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Introducing the Half-Logistic Marshall–Olkin XGamma (HLMOXG) Distribution with Enhanced Flexibility

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ABSTRACT

Over the past few years, some new probability distributions founded on composite and generalized frameworks have been suggested to offer more flexible statistical models that are able to capture a wide range of data behaviors. Even though the XGamma distribution is effective in the modeling of positively skewed lifetime information, it is not very effective in the analysis of heavy-tailed behavior and intricate pattern of hazard rates. In order to address these weaknesses, we propose a new distribution, built by the successive use of Half-Logistic generator and the extension with the Marshall-Olkin to XGamma distribution base, which transforms into Half-Logistic-Marshall-Olkin XGamma (HLMOXG) distribution. The proposed model offers the flexibility in shape offered by the MarshallOlkin transformation, even more shape control offered by the HalfLogistic generator, and the desirable characteristics of the XGamma base model. Statistical properties of the new distribution, such as PDF, CDF, survival and hazard rate functions, and major mathematical properties such as moments, moment-generating function, quantile function and order statistics are calculated and discussed. The estimation of parameters is done using the maximum likelihood estimation (MLE) technique. To test the performance of the proposed model, a simulation study and real data applications in the fields of reliability and environmental are employed. The relative performance shows that the HLMOXG distribution offers a better goodness-of-fit and more flexibility than the baseline and competing models that prove that the HLMOXG can be an effective instrument to analyze lifetime and reliability data

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1. INTRODUCTION

Probability distributions are important in statistical model-building as they allow scientists to describe and analyze the underlying processes of many aspects of real-life phenomena in engineering, environmental studies, biostatistics, and finance. Despite the wide use of classical families of distributions (e.g. the normal, exponential, gamma), it is found to be limited in modeling complex data structures, especially those with fat tails or non-monotonic hazard rate behavior. These constraints have encouraged the creation of more versatile distributional families which have the ability of modeling a broader data pattern.

Over the past few decades, a number of generalization methods have been suggested in order to increase the flexibility of probabilities distributions. Of these, the Marshall-Olkin (MO) transformation and generator-based approaches have attracted considerable interest because they both allow shape and tail behavior manipulation without affecting interpretability [1, 2]. An example of a popular base line distribution, which is used in this context, is the XGamma (XG) distribution, which fits well the positively skewed lifetime data [3]. Nevertheless, although useful, the XGamma distribution can be ineffective in cases where the available data have a greater skewness, kurtosis or more complicated hazard rate profiles.

In order to overcome such shortcomings, researchers have come up with several generalizations of the XGamma distribution based on transmutation maps, power transformations [4], and generator functions [5]. Of these, the Half-Logistic (HL) generator has been especially useful in enhancing model flexibility when used on the various base distributions [6]. The same way, the Marshall-Olkin (MO) transformation is a strong tool of independent control of tail weight and hazard rate behavior [7].

Even though we have considered HL and MO methods widely individually, to the best of our knowledge, no previous studies have combined the MO transformation and HL generator in series using the XGamma methodology. This literature gap impels the creation of a new model, which may be called Half-Logistic-Marshall-Olkin XGamma (HLMOXG) distribution, which combines the strengths of both mechanisms into one, highly versatile formulation.

The suggested HLMOXG distribution can capture a wide variety of shapes of hazard rate, such as increasing, decreasing, bathtub and inverted-bathtub (and other) shapes, and can provide better tail flexibility when dealing with heavy-tailed lifetime data.

Besides obtaining the major theoretical and statistical aspects of the proposed model, we also use Monte Carlo simulation study to determine the performance of the maximum likelihood estimators at various sample sizes. The results of the simulation also support the consistency and reliability of the estimation process.

The contributions of this paper can be summarized as follows:

- a) The HLMOXG distribution is constructed by sequentially applying the Marshall-Olkin and Half-Logistic transformations to the XGamma baseline.
- b) We obtain closed-form results of its probability density function (PDF), cumulative distribution function (CDF), survival function, hazard rate, quantile function, and moments.
- c) Our estimation of the parameters is based on the maximum likelihood estimation (MLE) methodology and also, we have evaluated its performance by simulation (Monte Carlo).

d) We prove the flexibility and superiority of the model by using real data in reliability and environmental analysis.

