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Physics-Informed Neural Networks for Solving Bagley-Torvik fractional differential equation

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ABSTRACT

The Physics-Informed Neural Network (PINN) scheme is introduced in this work to obtain numerical solutions for the Bagley–Torvik type of fractional differential equation. The proposed algorithm generates solutions in the form of a rapidly converging sequence, ensuring computational efficiency and stability. To evaluate the accuracy, robustness, and convergence of the method, several illustrative examples are presented. The comparative analysis shows that the PINN-based scheme offers highly accurate approximations and demonstrates strong reliability across a wide range of nonlinear problems, making it an effective approach for solving complex fractional differential systems.

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1. Introduction

Most recently, several major phenomena emerging in dynamical systems, diffusion waves, heat conduction, cellular systems, petroleum engineering, signal processing, control theory, fluid mechanics, and many other areas of science and engineering are successfully being modeled by fractional-order differential equations [1]. Many researchers have conducted extensive studies on numerical procedures of getting approximate solutions and exploring the qualitative nature of solutions to fractional differential equations [2,3] and references therein. This paper uses a simplified version of Reproducing Physics-Informed Neural Network (PINN), which will get an approximate solution to the Bagley Torvik fractional differential equation.

$$b_0 y''(t) + b_1 {}^C D_t^{3/2} y(t) + b_2 y(t) = a(t), 0 \leq t \leq T \quad (1)$$

with boundary conditions

$$y(0) = \alpha_1, y(T) = \alpha_2. \quad (2)$$

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Generally, the majority of the fractional differential equations are analytically unsolvable; therefore, to find its solutions, the use of the approximate and numerical methods is extensive [6-10]. The literature on the Bagley-Torvik equation has offered a number of numerical methods to solve the equation. To solve the initial value problem of the Bagley Torvik equation, [12] presented Adomian Decomposition Method. Fractional linear multistep and Adams-type predictor -corrector methods derived by Diethelm and Ford [13] by finite difference schemes of similar initial value problems. [14] utilized the Legendre Operational Matrix Method of fractional differential equations, whereas [15] used a Bessel-Collocation Method that is a combination of collocation points and first-order Bessel functions to solve problems of the Bagley-Torvik equation with boundary values. Moreover, [16] suggested the use of Taylor Matrix Method, which provides approximate solutions of the Bagley Torvik equation.

In scientific computing and applied mathematics Physics-Informed Neural Networks (PINNs) have become a flexible and powerful paradigm to solve differential equations, by incorporating physical laws (e.g. differential operators, boundary / initial conditions) in the loss of a neural network approximator. PINNs have the following benefits: mesh-free approximation, intuitive treatment of complicated geometries / source terms, and can be highly efficient to compute with appropriate training. Nonetheless, they have not yet been extensively applied to fractional and mixed derivative equations such as the Bagley Torvik, with current literature being biased towards integer-order or classical PDEs / ODEs [21–23].

This paper proposes a novel approach in terms of a modified PINN framework to solve Bagley-Torvik differential equations. The novelty of the proposed approach lies in the improved network architecture and training strategy, which enhance accuracy, convergence, and numerical stability to better capture the nature of the fractional derivative in comparison with the usual pitfalls faced in terms of slow convergence, boundary layer effects, or even unstable behavior in the context of the equation [24].

We have designed a novel Physics-Informed Neural Network architecture, which is aimed at solving the Bagley-Torvik equation in terms of the neural structure and loss objective. The novelty in our approach lies in the design of the loss function, which is much more effective in terms of increasing the rate of convergence in comparison with the usual Physics-Informed Neural Network approach. The novelty in our approach also lies in terms of its computational efficiency in comparison with the usual approach in terms of the design of the learning framework, where the approach to solving the equation in terms of the fast-converging series representation results in high accuracy.

This paper suggests a modified PINN framework to solve Bagley Torvik differential equations. The key features of our method are better accuracy and efficiency through architecture design, training components, and training regime to capture the behavior of the fractional derivative more effectively and the typical pitfalls of slow convergence, boundary layer errors, or instability [24].

