Journal of Al-Qadisiyah for Computer Science and Mathematics
3nd. Sinentific Conference 19-20/ APRIL -2011
Vol 3 No.2 Year 2011

Page 138-155

Chaos in a harvested prey-predator model with infectious disease in the prey

Riad Kamel Naji* and Hiba Abdullah Ibrahim

Department of Mathematics, College of Science, University of Baghdad, Baghdad, IRAQ.

E-mail: rknaji@gmail.com (R.K. Naji).

E-mail: haibrahim82@yahoo.com (H.A. Ibrahim)

(* refers to corresponding author)

Abstract:

A harvested prey-predator model with infectious disease in prey is investigated. It is assumed that the predator feeds on the infected prey only according to Holling type-II functional response. The existence, uniqueness and boundedness of the solution of the model are investigated. The local stability analysis of the harvested prey-predator model is carried out. The necessary and sufficient conditions for the persistence of the model are also obtained. Finally, the global dynamics of this model is investigated analytically as well as numerically. It is observed that, the model have different types of dynamical behaviors including chaos.

1. Introduction:

There has been growing interest in the study of diseases in prey-predator models. It is well know that, in nature species does not exist alone. In fact, any given habitat may contain dozens or hundreds of species, sometimes thousands. Since any species has at least the potential to interact with any other species in its habitat, the possibility of spreading of the disease in a community rapidly becomes astronomical as the number of infected species in the habitat increases. Therefore, it is more of biological significance to study the effect of disease on the dynamical behavior of interacting species. In the last two decades, some prey-predator models with infections diseases have been considered [1-5] and the references their in. All these studies, reached at the conclusion that disease may cause vital changes in the dynamics of an ecosystem.

On the other hand, harvesting has generally a strong impact on the population dynamics of a harvested species. The severity of this impact depends on the nature of the implemented harvesting strategy, which in turn may range from the rapid depletion to the complete preservation of a population. The study of population dynamics with harvesting is a subject of mathematical bio economics, and it is related to the optimal management of renewable resources [6]. The effect of constant rate of harvesting on the dynamical behavior of interacting species has been considered by many researchers [7-8] and the references their in. The conclusions of these studies can be summarized as follows: Harvesting may be used as a biological control for the coexistence of the species, but unregulated harvesting might lead to extinction in one or more species.

Keeping the above in view, the effect of disease on the dynamical behavior of the harvested prey-predator systems is important from economical viewpoint. Little attention has been paid so far in this direction Chattopadhyay et al [9], proposed and analyzed a mathematical model of a harvested prey-predator system with infection on prey population. They assumed that, the predator feeds on the susceptible prey population according to Holling type-II functional response, while it feeds on infected prey population according to Lotka-Volterra predation form. They reached to the following result, harvesting of infected prey may be used as a biological control for the persistence in an infected prey-predator system. In this chapter, Chattopadahyay et al model [9] modified by assuming that the predator feeds on the infected prey only according to the Holling type-II functional response. The possibility of occurrence of chaotic behavior is also considered, and then the effects of disease and harvesting on such chaotic behavior are studied.

2. The Mathematical Model

Let $S(t)$ and $I(t)$ be the numbers of the susceptible and infected prey population at time t respectively. Let $Z(t)$ be the number of the predator population at time t. The dynamics of a harvested prey-predator model with infection on prey population can be represented by the following set of differential equations:

$$\begin{aligned}
\frac{dS}{dt} &= rS\left(1-\frac{S+I}{K}\right) - cSI - E_1 S = Sf_1(S,I,Z);\ S(0)\ge 0 \\
\frac{dI}{dt} &= cSI - \frac{\alpha IZ}{\gamma + I} - \lambda I - E_2 I = If_2(S,I,Z);\ I(0)\ge 0 \qquad (1)\\
\frac{dZ}{dt} &= -\theta Z + \frac{hIZ}{\gamma + I} - E_3 Z = Zf_3(S,I,Z);\ Z(0)\ge 0
\end{aligned}$$

Here the positive constants r and K are, respectively, the intrinsic growth rate and carrying capacity of the prey species in the absence of predation and harvesting. The positive constants c, α, and γ represent the infection rate, maximum attack rate, and the half saturation coefficient, respectively. However, the positive constants λ and θ denote to the death rates of the infected prey and the predator, respectively. The positive constant h represents the growth rate of predator due to predation of infected prey and hence it can be written as $h = e\alpha$ with $0 < e < 1$. Finally, the non-negative constants E_1, E_2, and E_3 are the harvesting efforts for the susceptible prey, infected prey and predator, respectively.