The rest of the paper is structured in the following way. Section 2 proposes the formulation of suggested HLMOXG distribution. Its statistical properties are discussed in section 3. demonstrates the main structure of HLMOXG distribution with some important derivations and statistical properties of this distribution. section 4. Is devoted to discussing the model parameters estimation of the new distribution by using a method of maximum likelihood estimate. Section 5 gives Monte Carlo simulation study and the empirical application and a comparative analysis. Finally, we set several conclusions that highlight our work which is presented in section 6.

2. PRELIMINARIES AND BACKGROUND MODEL FORMULATION

This section entails the preliminaries and background that is required in connection with the baseline distribution and its extensions. The key mathematical tools and notations that should be used to develop the proposed model are also presented.

2.1 X-Gamma distribution

Let $G_{XG}(x)$ and $g_{XG}(x)$ represent the CDF, and the PDF of the X-Gamma distribution [3], respectively are:

$$g_{XG}(x; \gamma) = \frac{\gamma^2}{1+\gamma} \left(1 + \frac{\gamma x^2}{2}\right) e^{-\gamma x}, \quad x > 0, \gamma > 0, \quad (1)$$

$$G_{XG}(x; \gamma) = \frac{1+\gamma+\gamma x+\frac{\gamma^2 x^2}{2}}{1+\gamma} e^{-\gamma x}. \quad (2)$$

2.2 Marshall–Olkin Extension

With some given baseline CDF $G(x)$, the Marshall-Olkin (MO) extension [8] that adds a single additional parameter $\tau > 0$ transforms the CDF in the form:

$$G_{MO}(x; \tau) = \frac{G(x)}{\tau+(1-\tau)G(x)}, \quad \tau > 0, x > 0 \quad (3)$$

With respective PDF:

$$g_{MO}(x; \tau) = \frac{\tau g(x)}{[\tau+(1-\tau)G(x)]^2}, \quad \tau > 0, x > 0 \quad (4)$$

The following results from applying the MO transformation to the X-Gamma distribution:

$$G_{MOXG}(x; \gamma, \tau) = \frac{G_{XG}(x)}{\tau+(1-\tau)G_{XG}(x)}, \quad \gamma, \tau > 0, x > 0 \quad (5)$$

Then

$$G_{MOXG}(x; \gamma, \tau) = \frac{\frac{1+\gamma+\gamma x+\frac{\gamma^2 x^2}{2}}{1+\gamma} e^{-\gamma x}}{1-(1-\tau)\frac{1+\gamma+\gamma x+\frac{\gamma^2 x^2}{2}}{1+\gamma} e^{-\gamma x}}, \quad \gamma, \tau > 0, x > 0 \quad (6)$$

By driving equation (5) given us the PDF for MOXG distribution.

$$g_{MOXG}(x; \gamma, \tau) = \frac{\tau g_{XG}(x)}{[1+(\tau-1)S_{XG}(x)]^2}, \quad \tau > 0, x > 0 \tag{7}$$

Therefore

$$g_{MOXG}(x; \gamma, \tau) = \frac{\tau \frac{\gamma^2}{1+\gamma} \left(1 + \frac{\gamma x^2}{2}\right) e^{-\gamma x}}{\left[1 - (1-\tau) \left(\frac{1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} e^{-\gamma x}\right)\right]^2}, \quad \gamma, \tau > 0, x > 0 \tag{8}$$

And we immediately obtain the survival function

$$S_{MOXG}(x; \gamma, \tau) = \frac{\tau S_{XG}(x)}{1+(\tau-1)S_{XG}(x)} \tag{9}$$

$$S_{MOXG}(x; \gamma, \tau) = \frac{\frac{1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} \tau e^{-\gamma x}}{1+(\tau-1) \left(\frac{1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} e^{-\gamma x}\right)}, \quad \gamma, \tau > 0, x > 0 \tag{10}$$

2.3 Half-Logistic Generator

Let $S(x)$ be the survival function at baseline. $S(x)$ is converted into a new CDF using the Half-Logistic-Generator (HL-G). [6]:

$$G_{HL-G}(x; \alpha) = \frac{1-[S(x)]^\alpha}{1+[S(x)]^\alpha}, \quad \alpha > 0. \tag{11}$$

The corresponding PDF is:

$$g_{HL-G}(x; \alpha) = \frac{2\alpha[S(x)]^{\alpha-1}g(x)}{[1+[S(x)]^\alpha]^2}, \tag{12}$$

where $g(x)$ is the PDF of the baseline.

