 3:

Let $f \in WR(0, \beta, \alpha, \mu, \theta)$. Then

$$R_{\mu}^{\theta}((f * g)(z)) = \exp\left(\int_0^z \frac{\beta - \psi(t)\alpha}{t(\beta - \psi(t))} dt\right), |\psi(t)| < 1, z \in U.$$

proof: The case $\beta = 0$ is obvious . Let $\beta \neq 0$, for $f \in WR(0, \beta, \alpha, \mu, \theta)$ and

$$w = \frac{z(R_{\mu}^{\theta}((f * g)(z)))'}{R_{\mu}^{\theta}((f * g)(z))},$$

we have $\operatorname{Re} w > \beta|w - 1| + \alpha$. Therefore

$$\left| \frac{w - 1}{w - \alpha} \right| < \frac{1}{\beta} \quad \text{or equivalently} \quad \frac{w - 1}{w - \alpha} = \frac{\psi(z)}{\beta},$$

where $|\psi(z)| < 1, z \in U$.

so
$$\frac{(R_{\mu}^{\theta}((f * g)(z)))'}{R_{\mu}^{\theta}((f * g)(z))} = \frac{\beta - \psi(z)\alpha}{z(\beta - \psi(z))}$$

after integration , we get

$$\log(R_{\mu}^{\theta}((f * g)(z))) = \int_0^z \frac{\beta - \psi(t)\alpha}{t(\beta - \psi(t))} dt .$$

Thus
$$(R_{\mu}^{\theta}((f * g)(z))) = \exp\left[\int_0^z \frac{\beta - \psi(t)\alpha}{t(\beta - \psi(t))} dt\right].$$

This completes the proof .

Theorem 4:

Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ belong to $WR(\lambda, \beta, \alpha, \mu, \theta)$.

Then the Hadamard product of f and g given by $(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$

belongs to $WR(\lambda, \beta, \alpha, \mu, \theta)$.

Proof : Since f and $g \in WR(\lambda, \beta, \alpha, \mu, \theta)$, we have

$$\sum_{n=2}^{\infty} \left[\frac{(1 - \lambda + n\lambda)[n(1 + \beta) - (\alpha + \beta)]K(n, \mu, \theta)b_n}{1 - \alpha} \right] a_n \leq 1$$

and

$$\sum_{n=2}^{\infty} \left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)a_n}{1-\alpha} \right] b_n \leq 1$$

and by applying the (Cauchy- Schwarz) inequality , we have

$$\begin{aligned} & \sum_{n=2}^{\infty} \left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)\sqrt{a_n b_n}}{1-\alpha} \right] \sqrt{a_n b_n} \\ & \leq \left(\sum_{n=2}^{\infty} \left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)b_n}{1-\alpha} \right] a_n \right)^{\frac{1}{2}} \\ & \quad \times \left(\sum_{n=2}^{\infty} \left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)a_n}{1-\alpha} \right] b_n \right)^{\frac{1}{2}}. \end{aligned}$$

However , we obtain

$$\sum_{n=2}^{\infty} \left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)\sqrt{a_n b_n}}{1-\alpha} \right] \sqrt{a_n b_n} \leq 1.$$

Now, we want to prove

$$\sum_{n=2}^{\infty} \left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)}{1-\alpha} \right] a_n b_n \leq 1$$

Since

$$\begin{aligned} & \sum_{n=2}^{\infty} \left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)}{1-\alpha} \right] a_n b_n \\ & = \sum_{n=2}^{\infty} \left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)\sqrt{a_n b_n}}{1-\alpha} \right] \sqrt{a_n b_n} . \end{aligned}$$

Hence, we get the required result.