Obviously, the interaction functions in the right hand side of system (1) are continuously differentiable functions on $R_+^3 = \{(\mathrm{S}, \mathrm{I}, \mathrm{Z}) \in R^3, S \geq 0, I \geq 0, Z \geq 0\}$ and hence they are Lipschitzian functions. Therefore the solution of system (1) exists and is unique.

Note that, according to the form of $f_1(S, I, Z)$ in system (1), it is easy to verify that the necessary condition of coexistence of all species in system (1) is given by

$$r - E_1 > 0 \Leftrightarrow r > E_1 \tag{2}$$

Therefore, from now onward, we assume that condition (2) is always holds. Furthermore, the solution of system (1) with non-negative initial conditions is bounded as shown in the following theorem.

Theorem 1.

All the solutions of system (1), which initiate in R_+^3 are uniformly bounded.

Proof:

Let $(S(t), I(t), Z(t))$ be any solution of the system (1) with non-negative initial conditions. According to first equation of system (1) we have

$$\frac{dS}{dt} \leq rS\left(1 - \frac{S}{K}\right) - E_1 S.$$

Then due to the comparison theorem [10], we obtain

$$S(t) \leq \frac{K(r - E_1)}{r}; \forall t > 0. \tag{3}$$

Let $W(t) = S(t) + \frac{h}{\alpha} I(t) + Z(t)$, then by straight forward computations we get that

$$\frac{dW}{dt} \leq rS - N\left[S + \frac{h}{\alpha} I + Z\right]$$

where $N = \min\{E_1, \lambda + E_2, \theta + E_3\}$. Hence, by using Eq. (3) we obtains that:

$$\frac{dW}{dt} + NW \leq K(r - E_1)$$

Again, by applying the comparison theorem on the above differential inequality gives

$$W(t) \leq \frac{\beta}{L}; \forall \mathrm{t} > 0, \text{ where } \beta = K(r - E_1) > 0.$$

Journal of Al-Qadisiyah for Computer Science and Mathematics
3nd. Sinentific Conference 19-20/ APRIL -2011
Vol 3 No.2 Year 2011

Hence all the solutions of system (1) that initiate in R_+^3 are confined in the region $\Omega = \left\{(S, I, Z) \in R_+^3 : W \le \frac{\beta}{L} + \varepsilon \text{ for any } \varepsilon > 0\right\}$. Thus these solutions are uniformly bounded, and then the proof is complete.

Now, since an ecological system is said to be dissipative if the solution of the system, which initiate in R_+^3 is uniformly bounded as $t \to \infty$. Therefore, system (1) is dissipative.

3. Stability analysis with Persistence:

In this section, the existence and stability analysis of all possible equilibrium points of system (1) are discussed and the following results are obtained

1. The equilibrium points $F_0 = (0,0,0)$ and $F_1 = \left(\frac{K(r-E_1)}{r}, 0, 0\right)$ are always exist.

2. The planar equilibrium point $F_2 = (\bar{S}, \bar{I}, 0)$, where

$$\bar{S} = \frac{\lambda + E_2}{c} ; \quad \bar{I} = \frac{cK(r - E_1) - r(\lambda + E_2)}{c(r + cK)} \tag{4}$$

   exists in the *Int.* R_+^2 of the $SI-$ plane under the following condition

$$cK(r - E_1) > r(\lambda + E_2) \tag{5}$$

3. The positive equilibrium point $F_3 = (S^*, I^*, Z^*)$, where:

$$S^* = \frac{[K(r - E_1)(h - (\theta + E_3)) - \gamma(\theta + E_3)(r + cK)]}{r(h - (\theta + E_3))} \tag{6 a}$$

$$I^* = \frac{\gamma(\theta + E_3)}{h - \theta - E_3} \tag{6 b}$$

$$Z^* = \frac{(cS^* - \lambda - E_2)(\gamma + I^*)}{\alpha} \tag{6 c}$$

   exists in the *Int.* R_+^3 if and only if the following set of conditions hold.

$$\gamma < \frac{K(r-E_1)(h-(\theta+E_3))}{(\theta+E_3)(r+cK)} \tag{7 a}$$

$$h-\theta-E_3>0 \tag{7 b}$$

$$cS^*-\lambda-E_2>0 \Leftrightarrow S^* > \frac{\lambda+E_2}{c}=\bar{S} \tag{7 c}$$

Now to analyze the local stability of system (1) around each of these equilibrium points, the Jacobian matrix $J(F_i)$; $i=0,1,2,3$ of system (1) at each equilibrium point is computed and then the eigenvalues are determined. The following results are obtained.