3. CONSTRUCTION OF THE HLMOXG AND ITS STATISTICAL PROPERTIES

The Marshall-Olkin and Half-Logistic to the XGamma (HLMOXG) distribution is defined by using $S(x) = S_{MOXG}(x; \tau, \alpha)$ as the baseline in equation (10), we get CDF as follows:

$$G_{HLMOXG}(x; \gamma, \tau, \alpha) = \frac{1-[S_{MOXG}(x; \gamma, \tau)]^\alpha}{1+[S_{MOXG}(x; \gamma, \tau)]^\alpha}, \quad \gamma, \tau, \alpha > 0, x > 0 \tag{13}$$

Thus

$$G_{HLMOXG}(x; \gamma, \tau, \alpha) = \frac{1 - \left[\frac{\frac{1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} \tau e^{-\gamma x}}{1+(\tau-1) \left(\frac{1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} e^{-\gamma x} \right)} \right]^\alpha}{1 + \left[\frac{\frac{1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} \tau e^{-\gamma x}}{1+(\tau-1) \left(\frac{1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} e^{-\gamma x} \right)} \right]^\alpha} \tag{14}$$

where $\gamma, \tau, \alpha > 0, x > 0$

By differentiating the CDF in equation (14) with respect to x , we obtain the pdf of the HLMOXG distribution as given in (15):

$$f_{HLMOXG}(x; \gamma, \tau, \alpha) = \frac{2\alpha [S_{MOXG}(x; \gamma, \tau)]^{\alpha-1} g_{MOXG}(x; \gamma, \tau)}{[1 + [S_{MOXG}(x; \gamma, \tau)]^\alpha]^2} \tag{15}$$

Also, the survival, and hazard functions of the HLMOXG distribution are given below, respectively:

$$S_{HLMOXG}(x) = \frac{2[S_{MOXG}(x; \gamma, \tau)]^\alpha}{1 + [S_{MOXG}(x; \gamma, \tau)]^\alpha} \tag{16}$$

Therefore

$$S_{HLMOXG}(x) = \frac{2}{\left[\frac{(1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2})\tau e^{-\gamma x}}{\tau+(1-\tau) \left(1+\gamma - (1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2})e^{-\gamma x} \right)} \right]^{-\alpha} + 1} \tag{17}$$

$$h_{HLMOXG}(x; \gamma, \tau, \alpha) = \frac{\alpha [S_{MOXG}(x; \gamma, \tau)]^{-1} g_{MOXG}(x; \gamma, \tau)}{1 + [S_{MOXG}(x; \gamma, \tau)]^\alpha} \tag{18}$$

In a way that is no less important than the theoretical part, we display the graphs of the CDF, and the PDF of HLMOXG distribution as we can see in Figure 1 and Figure 2, respectively.