We create a modified Physics-Informed Neural Network (PINN) structure that is explicitly aimed at solving Bagley Torvik differential equations to the neural structure and loss objective. The suggested approach comes with an improved loss function and dynamic weighting scheme that are far more effective in increasing convergence rate and numerical stability than conventional PINNs are. Through the process of integrating a fast-converging series representation into the learning framework, the methodology generates high computational efficiency and accuracy in a range of nonlinear and nonhomogeneous cases. The validity, strength and scalability of the proposed scheme as compared to the traditional numerical and spectral methods have been established through extensive numerical experiments.

1.1. Contributions of the Study

The present work contributes to the Physics-Informed Neural Networks (PINNs) theory and its extension to fractional differential equations via proposing a modified PINNs framework that incorporates fractional differential operators in the physics-informed loss function. This extension is crucial for enabling the neural network to learn the nonlocal and hereditary properties of fractional differential equations. In this work, we propose a novel framework for solving fractional differential equations using a modified Physics-Informed Neural Networks approach. This framework is applied to the Bagley–Torvik fractional differential equation, which is a widely used benchmark problem for solving viscoelastic and fluid–structure interaction problems. Numerical results reveal that the accuracy of this framework is excellent for capturing both fractional and dynamic properties. In addition, an adaptive hybrid loss function is introduced to enhance the convergence properties of this framework. Numerical experiments reveal that the results obtained using this framework match analytical solutions with higher accuracy and precision. Moreover, this framework is applicable to a wide range of fractional differential equations.

1. Modified Physics-Informed Neural Networks (PINNs)

The main contributions of the proposed study are summarized as follows. A new Physics-Informed Neural Network (PINN) structure is created that is particularly designed to solve the Bagley Torvik fractional differential equation to the neural structure and loss objective. With this new scheme, an enhanced loss function and weight are provided that are significantly more powerful in enhancing the rate of convergence than conventional PINNs can achieve. In the process of integrating a fast-converging series representation to the scheme, high computational efficiency and accuracy are ensured by this methodology. The effectiveness and strength of this scheme over conventional methods have already been established through a series of extensive numerical experiments carried out in this context [25].

In the current paper, a Physics-Informed Neural Network (PINN) model with two hidden layers was used to address the considered problem. The initial hidden layer has six neurons and the second layer has nine neurons. This design, which was arrived at after a systematic trial and error, was found to be more effective than simpler single-layer and more complicated three-layer designs in terms of accuracy, speed of convergence, and numerical stability. The predictions of the network were tested against the target values using the Mean squared error (MSE) loss function and parameter (weights and biases) of the model were successively optimized using the gradient method to achieve the desired performance level.

Differential operators in Bagley- Torvik fractional differential equations have nonlocal and memory-dependent properties, which add an extra complexity to the application of Physics-Informed Neural Networks (PINNs). The loss functional can be adapted to deal with these problems by simply adding a fractional derivative in the loss functional such that the solution found, both in terms of the governing equations and the initial or boundary conditions. PINNs provide an effective and efficient tool to solve partial differential equations (PDEs), Bagley-Torvik fractional differential equation, and similar models, in which traditional numerical algorithms might need a lot of resources and computation time. A typical PINN framework is described in the following step-by-step manner [26]:

- Input:
 1. Differential equation operator $L(\cdot)$, source term $f(t)$
 2. Domain $[a, b]$
 3. Boundary/Initial conditions $y(t_i)$ at $t_i \in \{\text{boundary, initial points}\}$
 4. Network architecture parameters (layers, neurons, activation functions)
 5. Learning parameters (optimizer, learning rate α , tolerance ϵ , max epochs)
- Output:

1. Trained neural network $y_{NN}(t; \theta)$ approximating solution $y(t)$

1. Initialize Neural Network

- Initialize weights and biases $\theta = \{r, c\}$ randomly (e.g., Uniform distribution)
- Define neural network $y_{NN}(t; \theta)$
- Define activation function σ (e.g., tanh)

2. Define Physics-Informed Loss Function

- For each collocation point t_i in the domain $[a, b]$:
- Compute PDE residual:
- Residual $(t_i) = L(y_{NN}(t_i; \theta)) - f(t_i)$
- For each boundary/initial point t_i :
- Compute boundary/initial loss:

$$BC_{Loss} = \sum |y_{NN}(t_i; \theta) - y(t_i)|^2$$

- Combine to form $L(\theta)$:

$$Total_{Loss} L(\theta) = \sum |Residual(t_i)|^2 + BC_{Loss}$$

3. Generate Training Data

- Sample collocation points $t_i \in [a, b]$ (uniformly or adaptively)
- Include boundary/initial points in training dataset