The eigenvalues of $J(F_0)$ are given by $\lambda_{01}=r-E_1>0$, $\lambda_{02}=-(\lambda+E_2)<0$, $\lambda_{03}=-(\theta+E_3)<0$, and hence F_0 is a saddle point. However, the eigenvalues of $J(F_1)$ are $\lambda_{11}=-(r-E_1)<0$, $\lambda_{12}=\frac{cK(r-E_1)}{r}-\lambda-E_2$, and $\lambda_{13}=-(\theta+E_3)<0$. Therefore, F_1 is locally asymptotically stable provided that:

$$cK(r-E_1)<r(\lambda+E_2) \tag{8}$$

While, it is a saddle point, with locally stable manifold in the $SZ-$ plane and with locally unstable manifold in the $I-$direction, under condition (5). Obviously, if F_1 is locally asymptotically stable then F_2 does not exist. However, F_1 is a saddle when F_2 exists.

The eigenvalues of $J(F_2)$ satisfy the following relations

$$\lambda_{21}+\lambda_{22}=-\frac{r(\lambda+E_2)}{cK}<0 \tag{9 a}$$

$$\lambda_{21}.\lambda_{22}=\frac{(\lambda+E_2)[cK(r-E_1)-r(\lambda+E_2)]}{cK} \tag{9 b}$$

$$\lambda_{23}=-\theta+\frac{h\bar{I}}{(\gamma+\bar{I})}-E_3=\frac{\bar{I}(h-\theta-E_3)-\gamma(\theta+E_3)}{(\gamma+\bar{I})} \tag{9 c}$$

Where, λ_{2i} $(i=1,2,3)$ represent the eigenvalues in the $S-$, $I-$ and $Z-$direction respectively. Note that, from Eqs. (9 a)-(9 b) we obtain that, the eigenvalues λ_{21} and λ_{22}, which are describe the dynamics in the S- and I-direction respectively, have negative real parts under the condition (5). Therefore, F_2 is locally asymptotically stable in the $Int.R_+^2$ of the $SI-$ plane whenever it is

Journal of Al-Qadisiyah for Computer Science and Mathematics
3nd. Sinentific Conference 19-20/ APRIL -2011
Vol 3 No.2 Year 2011

exists. Further, F_2 is locally asymptotically stable or saddle point in the $Int.R_+^3$, depending on whether the eigenvalue λ_{23} is negative or positive respectively.

Theorem 2.

The equilibrium point F_2 is a globally asymptotically stable in the $Int.R_+^2$ of the $SI-$ plane.

Proof:

Clearly in the $Int.R_+^2$ of the $SI-$ plane, system (1) reduces to the following subsystem

$$\frac{dS}{dt} = rS\left(1-\frac{S+I}{K}\right) - cSI - E_1 S = h_1(S,I)$$

$$\frac{dI}{dt} = cSI - \lambda I - E_2 I = h_2(S,I)$$

for which ($\bar{S},\bar{I}$) is a unique positive equilibrium point.

Let $H(S,I)=\frac{1}{SI}$ be a C^1 positive function in the $Int.R_+^2$ of the $SI-$ plane. Now, since

$$\Delta(S,I)=\frac{\partial(Hh_1)}{\partial S}+\frac{\partial(Hh_2)}{\partial I}=\frac{-r}{IK}<0$$

is not change sign and does not identically zero. Hence, according to Bendixson-Dulic criterion [11], there is no closed curve in the $Int.R_+^2$ of the $SI-$ plane. Therefore, F_2 is a globally asymptotically stable in the $Int.R_+^2$ of the $SI-$ plane.

Finally, the Jacobian matrix of system (1) at the positive equilibrium point $F_3=\left(S^*,I^*,Z^*\right)$ is given by $J(F_3)=\left\lfloor a_{ij}\right\rfloor_{3\times 3}$; $i,j=1,2,3$ where:

$a_{11}=-\frac{rS^*}{K}<0$; $a_{12}=-\left(\frac{r}{K}+c\right)S^*<0$; $a_{13}=0$; $a_{21}=cI^*>0$; $a_{22}=\frac{\alpha I^* Z^*}{B^2}>0$; $a_{23}=-\frac{\alpha I^*}{B}<0$; $a_{31}=0$; $a_{32}=\frac{h\gamma Z^*}{B^2}>0$; $a_{33}=0$ and $B=\gamma+I^*$.

Then the characteristic equation of $J(F_3)$ can be written as

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \qquad \text{(10 a)}$$

with $A_1=-(a_{11}+a_{22})$; $A_2=-(a_{23}a_{32}+a_{12}a_{21}-a_{11}a_{22})$ and $A_3=a_{11}a_{23}a_{32}$.