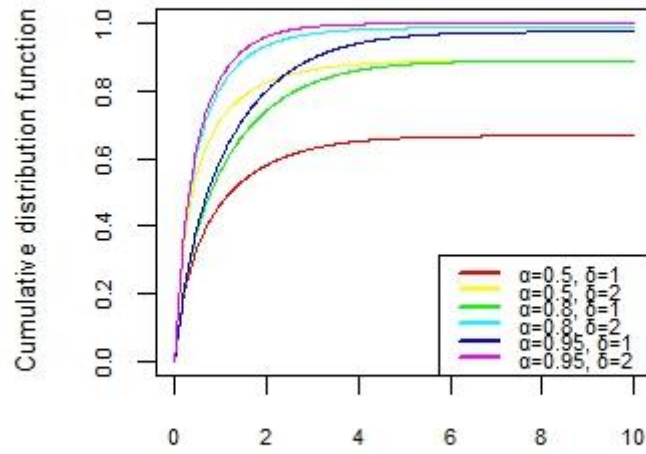


Figure 1. The CDF of HLMOXG distribution with different parameters

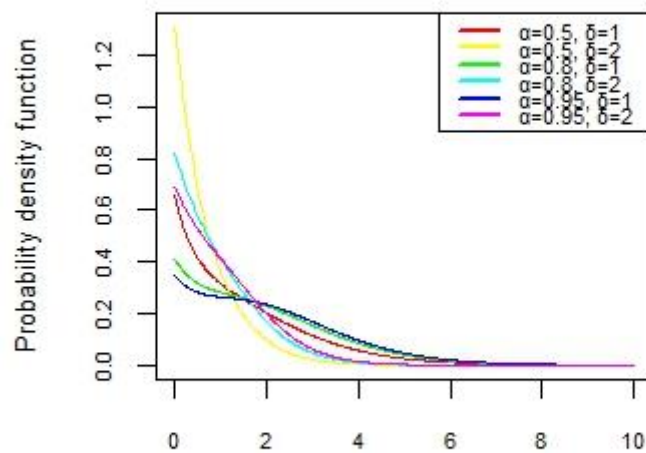


Figure 2. The PDF of HLMOXG distribution with different parameters

3.1 Quantile Function

Let $u \in (0,1)$ be a probability value [9]. From the CDF definition:

$$u = \frac{1-[S_{MOXG}(x; \gamma, \tau)]^\alpha}{1+[S_{MOXG}(x; \gamma, \tau)]^\alpha}, \tag{19}$$

we solve for $S_{MOXG}(x; \gamma, \tau)$:

$$[1 - G_{MOXG}(z; \gamma, \tau)]^\alpha = \frac{1-u}{1+u} \tag{20}$$

$$G_{MOXG}(x; \gamma, \tau) = 1 - \left(\frac{1-u}{1+u}\right)^{\frac{1}{\alpha}}. \tag{21}$$

Given $G_{MOXG}(x; \gamma, \tau)$ has no closed-form inversion in general (due to the XGamma baseline), the quantile function $Q(u)$ can be obtained numerically by solving:

$$\frac{G_{XG}(x; \gamma)}{\tau+(1+\tau)G_{XG}(x; \gamma)} = 1 - \left(\frac{1-q}{1+q}\right)^{\frac{1}{\alpha}}. \tag{22}$$

This numerical inversion allows for random sample generation via the inverse transform method.

3.2 Moments

Theorem 3.1 let X be a random variable from HLMOXG distribution. The moments [10] of X is given by:

$$\mu_r \approx A \sum_{l=0}^L M_l \left[B \frac{\Gamma(r+l+1)}{(\gamma(1+z))^{r+l+1}} + P \frac{\Gamma(r+l+3)}{(\gamma(1+z))^{r+l+3}} \right] \tag{23}$$

Proof The moments of a random variable X with respect to the pdf of HLMOXG distribution is defined by

$$\begin{aligned} \mu_r &= 2\alpha \int_0^\infty x^r \frac{[1 - G_{MOXG}(x; \gamma, \tau)]^{\alpha-1} g_{MOXG}(x; \gamma, \tau)}{[1 + [1 - G_{MOXG}(x; \gamma, \tau)]^\alpha]^2} dx . \\ \mu_r &= 2\alpha \int_0^\infty x^r \frac{[S_{MOXG}(x; \gamma, \tau)]^{\alpha-1}}{[1 + [S_{MOXG}(x; \gamma, \tau)]^\alpha]^2} g_{MOXG}(x; \gamma, \tau) dx . \end{aligned}$$

According to the negative series expansion, we obtain

$$\mu_r = 2\alpha \sum_{k=0}^\infty (k+1)(-1)^k \int_0^\infty x^r [S_{MOXG}(x; \gamma, \tau)]^{\alpha(k+1)-1} g_{MOXG}(x; \gamma, \tau) dx .$$

Substituting equations (5),and (6) into the equation above, then

$$\mu_r = 2\alpha \sum_{k=0}^\infty (k+1)(-1)^k \int_0^\infty x^r \left(\frac{\tau S_{XG}(x)}{1 + (\tau - 1)S_{XG}(x)} \right)^{\alpha(k+1)-1} \frac{\tau g_{XG}(x)}{[1 + (\tau - 1)S_{XG}(x)]^2} dx .$$

Once again, we use the negative series expansion, and the exponential series expansion, the formulas $\frac{1}{(1+(\tau-1)S_{XG}(x))^{\alpha(k+1)+1}}$ can be written, as

$$\frac{1}{(1+(\tau-1)S_{XG}(x))^{\alpha(k+1)+1}} = \sum_{m=0}^{\infty} \binom{\alpha(k+1)+m}{m} (1-\tau)^m (S_{XG}(x))^m$$