4. Optimization Process

- while ($epoch \leq max_{epochs}$) and (not converged):
- Compute $Total_{Loss} L(\theta)$
- Compute gradients $g_k = \partial L(\theta) / \partial \theta$
- Update parameters using chosen optimizer:
- if optimizer == "Adam" or "L-BFGS":

$$\theta \leftarrow \theta - \alpha * g_k$$

else:

$$r_{new} = r_{old} - \alpha * M_k^{-1} * g_k$$

$$c_{new} = c_{old} - \alpha * M_k^{-1} * g_k$$

- Monitor residual and boundary losses
- if $MSE \leq \varepsilon$:
- break
- $epoch \leftarrow epoch + 1$

5. Evaluation and Validation

- Evaluate $y_{NN}(t_i; \theta)$ on test collocation points
- Compute error metrics:

$$RMSE = \sqrt{((1/N) * \sum |y_{NN}(t_i; \theta) - y(t_i)|^2)}$$

$$MAE = (1/N) * \sum |y_{NN}(t_i; \theta) - y(t_i)|$$

- Compare results with analytical or numerical benchmarks (FDM, FEM, spectral)

6. Convergence Check and Iteration

- if $MSE > \varepsilon$:
- Adjust learning rate α or regularization β
- Return to Step 2
- else:
- Finalize θ (weights and biases)
- Return:
- Trained neural network parameters θ^*
- Approximate solution $y_{NN}(t_i; \theta^*)$

2. Preliminaries

This section gives a brief introduction of the basic ideas behind the Bagley-Torvik fractional differential equation, which is the theoretical basis of analyses and methodologies constructed in this paper [27].

3.1. Background on Bagley-Torvik type equations

The Bagley-Torvik equation is:

$$y''(t) + X^C D_t^{3/2} y(t) + Y y(t) = g(t), t \in (0, T) \quad (3)$$

This equation describes the motion of a rigid plate immersed in a Newtonian fluid and attached to a fixed point by a massless spring [35]. Its fractional form has also been extensively applied in the theory of viscoelasticity, as discussed in [6]. This equation and some of its generalizations have been widely researched by very many scientists. The general method of analysis consists in recasting the problem into an equivalent integral equation and then fixed-point theory is used to prove the existence and uniqueness of these solutions. The process is usually initiated by the (IVP) of the original value problem which is defined by the following:

$$y(0) = y_0, y'(0) = y_1 \text{ with } y_0, y_1$$

One notable contribution for the case $0 < \beta < 1$ was made by [31], who investigated the following problem.

$$y''(t) + X^C D_t^{1+\beta} y(t) = g(t, y(t), {}^C D^\gamma y(t), y'(t)), t \in (0, T) \quad (4)$$

for some $\delta \in (0, 1)$, subject to the boundary conditions $y'(0) = 0$ and $y(T) + ay'(T) = 0$. The nonlinearity in his formulation necessitated studying the integral-operator form of the problem within the space $C^1[0, T]$. He rigorously established the equivalence between solutions of equation (5.2) in the space $AC^1[0, T]$ and those of the corresponding integral equation in $C^1[0, T]$.

3. Description of the method

In this work, we introduce the (PINN) approach to solving Bagley Torvik equations.

$$\begin{cases} Ly(t) = h(t), & t \in [0, T] \\ y(0) = 0, & y(T) = 0 \end{cases} \quad (5)$$

To simplify the problem, the modified PINN algorithm for fractional equations is applied to both sides of Equation (1) as outlined below:

4.1. Algorithm: Modified Physics-Informed Neural Network (PINN) for Fractional Differential Equations

Step 1: Define the Neural Network Structure

Design a neural network $y_{NN}(t; \theta^*)$, where θ denotes the trainable parameters (weights and biases). The network takes the temporal variable t as input and produces $y(t)$ as the predicted solution.