Then, by substituting the values of a_{ij}, and then simplifying the resulting terms we obtain:

$$A_1 = \frac{rB^2S^* - \alpha KI^*Z^*}{KB^2} \tag{10 b}$$

$$A_3 = \frac{r\alpha\gamma\, hS^*I^*Z^*}{KB^3} > 0 \text{ always} \tag{10 c}$$

And $\Delta = A_1A_2 - A_3 = -(a_{11}+a_{22})(a_{11}a_{22} - a_{23}a_{32} - a_{12}a_{21}) - a_{11}a_{23}a_{32}.$

$$= \frac{I^*}{K^2B^5}\left[S^*B\left(rB^2S^* - \alpha KI^*Z^*\right)\left(cB^2(r+cK) - r\alpha Z^*\right) - \alpha^2 h\gamma K^2 I^* Z^{*2}\right] \tag{10 d}$$

Now, it is easy to verify that $A_1 > 0$ and $\Delta > 0$ under the following set of conditions

$$rB^2S^* > \alpha KI^*Z^* \tag{11 a}$$

$$cB^2(r+cK) > r\alpha\, Z^* \tag{11 b}$$

$$h < \frac{S^*B\left(rB^2S^* - \alpha KI^*Z^*\right)\left(cB^2(r+cK) - r\alpha Z^*\right)}{\alpha^2\gamma\, K^2I^*Z^{*2}} = \frac{H_1}{H_2} \tag{11 c}$$

Therefore, according to the above analysis, the following theorem can be easily proved.

Theorem 3.

Assume that the positive equilibrium point F_3 exists in the $Int.R_+^3$. Then, F_3 is locally asymptotically stable if and only if conditions (11 a)-(11 c) hold.

Keeping the above in view the persistence conditions of system (1) are established in the following theorem.

Theorem 4.

Suppose that, F_2 exists in the $Int.R_+^2$ of the $SI-$ plane. Then the necessary condition for the persistence of system (1) is

$$\bar{I}(h-\theta-E_3) - \gamma(\theta+E_3) \geq 0 \tag{12}$$

However, the sufficient condition for the persistence of the system (1) is

$$\bar{I}(h-\theta-E_3) - \gamma(\theta+E_3) > 0 \tag{13}$$

Proof:

Clearly the solution of system (1) is bounded as shown in theorem 1. Now, since $\lambda_{23} = \frac{\bar{I}(h-\theta-E_3)-\gamma(\theta+E_3)}{(\gamma+\bar{I})}$ is the eigenvalue, which gives the stability in the positive direction orthogonal to the $SI-$plane. In addition, F_2 is a globally asymptotically stable in the $Int.R_+^2$ of the $SI-$ plane whenever it exists, therefore if condition (12) violates then $\lambda_{23} < 0$ and there are orbits in the positive cone approach F_2. Hence condition (12) is the necessary condition for the persistence. For the sufficient condition (13), it is easy to verify that; system (1) satisfies the following hypotheses:

(M1) $\frac{\partial f_1}{\partial I} = -\frac{r}{K} - c < 0$; $\frac{\partial f_1}{\partial Z} = 0$; $\frac{\partial f_2}{\partial S} = c > 0$; $\frac{\partial f_2}{\partial Z} = -\frac{\alpha}{(\gamma + I)} < 0$; $\frac{\partial f_3}{\partial S} = 0$; $\frac{\partial f_3}{\partial I} = \frac{\gamma h}{(\gamma + I)^2} > 0$; $f_2(0, I, Z) < 0$ and $f_3(0,0,Z) < 0$.

(M2) The prey grows to carrying capacity in the absence of predation, infection and harvesting, that is $f_1(0,0,0) = r > 0$, $\frac{\partial f_1}{\partial S}(S, I, Z) = -\frac{r}{K} < 0$. However, the predator population dies in the absence of the prey (i.e $f_3(0,0,0) = -(\theta + E_3) < 0$).

(M3) There are no equilibria in the IZ-plane and $SZ-$ plane.

(M4) In the absence of the predator the susceptible prey and then the infected prey can survive in the interior of positive quadrant of $SI-$ plane. Therefore, there exists an equilibrium point F_2 in the $SI-$ plane, which is globally asymptotically stable.

Hence, an application to the Freedman and Waltman persistence theorem [12], system (1) persists provided that condition (13) satisfied, and that completes the proof.

Finally, the global dynamics of system (1) in the $Int.R_+^3$ is investigated in the following theorem.

Theorem 5.

Assume that, the positive equilibrium point F_3 is locally asymptotically stable with

$$\frac{\alpha Z^*}{R}\left(I - I^*\right)^2 < \frac{cr}{r + cK}\left(S - S^*\right)^2 \tag{14}$$

Here $R = (\gamma + I)\left(\gamma + I^*\right)$. Then F_3 is a globally asymptotically stable in the $Int.R_+^3$.