Theorem 5:

Let the function f defined by (1) and g given by $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ be in the class $WR(\lambda, \beta, \alpha, \mu, \theta)$. Then the function h defined by $h(z) = z - \sum_{n=2}^{\infty} a_n^2 b_n^2 z^n$ is in the class $WR(\lambda, \beta, \gamma, \mu, \theta)$, where $0 \leq \lambda \leq 1, 0 \leq \alpha < 1, 0 \leq \gamma < 1, \beta \geq 0, z \in U, 0 \leq \mu < 1, 0 \leq \theta \leq 1$ and

$$\gamma \leq 1 - \frac{(1-\alpha)^2(1+\beta)}{(1+\lambda)(2+\beta-\alpha)^2(1-\mu)(1+\theta) - (1-\alpha)^2}.$$

Proof: We must find the largest γ such that

$$\sum_{n=2}^{\infty} \frac{(1-\lambda+n\lambda)[n(1+\beta) - (\beta+\gamma)]K(n, \mu, \theta)}{1-\gamma} a_n^2 b_n^2 \leq 1. \quad (11)$$

Since f and g are in the class $WR(\lambda, \beta, \alpha, \mu, \theta)$, we see that

$$\sum_{n=2}^{\infty} \left\{ \left[\frac{(1-\lambda+n\lambda)[n(1+\beta) - (\beta+\alpha)]K(n, \mu, \theta)b_n}{1-\alpha} \right] a_n \right\}^2 \leq 1 \quad (12)$$

and

$$\sum_{n=2}^{\infty} \left\{ \left[\frac{(1-\lambda+n\lambda)[n(1+\beta) - (\beta+\alpha)]K(n, \mu, \theta)a_n}{1-\alpha} \right] b_n \right\}^2 \leq 1 \quad (13)$$

Combining the inequalities (12) and (13), gives

$$\sum_{n=2}^{\infty} \left\{ \frac{(1-\lambda+n\lambda)[n(1+\beta) - (\beta+\alpha)]K(n, \mu, \theta)}{1-\alpha} \right\}^2 a_n^2 b_n^2 \leq 1,$$

but $h \in WR(\lambda, \beta, \gamma, \mu, \theta)$, if and only if

$$\sum_{n=2}^{\infty} \left\{ \frac{(1-\lambda+n\lambda)[n(1+\beta) - (\beta+\gamma)]K(n, \mu, \theta)}{1-\gamma} \right\} a_n^2 b_n^2 \leq 1. \quad (14)$$

The inequality (14) would obviously imply (11) if

$$\frac{(1-\lambda+n\lambda)[n(1+\beta) - (\beta+\gamma)]K(n, \mu, \theta)}{1-\gamma} \leq \left[\frac{(1-\lambda+n\lambda)[n(1+\beta) - (\beta+\alpha)]K(n, \mu, \theta)}{1-\alpha} \right]^2 = u^2,$$

then

$$\frac{(1-\lambda+n\lambda)[n(1+\beta) - (\beta+\gamma)]K(n, \mu, \theta)}{1-\gamma} \leq u^2$$

or

$$\frac{1-\gamma}{1+\beta} \geq \frac{(1-\lambda+n\lambda)(n-1)K(n,\mu,\theta)}{u^2 - (1-\lambda+n\lambda)K(n,\mu,\theta)}$$

The right hand is a decreasing function of n and its maximum if $n = 2$.
 Now

$$\frac{1-\gamma}{1+\beta} \geq \frac{(1-\lambda+n\lambda)(n-1)(1-\alpha)^2}{[(1-\lambda+n\lambda)[n(1+\beta) - (\beta+\gamma)]]^2 K(n,\mu,\theta) - (1-\lambda+n\lambda)(1-\alpha)^2} \quad (15)$$

Simplifying (15) we get

$$\frac{1-\gamma}{1+\beta} \geq \frac{(1-\alpha)^2}{(1+\lambda)(2+\beta-\alpha)^2(1-\mu)(\theta+1) - (1-\alpha)^2}$$

or

$$\gamma \leq 1 - \frac{(1-\alpha)^2(1+\beta)}{(1+\lambda)(2+\beta-\alpha)^2(1-\mu)(1+\theta) - (1-\alpha)^2}$$

This completes the proof of theorem.

Next, we obtain the inclusive properties of the class $WR(\lambda, \beta, \alpha, \mu, \theta)$.