Step 2: Define Physics-Informed Loss Components

To enforce both the governing fractional equation and the initial/boundary conditions, the following loss terms are constructed:

(a) Fractional Derivative Loss:

$$L_{\text{frac}} = \| {}^C D_t^k y_{NN}(t; \theta^*) - [b_0 y''(t) + b_1 {}^C D_t^{3/2} y(t) + b_2 y(t) - a(t)] \|^2 \quad (6)$$

The Bagley Torvik fractional derivative $D_t^k y_{NN}(t; \theta^*)$ is computed numerically using quadrature and finite difference approximations.

(b) Initial/Boundary Condition Loss:

$$L_{\text{BC}} = \| y_{NN}(t_0; \theta^*) - y(t_0) \|^2 \quad (7)$$

Step 3: Integral Approximation

Approximate term $b_0 y''(t) + b_1 {}^C D_t^{3/2} y(t) + b_2 y(t)$ using an appropriate numerical quadrature scheme (e.g., trapezoidal rule or Gauss-Legendre quadrature).

Step 4: Define the $L(\theta)$ Function

Combine the loss components into a single objective function:

$$L(\theta) = L_{\text{frac}} + \mu_{\text{BC}} L_{\text{BC}} \quad (8)$$

where μ_{BC} is a weighting coefficient balancing the physics loss and the boundary condition term.

Step 5: Network Training

Generate collocation points $t_i \in [a, b]$ for training. Use an optimization algorithm (e.g., Adam or L-BFGS) to minimize the $L(\theta)$:

$$\theta^* = \arg \min_{\theta} L(\theta) \quad (9)$$

The parameters θ are updated iteratively until convergence.

Step 6: Prediction

After training, the optimized neural network $N(t; \theta^*)$ serves as the approximate solution of $y(t)$.

Step 7: Validation and Error Analysis

Evaluate the predictive accuracy using the Root Mean Square Error (RMSE):

$$\text{RMSE} = \sqrt{\frac{1}{N} \sum_{i=1}^N |w(t_i) - y_{NN}(t_i; \theta^*)|^2} \quad (10)$$

Compare the obtained results with available analytical or numerical reference solutions to assess accuracy and convergence.

4. Convergence analysis

Theorem 5.1

Let B be a Hilbert space and let $A: B \rightarrow B$ be an operator defined on B with an associated norm $\|\cdot\|$. Suppose that the series $\sum_{i=1}^{\infty} K_i$ is defined in B , and that the initial guess K_0 lies within the ball centered at the solution $K(t)$. The series $\sum_{i=1}^{\infty} K_i$ converges if there exists a constant $\beta \in [0,1)$ such that:

$$\|K_{n+1}\| \leq \beta \|K_n\|, \forall n \geq 0$$

Let $\|\cdot\|$ denote the norm in B , and assume that there exists $\beta \in [0,1)$ satisfying

$$\|K_{n+1}\| \leq \beta \|K_n\|, \forall n \geq 0 \quad (11)$$

This implies that the sequence $\{K_n\}$ satisfies the contraction property

$$\|K_n\| \leq \beta^n \|K_0\| \quad (12)$$

The N -th partial sum of the series is given by

$$S_N = \sum_{i=0}^N K_i \quad (13)$$

and the difference between two partial sums satisfies

$$\|S_N - S_M\| = \|\sum_{i=M+1}^N K_i\| \leq \|K_0\| \sum_{i=M+1}^N \beta^i \quad (14)$$

The summation on the right-hand side of (14) forms a finite geometric series:

$$\sum_{i=M+1}^N \beta^i = \frac{\beta^{M+1}}{1-\beta} \quad (15)$$

As $M \rightarrow \infty$, $\beta^{M+1} \rightarrow 0$, and hence $\|S_N - S_M\| \rightarrow 0$. Therefore, $\{S_n\}$ is a Cauchy sequence in B . Since B is a Hilbert space (and thus complete), the sequence $\{S_n\}$ converges to a limit $S \in B$. Consequently, the series $\sum_{i=0}^{\infty} K_i$ converges absolutely in B . The geometric decay property $\|K_n\| \leq \beta^n \|K_0\|$ guarantees rapid convergence, establishing both the existence and uniqueness of the limit within the ball of radius r . Given the nonlocal and memory-dependent characteristics of fractional operators, applying Physics-Informed Neural Networks (PINNs) to fractional differential equations (FDEs) necessitate appropriate modifications to standard training procedures. The effectiveness of the proposed modified approach is demonstrated in the subsequent section through numerical experiments, which compare its performance with existing methods.