Proof: Consider the following positive definite function

$$V(S,I,Z)=C_1\int_{S^*}^{S}\frac{(S-S^*)}{S}dS+C_2\int_{I^*}^{I}\frac{(I-I^*)}{I}dI+C_3\int_{Z^*}^{Z}\frac{(Z-Z^*)}{Z}dZ$$

Where C_1, C_2 and C_3 are positive constants to be determined. Now, straight forward calculations give that

$$\begin{aligned}\frac{dV}{dt}&=C_1\left[\frac{-r}{K}\right](S-S^*)^2+\left[\frac{-C_1 r}{K}-cC_1+cC_2\right](S-S^*)(I-I^*)\\&+\left[\frac{-C_2\alpha\gamma}{R}-\frac{C_2\alpha\, I^*}{R}+\frac{C_3 h\gamma}{R}\right](I-I^*)(Z-Z^*)\\&+\left[\frac{C_2\alpha\, Z^*}{R}\right](I-I^*)^2\end{aligned}$$

By choosing the positive constants as $C_1=\frac{cK}{(r+cK)}$; $C_2=1$; $C_3=\frac{\alpha(\gamma+I^*)}{h\gamma}$, then

$$\frac{dV}{dt}=-\frac{cr}{(r+cK)}(S-S^*)^2+\frac{\alpha\, Z^*}{R}(I-I^*)^2$$

Hence $\frac{dV}{dt}<0$ provided that condition (14) holds, and then V is a Lyapunov function. Therefore, F_3 is a globally asymptotically stable in the *Int.* R_+^3.

4. Numerical Simulation:

The globally dynamical behavior of the prey-predator system (1) with infectious disease in prey species is studied numerically. The solution of the system with a positive initial condition is obtained for biologically feasible range of parametric values. In all the cases being considered here the data is chosen depending on two factors: first, we wanted to investigate biologically reasonable harvested prey-predator system with disease, and the second, we wanted to determine if chaotic dynamics were likely. Therefore, as the solution of the system is bounded we expect that system (1) have a rich dynamic including limit cycle, and chaos. Consequently, system (1) is solved numerically, and then number of bifurcation diagrams are drawn between the Maximum value of predator and the control parameter.

The first bifurcation diagram, Fig. 1, shows the dynamical behavior of system (1) as a function of the intrinsic growth rate $(i.e \text{ parameter } r)$ in the range $15\le r\le 25$ keeping other parameters fixed at

$$\begin{gathered}K=400,\ c=0.06,\ \gamma=15.0,\ E_1=0.0,\ \alpha=15.5,\\ \lambda=3.4,\ E_2=0.0,\ \theta=8.0,\ h=10.0,\ E_3=0.0\end{gathered}\qquad(15)$$

The evidence for the existence of cascade of periodic doubling leading to chaos can be seen clearly in Fig. 1, and then the solution becomes chaotic in between there are periodic windows too. Finally, the predator species, still survive for $r > 23.7$ and the solution returns to periodic dynamic.

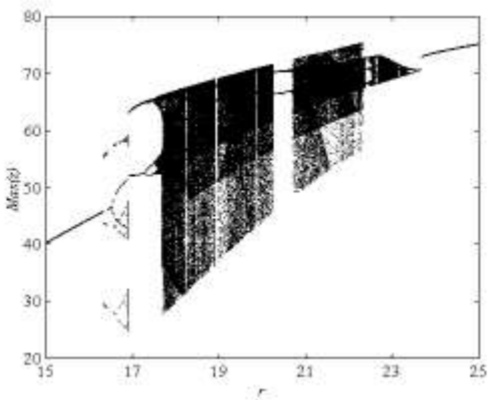


Fig. 1 Bifurcation diagram as a function of r in the range $15 \leq r \leq 25$ keeping other parameters fixed as in Eq. (15).

Now the projection of the attracting set of the solution of the system (1) in the $SI-$ plane is drawn in Fig. 2 (a-d) for the parametric values given in Eq. (15) with r=15, 16.55, 17, and 18 respectively. The figures show the evidence of periodic doubling leading to chaos.

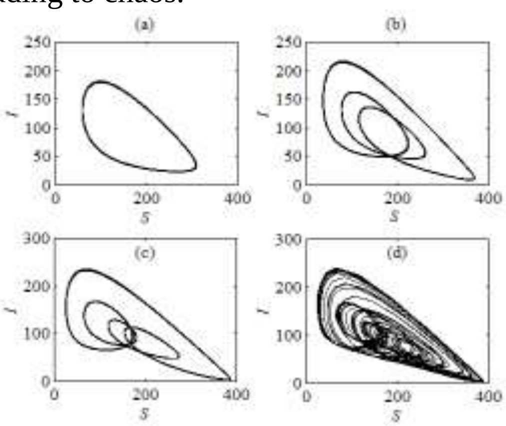


Fig. 2 The projection of the attracting set of the solution of system (1) in the S-I plane, at the parametric values given in Eq. (15). (a) $r = 15$. (b) $r = 16.55$. (c) $r = 17$. (d) $r = 18$.