Theorem 6 :

Let $\beta \geq 0, 0 \leq \alpha < 1, 0 \leq \lambda \leq 1, \gamma \geq 0, 0 \leq \mu < 1$ and $0 \leq \theta \leq 1$. Then

$WR(\lambda, \beta, \alpha, \mu, \theta) \subseteq WR(0, \beta, \gamma, \mu, \theta)$, where

$$\gamma \leq 1 - \frac{(n-1)(1-\alpha)(1+\beta)K(n,\mu,\theta)}{(1-\lambda+n\lambda)[n(1+\beta) - (\beta+\gamma)]K(n,\mu,\theta) - (1-\alpha)K(n,\mu,\theta)}$$

$n \in \mathbb{N}, n \geq 2$.

Proof: Let $f \in WR(\lambda, \beta, \alpha, \mu, \theta)$. Then in view of Theorem 1, we have

$$\sum_{n=2}^{\infty} \frac{(1-\lambda+n\lambda)[n(1+\beta) - (\alpha+\beta)]K(n,\mu,\theta)}{1-\alpha} a_n b_n \leq 1 \quad (16)$$

we wish to find the value γ such that

$$\sum_{n=2}^{\infty} \frac{[n(1+\beta) - (\beta+\gamma)]K(n,\mu,\theta)}{1-\gamma} a_n b_n \leq 1. \quad (17)$$

The inequality (16) would obviously imply (17) if

$$\frac{[n(1+\beta) - (\beta + \gamma)]K(n, \mu, \theta)}{1 - \gamma} \leq \frac{(1 - \lambda + n\lambda)[n(1+\beta) - (\alpha + \beta)]K(n, \mu, \theta)}{1 - \alpha} = u.$$

Therefore

$$\frac{[n(1+\beta) - (\beta + \gamma)]K(n, \mu, \theta)}{1 - \gamma} \leq u. \quad (18)$$

Now (18) gives on simplification

$$\frac{1 - \gamma}{1 + \beta} \geq \frac{(n-1)K(n, \mu, \theta)}{u - K(n, \mu, \theta)}, \quad (n \geq 2, n \in \mathbb{N}). \quad (19)$$

The right hand side of (19) decreases as n increases and so is maximum for $n = 2$.

So (19) is satisfied provided

$$\frac{1 - \gamma}{1 + \beta} \geq \frac{(n-1)(1 - \alpha)K(n, \mu, \theta)}{(1 - \lambda + n\lambda)[n(1+\beta) - (\beta + \alpha)]K(n, \mu, \theta) - (1 - \alpha)K(n, \mu, \theta)} = d.$$

Obviously $d < 1$ and

$$\gamma \leq 1 - \frac{(n-1)(1 - \alpha)(1 + \beta)K(n, \mu, \theta)}{(1 - \lambda + n\lambda)[n(1+\beta) - (\beta + \alpha)]K(n, \mu, \theta) - (1 - \alpha)K(n, \mu, \theta)}.$$

This completes the proof of theorem.

Theorem 7 :

Let $\beta \geq 0, 0 \leq \alpha < 1, \lambda_1 \geq \lambda_2 \geq 0, 0 \leq \theta \leq 1, 0 \leq \mu < 1$. Then

$$WR(\lambda_1, \beta, \alpha, \mu, \theta) \subseteq WR(\lambda_2, \beta, \alpha, \mu, \theta).$$

The proof of theorem follows also from Theorem 6 .

Now, we determine a set of inclusion relations involving (n, τ) -neighborhoods.

Following [6], we define the (n, τ) -neighborhoods

of a function $f \in R$ by

$$N_{n,\tau}(f) = \left\{ g \in R : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} n|a_n - b_n| \leq \tau, 0 \leq \tau < 1 \right\} \quad (20)$$

We need the following definition:

Definition 2:

The function f defined by (1) is said to be member of the class

$WR^\ell(\lambda, \beta, \alpha, \mu, \theta)$ if there exists a function $g \in WR(\lambda, \beta, \alpha, \mu, \theta)$

such that $\left| \frac{f(z)}{g(z)} - 1 \right| \leq 1 - \ell, \quad (z \in U, 0 \leq \ell < 1).$

Theorem 8:

Let $g \in WR(\lambda, \beta, \alpha, \mu, \theta)$ and

$$\ell = 1 - \frac{\tau(1+\lambda)(2+\beta-\alpha)(1-\mu)(\theta+1)a_2}{2\{(1+\lambda)(2+\beta-\alpha)(1-\mu)(\theta+1)a_2 - (1-\alpha)\}} \quad (21)$$