Therefore

$$\begin{aligned} \mu'_r &= 2\alpha \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{\alpha(k+1)+m}{m} (1-\tau)^m (k+1) (-1)^k \tau^{\alpha(k+1)} \int_0^{\infty} x^r (S_{XG}(x))^{\alpha(k+1)+m-1} \times g_{XG}(x; \gamma) dx . \\ \mu'_r &= A \int_0^{\infty} x^r (S_{XG}(x))^{\alpha(k+1)+m-1} g_{XG}(x) dx . \end{aligned}$$

Where

$$A = 2\alpha \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{\alpha(k+1)+m}{m} (1-\tau)^m (k+1) (-1)^k \times \tau^{\alpha(k+1)}$$

By substituting equations (1), and (3) in the equation above, we get

$$\mu'_r = A \int_0^{\infty} (x^r B e^{-\gamma x(1+z)} (C(x))^z + (Px^2)(C(x))^z e^{-\gamma x(1+z)}) dx .$$

$$z = \alpha(k+1) + m - 1, \quad B = \frac{\gamma^2}{1+\gamma}, \quad P = \frac{\gamma^3}{2(1+\gamma)}, \quad C(x) = \frac{1+\gamma+\gamma x+\frac{\gamma^2 x^2}{2}}{1+\gamma} \approx \sum_{l=0}^L M_l x^l$$

$$\mu'_r \approx A \sum_{l=0}^L M_l \int_0^{\infty} (x^{r+l} B e^{-\gamma x(1+z)} + P x^{2+r+l} e^{-\gamma x(1+z)}) dx .$$

Therefore, with the help of gamma distribution, we obtain that

$$\mu'_r \approx A \sum_{l=0}^L M_l \left[B \frac{\Gamma(r+l+1)}{(\gamma(1+z))^{r+l+1}} + P \frac{\Gamma(r+l+3)}{(\gamma(1+z))^{r+l+3}} \right]$$

Table 1 shows the values of the first four moments at some values of selected parameters.

Table 1 The values of the first four moments, where $\gamma = 2.0, \tau = 0.7,$ and $\alpha = 1.5$.

r	μ'_r
1	0.6370
2	0.7424
3	1.2038
4	2.4501

3.3 Moment Generating Function

The moment-generating function (m.g.f) [10] of a random variable X from HLMOXG distribution can be derived by presenting its proof in the following theorem.

Theorem :

If X is a random variable from HLMOXG distribution, then its mgf is given by

$$E[e^{xt}] \approx A \sum_{l=0}^L M_l \left[B \frac{\Gamma(l+1)}{(\gamma(1+z)+t)^{l+1}} + P \frac{\Gamma(l+3)}{(\gamma(1+z)+t)^{l+3}} \right] \tag{24}$$

Proof the mgf for X is defined by:

$$M_X(t) = E[e^{xt}] = \int_0^\infty e^{xt} f_{MO-HL-XG}(x; \gamma, \tau, \alpha) dx$$

Similar to the proof of the theorem (1) and by solving the above integral, we obtain that

$$E[e^{xt}] \approx A \sum_{l=0}^L M_l \int_0^\infty (x^l B e^{-x(\gamma(1+z)+t)} + P x^{2+l} e^{-x(\gamma(1+z)+t)}) dx .$$

$$E[e^{xt}] \approx A \sum_{l=0}^L M_l \left[B \frac{\Gamma(l+1)}{(\gamma(1+z)+t)^{l+1}} + P \frac{\Gamma(l+3)}{(\gamma(1+z)+t)^{l+3}} \right]$$

Where

$$z = \alpha(k + 1) + m - 1, B = \frac{\gamma^2}{1+\gamma}, P = \frac{\gamma^3}{2(1+\gamma)}, C(x) = \frac{1+\gamma+\gamma x+\frac{\gamma^2 x^2}{2}}{1+\gamma} \approx \sum_{l=0}^L M_l x^l$$

$$A = 2\alpha \sum_{k=0}^\infty \sum_{m,j=0}^\infty \binom{\alpha(k+1)+m-2}{m} (\tau - 1)^{m+j} (k + 1)(j + 1)(-1)^{k+m+j} \times \tau^{\alpha(k+1)}$$