Fig. 3 shows the bifurcation diagram as a function of the natural death rate of infected species $(i.e\ \ \lambda)$ in the range $1.5 \leq \lambda \leq 6.5$, keeping other parameters fixed as in Eq. (15) with $r = 18$. It is observed that there are number of periodic regions followed by the chaotic regions, and then the system return to periodic for $(4.8 < \lambda < 6.5)$. Finally the infected prey species and then the predator species approaches to extinction for $\lambda > 6.5$ due to the effect of increasing in the natural death rate of the infected species, which is the sole food for predator.

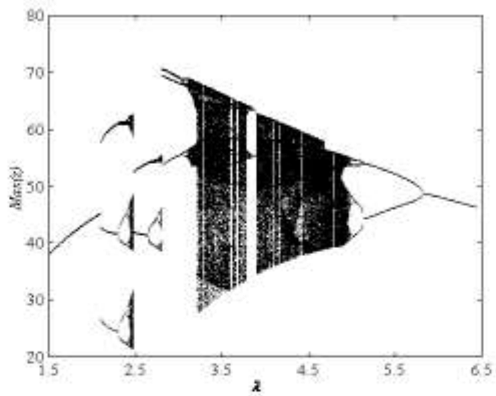


Fig. 3 Bifurcation diagram as a function of λ in the range $1.5 \le \lambda \le 6.5$, keeping the other parameters fixed as in Eq. (15) with $r = 18$.

Again, the projection of the attracting set of the solution of the system (1) in the $SI-$ plane is drawn in Fig. 4 (a-d) for the parametric values given in Eq. (15); $r = 18$, with λ =4.5, 5.0, 5.5, and 6.0 respectively. The figures show the evidence of return to periodic attractor from chaotic attractor through cascade of periodic halving.

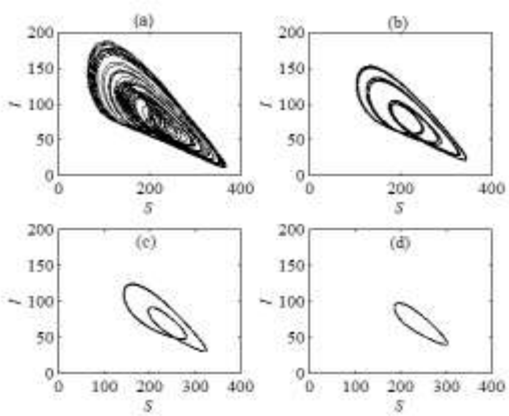


Fig. 4. The projection of the attracting sets of the solution of system (1) in the S-I plane, at the parametric values given in the Eq. (15) and $r = 18$. (a) $\lambda = 4.5$. (b) $\lambda = 5.0$. (c) $\lambda = 5.5$. (d) $\lambda = 6.0$.

Now the dynamical behavior of system (1), as a function of varying in the infection rate in the range $0.03 \le c \le 0.1$ keeping other parameters fixed as in Eq.

(15) with $r = 18$, is investigated in the bifurcation diagram given by Fig. 5. It is observed that, the solution start with periodic and then periodic doubling leading to chaos. The figure also shows the existence of narrow periodic windows within the chaotic region and decline of predator species for $c > 0.063$, approaching to extinction, due to the effect of increasing in the infection rate.

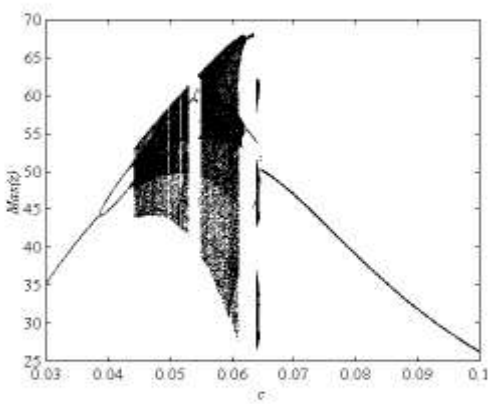


Fig. 5. Bifurcation diagram as a function of c in the range $0.03 \le c \le 0.1$, keeping other parameters fixed as in Eq. (15) with $r = 18$.