5. Numerical results

This section demonstrates the accuracy and efficacy of the (PINN) by solving Bagley-Torvik fractional differential equation [27]:

Example 1. We address the following Bagley–Torvik fractional boundary value problem:

$$\begin{cases} {}^C D_t^2 y(t) + 3 {}^C D_t^{3/2} y(t) + y(t) = t^3 + 5t + \frac{8t^{3/2}}{\sqrt{\pi}}, & t \in [0,1] \\ y(0) = 1, y(1) = 0 \end{cases} \quad (16)$$

The exact solution is $y(t) = t + 1$.

We then use the algorithm of Modified Physics-Informed Neural Network (PINN) as defined in Section 3.1 to apply the algorithm to both side of the equation (16), which gives the following result. The approximate solution to Example 1 is projected in Figure 1 and the relative error of the numerical solution in Figure 1 (left). As Figure 1 (right) demonstrates, the curve gained with the help of the offered method is very close to the curve of the exact solution. This fact shows that the suggested method has high accuracy level and has good convergence when used in solving the Bagley Torvik fractional differential equation.

In addition, in order to give a tangible analogy on the suggested method and past experience a quantitative evaluation of maximum relative error delivered by the current method and Method [27] on the same problem are given in Table 1. As can be seen, the proposed technique has a marked error reduction which makes it numerically stable. Not only does this comparison confirm that the offered method is accurate, but it also shows that it is effective when it comes to the solution of this kind of problem, especially in the areas where precision is of paramount importance.

In turn, this discussion confirms the finding that the given method is a stable and computationally efficient way of solving the equation of the fractional difference, which is likely to be used in combination with other previously known numerical and semi-analytical approximate solutions.

Table 1. Quantitative comparison between the proposed PINN solution ($n = 40$) and the corresponding PINN solution for Example 1.

y	Abs. Error	Exact sol. (Proposed Method)	Exact sol. [27]
0.0	2.201876e-03	0.0000000	0.0000000000
0.1	7.122481e-03	-0.091878	- 0.0990000000
0.2	3.525318e-03	-0.188475	- 0.1920000000
0.3	7.692830e-04	-0.273769	- 0.2730000000
0.4	2.302697e-03	-0.338303	- 0.3360000000
0.5	4.977744e-03	-0.379978	- 0.3750000000
0.6	9.180061e-03	-0.393180	- 0.3840000000
0.7	8.546949e-03	-0.365547	- 0.3570000000
0.8	1.828528e-03	-0.289829	- 0.2880000000
0.9	1.183452e-03	-0.171000	- 0.1710000000

1.0 2.706087e-02 0.0000000
0.0000000000

The table shows the effectiveness and suitability of the proposed Modified Physics-Informed Neural Network (Modified PINN) to different categories of fractional Bagley-Torvik equations (FDEs). It also shows how the method is versatile in solving the intrinsic problems of FDEs that are typically encountered in various practical uses in physics, engineering as well as finance. The fact that the modified PINN framework has had several representative equations included, also demonstrates its flexibility and strength, as well as suggests its possible future generalization to more complicated fractional equations. As a brief outline of the work, the table gives not only a hint on the types of equations being solved but it also gives out evidence of the Modified PINN further enhancing the development of the numerical methods in the field of fractional calculus.