3.4 Probability Density Function of Order Statistics for HLMOXG distribution

The j^{th} order statistics of a random variable X from the HLMOXG distribution is given by recalling the general pdf of the j^{th} order statistics and substitute both the pdf and cdf for the desired distribution [10], as follows

$$g_{x_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} (G_{MO-HL-XG}(x))^{j-1} (1 - G_{MO-HL-XG}(x))^{n-j} \times g_{MO-HL-XG}(x) \tag{25}$$

In particular, the smallest order statistic for HLMOXG distribution is given in equation (26)

$$g_{x_{(1)}}(x) = n(G_{MO-HL-XG}(x))^{n-1} g_{MO-HL-XG}(x) \tag{26}$$

While the greatest order statistics of HLMOXG distribution can be written as

$$g_{x_{(n)}}(x) = n(1 - G_{MO-HL-XG}(x))^{n-1} g_{MO-HL-XG}(x) \tag{27}$$

This facilitates applications in reliability theory and life testing experiments.

4. ESTIMATION OF MODELS (HLMOXG) PARAMETERS

In this part, we focus on the estimation of the model parameters of **HLMOXG distribution** by utilizing the MLE method one of the non-Bayesian methods. Mathematically speaking, let X_1, X_2, \dots, X_n be a random sample of size (n) from the HLMOXG distribution, we can express the likelihood function as:

$$L(\gamma, \tau, \alpha; \underline{x}) = \prod_{i=1}^n g_{HLMOXG}(x_i; \gamma, \tau, \alpha)$$

$$L(\gamma, \tau, \alpha; \underline{x}) = \prod_{i=1}^n \left[\frac{2\alpha [S_{MOXG}(x_i; \gamma, \tau)]^{\alpha-1} g_{MOXG}(x_i; \gamma, \tau)}{[1 + [S_{MOXG}(x_i; \gamma, \tau)]^\alpha]^2} \right] \tag{28}$$

While the log-likelihood is written as

$$\ell(\gamma, \tau, \alpha; \underline{x}) = \log(L(\gamma, \tau, \alpha; \underline{x}))$$

Thus

$$\ell(\gamma, \tau, \alpha; \underline{x}) = n \ln(2\alpha) + (\alpha - 1) \sum_{i=1}^n \ln[S_{MOXG}(x_i; \gamma, \tau)] - 2 \sum_{i=1}^n \ln[1 + [S_{MOXG}(x_i; \gamma, \tau)]^\alpha] + \sum_{i=1}^n \ln g_{MOXG}(x_i; \gamma, \tau). \tag{29}$$

To estimate the desired parameters, we differentiate the log-likelihood function with respect to each parameter partially, above equation with respect to parameters $\theta = (\gamma, \tau, \alpha)^T$, we obtain that:

$$\begin{aligned} \frac{\partial \ell}{\partial \alpha} = & \frac{n}{\alpha} \ln(2\alpha) + (\alpha - 1) \sum_{i=1}^n \ln(S_{MOXG}(x_i; \gamma, \tau)) - 2 \sum_{i=1}^n \ln(1 + (S_{MOXG}(x_i; \gamma, \tau))^\alpha) + \\ & \sum_{i=1}^n \ln(g_{MOXG}(x_i; \gamma, \tau)) = \end{aligned} \tag{30}$$

$$\begin{aligned} \frac{\partial \ell}{\partial \tau} = & (\alpha - 1) \sum_{i=1}^n \left(1 + \frac{(1-S_{XG}(x))}{\tau S_{XG}(x)}\right) \frac{(S_{XG}(x)G_{XG}(x))}{(\tau S_{XG}(x)+G_{XG}(x))^2} - \\ & 2 \sum_{i=1}^n \frac{\alpha \left[\frac{\tau S_{XG}(x)}{\tau S_{XG}(x)+G_{XG}(x)}\right]^{\alpha-1}}{1 + \left[\frac{\tau S_{XG}(x)}{\tau S_{XG}(x)+G_{XG}(x)}\right]^\alpha} \times \left(1 + \frac{G_{XG}(x)}{\tau S_{XG}(x)}\right) \frac{(S_{XG}(x)G_{XG}(x))}{(\tau S_{XG}(x)+G_{XG}(x))^2} + \frac{1}{\tau} \sum_{i=1}^n \left(1 - \frac{2\tau S_{XG}(x)}{(G_{XG}(x)-\tau S_{XG}(x))}\right) = \end{aligned} \tag{31}$$