In the following, bifurcation diagrams as a function of harvest rate $0 \le E_1 \le 2$, $0 \le E_2 \le 3$, and $0 \le E_3 \le 0.6$ are drawn in Figs. 6(a-c) respectively keeping other parameters fixed as in Eq. (15) with $r = 18$. All these figures show the alternate between the chaotic and periodic dynamic, and then the solution approaches to periodic attractors through sequence of periodic halving. Moreover, increasing the bifurcation parameter E_i; $i = 1,2,3$ further, will leads to decaying in the predator species Z approaching to extinction.

Finally, the effect of infection rate on the dynamics of system (1), in case of existence harvesting, is investigated in Figs. 7(a-e) for the range $0.03 \le c \le 0.1$ and $E_2 = 0.2, 0.4, 1.0, 1.25, 1.5$ respectively holding the rest of parameters as:

$$r = 18, K = 400, \gamma = 15, E_1 = 0.4, \alpha = 15.5, \lambda = 3.4, \theta = 8, h = 10, E_3 = 0.02 \quad (16)$$

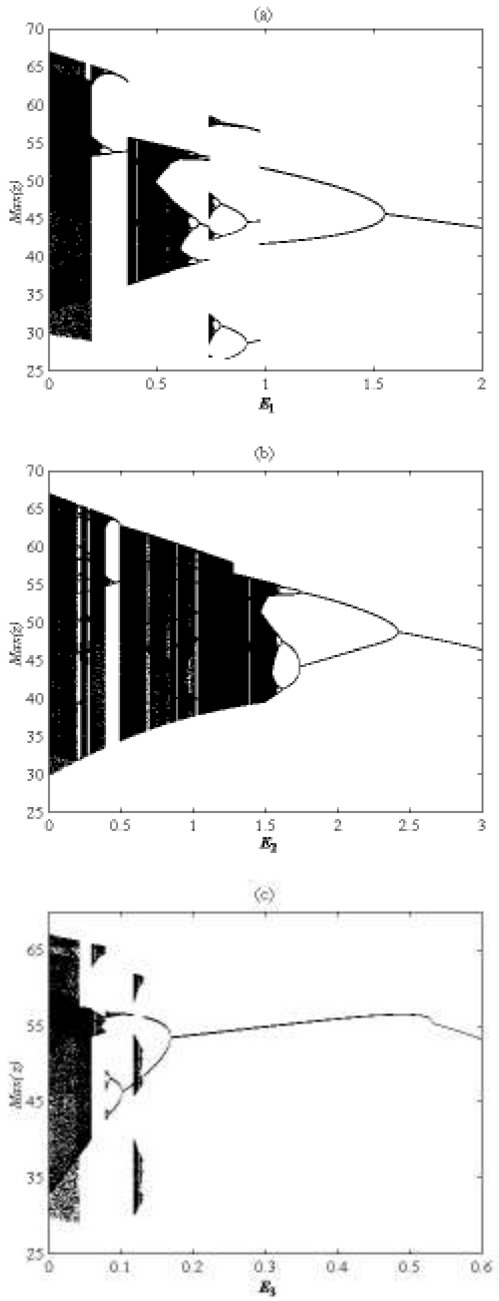
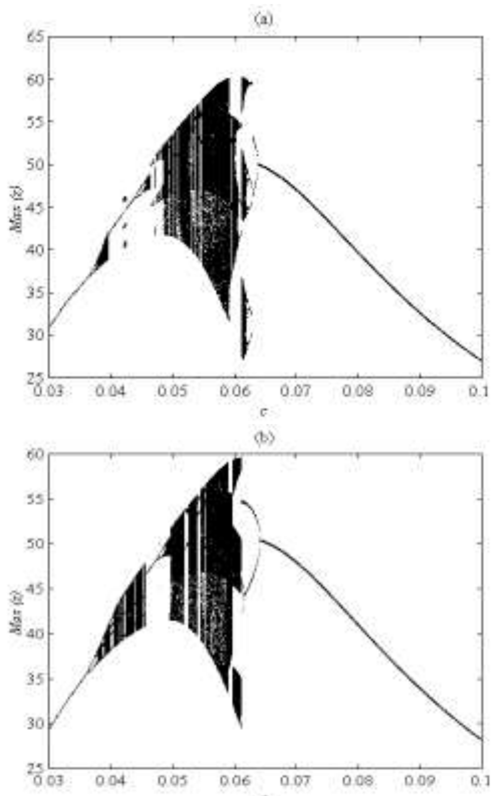


Fig. 6. Bifurcation diagram as a function of harvest rate keeping other parameters fixed as in Eq. (15) with $r = 18$: (a) $0 \le E_1 \le 2$; (b) $0 \le E_2 \le 3$; (c) $0 \le E_3 \le 0.6$.