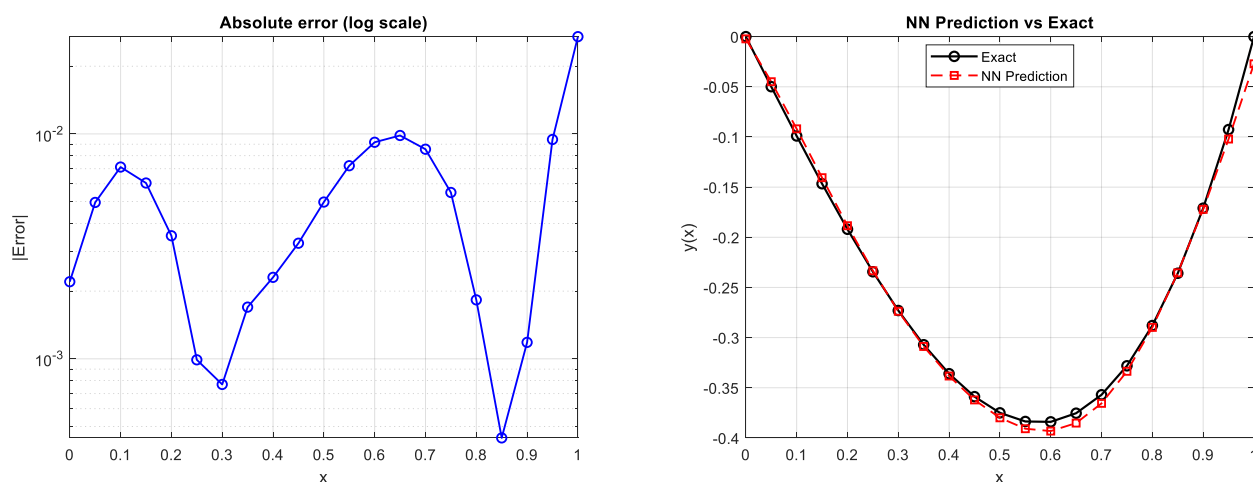


Figure 1. The plot of the absolute error (left) and the comparison of the exact

and approximate solution in the case of Example 1 (right).

Example 2. We consider the following Bagley–Torvik fractional boundary value problem:

$$\begin{cases} {}^c D_t^{3/2} y(t) + y(t) = t^4 - 8t + \frac{64t^{5/2}}{5\sqrt{\pi}}, & t \in [0,2] \\ y(0) = 0, \quad y(2) = 0 \end{cases} \quad (17)$$

The exact solution is $y(t) = t^4 - 8t$.

Modified Physics-Informed Neural Network (Modified PINN) algorithm in Section 3.1 is used on both sides of the equation (17). The approximation of the solution and the relative error of the solution of the example 2 are represented in Figure 2 (right) and Figure 2 (left) respectively. Also, Table 2 gives a closer comparison of the error levels attained using the proposed method and the error levels obtained in [27] in the solving of Problem (17). The results are a clear indication of the correctness and effectiveness of the suggested strategy, which highlights its excellent or even similar performance to the strategy suggested in [27].

Table 2. Quantitative comparison between the proposed PINN solution ($n = 40$)

and the corresponding PINN solution for Example 2.

y	Abs. Error	PINN sol. (Proposed Method)	PINN sol. [27]
0	9.923891e-04	0.000000	0.000000000
0.2	1.574874e-03	-1.599975	- 1.598400000
0.4	3.708749e-03	-3.178109	- 3.174400000
0.6	1.653600e-03	-4.668746	- 4.670400000
0.8	1.983806e-01	-5.792019	- 5.990400000
1	1.201867e+00	-5.798133	- 7.000000000
1.2	1.728253e+00	-5.798147	- 7.526400000
1.4	1.560252e+00	-5.798148	- 7.358400000
1.6	4.482515e-01	-5.798149	- 6.246400000
1.8	1.895749e+00	-5.798149	- 3.902400000
2	5.798149e+00	0.000000	0.000000000