$$\begin{aligned} \frac{\partial \ell}{\partial \gamma} = & (\alpha - 1) \sum_{i=1}^n \frac{1}{(S_{MOXG}(x_i; \gamma, \tau))} \times \frac{\left(1 - (1-\tau) \left(\frac{1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} e^{-\gamma x}\right)\right)}{\left(1 - (1-\tau) \left(\frac{1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} e^{-\gamma x}\right)\right)^2} \times \frac{1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} x \tau e^{-\gamma x} + \tau e^{-\gamma x} \frac{(1+\gamma)(1+x+\gamma x^2)}{(1+\gamma)^2} + \\ & \frac{\left(\frac{1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} \tau e^{-\gamma x}\right)}{\left(1 - (1-\tau) \left(\frac{1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} e^{-\gamma x}\right)\right)^2} \times (\tau - 1) \times \left(\frac{1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} x e^{-\gamma x} - e^{-\gamma x} \frac{(1+\gamma)(1+x+\gamma x^2)}{(1+\gamma)^2}\right) \\ & - 2 \sum_{i=1}^n \frac{1}{1 + [S_{MOXG}(x; \gamma, \tau)]^\alpha} \times \alpha [S_{MOXG}(x; \gamma, \tau)]^{\alpha-1} \times \frac{\left(1 - (1-\tau) \left(\frac{1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} e^{-\gamma x}\right)\right)}{\left(1 - (1-\tau) \left(\frac{1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} e^{-\gamma x}\right)\right)^2} \times \frac{1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} x \tau e^{-\gamma x} + \\ & \tau e^{-\gamma x} \frac{(1+\gamma)(1+x+\gamma x^2)}{(1+\gamma)^2} + \frac{\left(\frac{1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} \tau e^{-\gamma x}\right)}{\left(1 - (1-\tau) \left(\frac{1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} e^{-\gamma x}\right)\right)^2} \times (\tau - 1) \times \left(\frac{1+\gamma+\gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} x e^{-\gamma x} - e^{-\gamma x} \frac{(1+\gamma)(1+x+\gamma x^2)}{(1+\gamma)^2}\right) \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{i=1}^n \frac{1}{g_{MOXG}(x_i; \gamma, \tau)} \times \frac{\left(\frac{-\gamma^2}{1+\gamma} \tau x e^{-\gamma x} + \tau e^{-\gamma x} \frac{2\gamma + \gamma^2}{(1+\gamma)^2}\right)}{\left[1 - (1-\tau) \left(\frac{1+\gamma + \gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} e^{-\gamma x}\right)\right]^2} + \frac{2\left(\frac{\gamma^2}{1+\gamma} \tau e^{-\gamma x} + \frac{\gamma^3 \tau x^2}{2(1+\gamma)} e^{-\gamma x}\right)}{\left[1 - (1-\tau) \left(\frac{1+\gamma + \gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} e^{-\gamma x}\right)\right]^3} \times \\
 &\left((\tau - 1) \times \left(\frac{1+\gamma + \gamma x + \frac{\gamma^2 x^2}{2}}{1+\gamma} x e^{-\gamma x} - e^{-\gamma x} \frac{(1+\gamma)(1+x+\gamma x^2)}{(1+\gamma)^2} \right) \right) = 0 \tag{32}
 \end{aligned}$$

Equations (30), (31), and (32) are non-linear equations and cannot solve analytically. Then, numerical methods like the Newton –Raphson method are employed to find the estimated values of the parameters $\alpha, \tau,$ and γ .

5. APPLICATIONS

This section contains two parts. First of all, we discuss the simulation part, and then we show an application of real data set.

5.1 Simulation

A Monte Carlo simulation study was conducted on the R software to assess the performance and efficiency of the maximum likelihood estimators (MLEs) of parameters of the proposed HLMOXG distribution. The suggested model was used to draw random samples of various sizes 25, 50, 100, 150, and 200. In each case, the simulation experiment was conducted 1000 times in order to get the average estimates of the parameters, as well as the mean square errors (MSEs) [10]. The findings have been provided in Table 2, which shows the estimated parameters and MSE values used in the sample sizes considered. It is evident that the larger is the sample size, the smaller the mean square errors of the estimators, and the closer the average estimates to the true parameters. The consistency and asymptotic efficiency of the MLEs of the proposed distribution are confirmed by this result. Specifically, in the case of the largest sample size (n = 200), the mean estimation error was close to zero, meaning that the estimation is superb in large samples and the MLE technique is very suitable to estimate the parameters under the proposed model.