Then $N_{n,\tau}(g) \subset WR^\ell(\lambda, \beta, \alpha, \mu, \theta).$

Proof: Let $f \in N_{n,\tau}(g)$. Then we have from (20) that

$$\sum_{n=2}^{\infty} n|a_n - b_n| \leq \tau,$$

which readily implies the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - b_n| \leq \frac{\tau}{2}.$$

Also, since $g \in WR(\lambda, \beta, \alpha, \mu, \theta)$, we have from Theorem 1

$$\sum_{n=2}^{\infty} b_n \leq \frac{(1-\alpha)}{(1+\lambda)(2+\beta-\lambda)(1-\mu)(\theta+1)a_2},$$

so that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < \frac{\sum_{n=2}^{\infty} |a_n - b_n|}{1 - \sum_{n=2}^{\infty} b_n} \leq \frac{\tau}{2} \cdot \frac{(1+\lambda)(2+\beta-\alpha)(1-\mu)(\theta+1)a_2}{(1+\lambda)(2+\beta-\alpha)(1-\mu)(\theta+1)a_2 - (1-\alpha)}$$

$$= 1 - \ell .$$

Thus by definition , $f \in WR^\ell(\lambda, \beta, \alpha, \mu, \theta)$ for ℓ given by (21).

This completes the proof.

We further define the integral operator , in the following discussion :

Theorem 9:

Let c be a real number such that $c > -1$. If $f \in WR(\lambda, \beta, \alpha, \mu, \theta)$. Then the function F_c defined by

$$F_c(z) = \frac{c+1}{z^c} \int_0^z s^{c-1} f(s) ds \tag{22}$$

also belongs to $WR(\lambda, \beta, \alpha, \mu, \theta)$.

Proof: Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$. Then

$$\begin{aligned} F_c(z) &= \frac{c+1}{z^c} \int_0^z s^{c-1} (s - \sum_{n=2}^{\infty} a_n s^n) ds \\ &= \frac{c+1}{z^c} \int_0^z (s^c - \sum_{n=2}^{\infty} s^{c-1+n} a_n) ds \\ &= \frac{c+1}{z^c} \left[\frac{s^{c+1}}{c+1} - \sum_{n=2}^{\infty} \frac{s^{c+n}}{c+n} a_n \right]_0^z \\ &= z - \sum_{n=2}^{\infty} \frac{c+1}{c+n} a_n z^n . \end{aligned}$$

Hence $F_c(z) = z - \sum_{n=2}^{\infty} \frac{c+1}{c+n} a_n z^n$.

Therefore ,

$$\sum_{n=2}^{\infty} \frac{(c+1)(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)}{(c+n)} a_n b_n$$

$$\leq (1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta) a_n b_n \leq 1-\alpha.$$

Hence $F_c \in WR(\lambda, \beta, \alpha, \mu, \theta)$.

This completes the proof.

REFERECCES

- [1] E. S. Aqlan , *Some Problems Connected with Geometric Function Theory* , Ph. D. Thesis (2004), Pune University, Pune .
- [2] W. G. Atshan and S. R. Kulkarni ,*Some applications of generalized Ruscheweyh derivatives involving a general fractional derivative operator to a class of analytic functions with negative coefficients I*, J. Surveys in Mathematics and its Applications, 5(2010),35-47.
- [3] S. Kanas and A. Wisniowska, *Conic regions and k –uniform convexity*, J. Comput. Appl. Math. 105(1999) 327-336.
- [4] Y. Komatu, *On analytic Prolongation of a family of operators*, Mathematica (Cluj), 39 (55) (1990) , 141-145.
- [5] T. Rosy, K. G. Subramanian and G. Murugusundaramoorthy ,*Neighborhoods and partial sums of starlike functions based on Ruscheweyh derivatives* , J. Inequal. Pure and Appl. Math. ,4(4) (2003), Art. 64,1-8.
- [6] S. Ruscheweyh, *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc. ,81 (1981), 521-527.