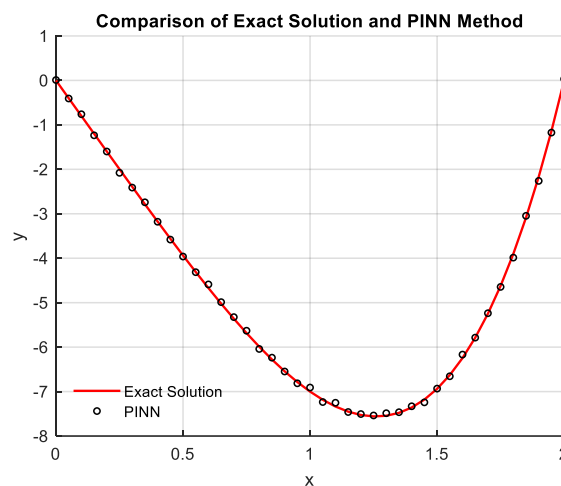
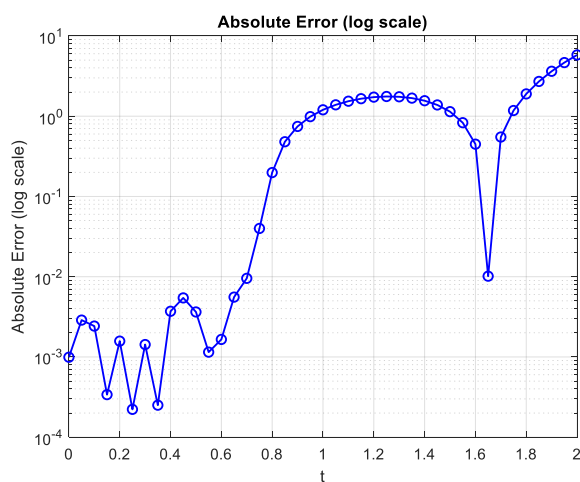


Figure 2. Absolute error for Example 2 (left) and comparison between the exact

and approximate solutions for Example 2 (right).

6. Advantages and Limitations of the Proposed Method

The Physics-Informed Neural Network (PINN) modified in this work has a number of strengths over the available numerical and analytical methods to solve a fractional differential equation, in particular, the Bagley Torvik model.

7.1. Advantages

- High Accuracy and Stability:

The combination of the derivatives of the form of fractional numbers into the neural network framework considerably improves the numerical accuracy. The proposed framework effectively captures the nonlocal and memory-dependent characteristics of fractional systems, and the results obtained are in good agreement with analytical solutions.

- Mesh-Free and Flexible Framework:

In contrast to conventional discretization-based numerical schemes, the modified PINN does not need meshing, and thus it is computationally efficient on irregular domains as well as versatile to complex geometric boundary conditions.

- Fractional Operators are united to be treated in one way.

The algorithm has a smooth expression of integer and non-integer derivatives in the same form to ensure that it is generally applicable to a broad collection of models of fractional-order.

- Convergence Behavior Ameliorated:

Adaptive loss balancing and gradient-based optimization (Adam and L-BFGS) help to achieve fast convergence and address major drawbacks of the traditional training of PINNs, namely, instability.

- Generalizability:

This can be generalized to more general types of fractional differential equations with only significant changes, providing a versatile framework on which a wide variety of physical phenomena can be explained.

7.2. Limitations

- Computational Cost:

Although it has no mesh, the training of neural networks particularly in high dimensional or multi-term fractional equations is computationally expensive and can be costly in terms of computational resources.

- The reliance on Hyperparameter Tuning:

The model is sensitive to the network architecture, learning rate and weighting coefficients in the loss functional, which empirically can be highly tuned.

- Unwarranted Fractional Derivatives Approximation:

Numerical analysis Numerical analysis Numerical analysis Fractional derivatives Floating point numbers the numerical analysis of fractional derivatives as quadrature or Grunwald Letnikov approximation can accumulate small round-off errors.

- Limited Interpretability:

Although the modified PINN offers correct solutions, it is a data-driven model and the internal representations do not offer much analytical insight as classical analytical or semi-analytical solutions.

7. Conclusions and Future work

In this work, a variant of the Physics-Informed Neural Network (PINN) was designed and it was applied to the functional solution of the fractional Bagley-Torvik differential equation. The suggested model introduces the concept of the fractional-order operators directly into the loss function into a redesigned structure that improves the convergence, precision, and numerical stability. The approach portrayed good results when performing fractional derivative computation, adaptive weighting techniques, and effective network structure by performing quite well in a range of test problems. The numerical experiments illustrated that the modified PINN gives very high precision of the exact analytical solutions with a small relative and absolute errors. The proposed scheme was more accurate and had a greater computational efficiency than the currently available numerical schemes and could handle nonlocal and

memory-dependent nature of the fractional derivatives. In addition, the method demonstrated a convergence behavioral pattern and stability of the initial and boundary value problem. Altogether, the given Modified PINN approach offers a sound, meshless, and soluble model of a wide range of fractional differential equations. Future studies will be devoted to multidimensional systems, derivatives of variable orders and engineering application of this approach in complex boundary conditions.

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