Table 2. MLEs of parameters and MSEs

sample sizes	Parameter estimate ($\hat{\gamma}, \hat{\tau}, \hat{\alpha}$)	$\hat{\rho}$	MSE
n=25	0.50190		3.594295603e-06
	0.49906	-76.63997	8.796084891e-07
	0.03712		0.00137802605
n=50	0.50001		6.906220802e-11
	0.49973	-130.753371	7.42186322e-08
	0.01915		0.000366596919
n=100	0.5001118		1.250423037e-08
	0.5000001	-325.495304	2.24869935e-14
	0.0072170		5.208514364e-05

	0.500001		1.344063189e-12
n=150	0.500000	-346.828403	1.244655464e-16
	0.005395		2.91075882e-05
	1.094160		0.0000000
n=200	8.529716	-627.508319	0.0000000
	0.085608		0.0000000

Based on Table 2, it is clear that the values of MSE decreases with the increase in the sample size and the estimated values of the parameters are brought nearer to the true values which proves the consistency of the MLEs.

5.2 Real Data

We analyzed a real dataset concerning the Time to Failure (TTF) of disk drives to demonstrate the flexibility and applicability of the proposed Half-Logistic-Marshall-Olkin-XGamma (HLMOXG) distribution. The dataset which was sourced from the study titled “Disk Failures in the Real World: What Does an MTTF of ... Mean to You?” [12], is: 15.2, 12.5, 18.7, 14.3, 21, 16.8, 19.5, 13.4, 15.9, 20.1, 17.3, 14.8, 19.9, 16.1, 22.4, 13.9, 17.8, 18.5, 20.8, 19.1, 15.6, 21.5, 14.6, 18.3, 16.4, 21.9, 13.7, 19.7, 20.5, 15.4, 0.306, 0.285, 0.144, 0.124, 0.049, 0.029, 0.017, 0.016, 0.01, 0.003. Using this dataset, we estimated the parameters of the HLMOXG distribution, along with several contesting lifetime models, including the distributions Gamma, MOXG, and XGamma. Then we implementation of the Newton-Raphson algorithm in the R programming language to solve the MLE problem yields the results displayed in Table 3. These results with the help of goodness of fit measures (AIC, and BIC) [9] show that the proposed distribution can be considered as an alternative approach to fitting the desired data set.

Table 3. Log-likelihood, AIC, BIC, and KS statistic for the proposed and competing models based on the Time to Failure data set.

Model	MLE	BIC	AIC
HLMOXG	-130.4275	271.9217	266.855
MOXG	-136.1935	279.7648	276.387
X Gamma	-138.319	280.3269	278.638
Gamma	-135.2305	277.8387	274.461

Based on Table 3, it is clear that the proposed MOHLXG distribution gives the highest log-likelihood, the least AIC and BIC compared to all other competing models and thus the best overall fit of the model to the data. The findings substantiate the effectiveness and versatility of the proposed distribution in modeling lifetime data over the classical distributions like Gamma and XGamma .

6. CONCLUSION

In this study, a new and flexible probability model, namely the Half-Logistic-Marshall-Olkin XGamma (HLMOXG) distribution, was proposed. The model was developed through the sequential application of the Marshall-Olkin extension and the Half-Logistic generator to the XGamma baseline distribution. Several important statistical and mathematical properties of the HLMOXG distribution were derived and analyzed in detail. The parameters of the proposed model were estimated using the (MLE) method, and

the performance of these estimators was assessed through a comprehensive Monte Carlo simulation under different sample sizes. The simulation findings confirmed the efficiency and consistency of the MLE estimates. Moreover, the practical applicability of the HLMOXG distribution was demonstrated through real lifetime data analysis. The comparative analysis revealed that the proposed HLMOXG distribution provides a superior fit compared to related models such as the Gamma, XGamma, and MOXG distributions, highlighting its greater flexibility in modeling lifetime and reliability data. Therefore, the proposed HLMOXG distribution can be considered a promising alternative for future applications in reliability engineering, survival analysis, and other fields requiring flexible probabilistic modeling. All computations were carried out using the R statistical software

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