According to Figs. 7(a-e), it is observed that, system (1) has rich dynamics including periodic, period doubling leading to chaos, chaos, and periodic halving. The predator species Z start increasing as c increases reaching its maximum, in the range $0.05 < c < 0.065$, due to abundance of its sole food, and then declines for $c > 0.07$, approaching to extinction due to rarity of its sole food. The chaotic regions become wider as the harvest rate E_2 increases from 0.2 to 0.4 keeping other parameters fixed as in Eq. (16). However these regions become narrower, as E_2 increases further, and they are fully disappear for $E_2 = 1.5$ and the system becomes periodic.



Journal of Al-Qadisiyah for Computer Science and Mathematics
3nd. Sinentific Conference 19-20/ APRIL -2011
Vol 3 No.2 Year 2011

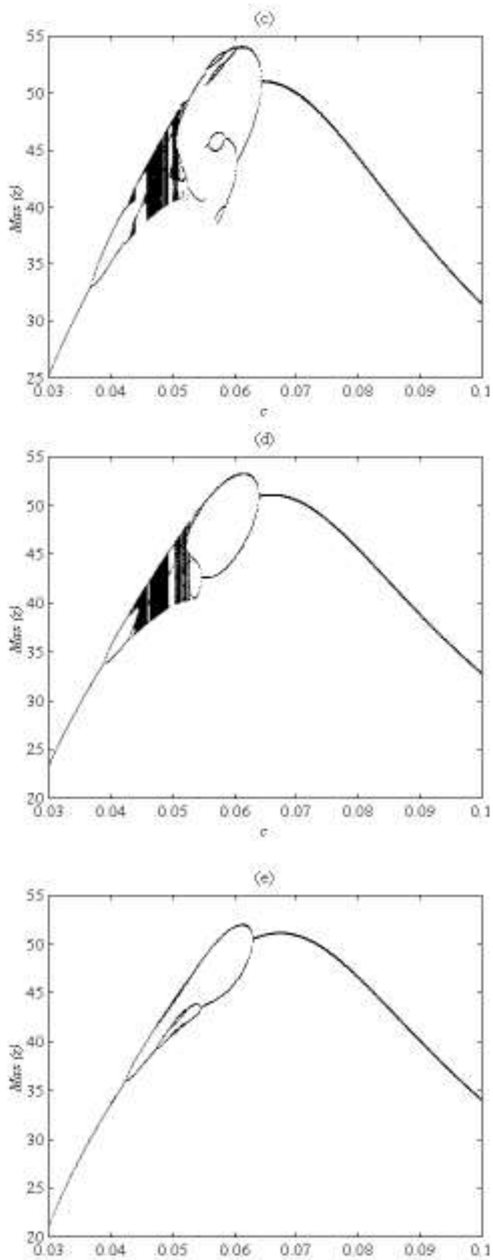


Fig. 7. Bifurcation diagram as a function of c in the range $0.03 \le c \le 0.1$, keeping other parameters fixed as in Eq. (16). (a) $E_2 = 0.2$.(b) $E_2 = 0.4$. (c) $E_2 = 1.0$. (d) $E_2 = 1.25$. (e) $E_2 = 1.5$.

The projection of the attracting set of the solution of the system (1) in the $SI-$ plane is drawn in Fig. 8(a-d) for the parametric values given in Eq. (16) with

$c = 0.06$ and E_2 =0.2, 0.4, 1.0, and 1.5 respectively. The figures show the evidence of return to periodic attractor from chaotic attractor as the harvest rate of infected prey increases.

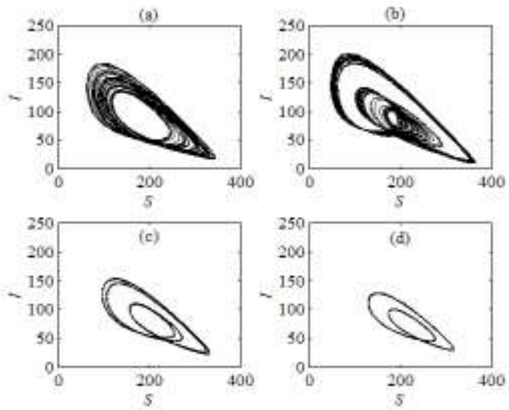


Fig. 8. The projection of the attracting set of the solution of system (1) in the S-I plane, at the parametric values given in Eq. (16) with $c = 0.06$. (a) $E_2 = 0.2$. (b) $E_2 = 0.4$. (c) $E_2 = 1.0$. (d) $E_2 = 1.5$.

5. Discussion and Conclusion:

A harvested prey-predator model with disease in prey population is proposed and analyzed. It is assumed that, in the absence of predation and harvesting, the prey species grows logistically. However, the predator species feeds on the infected preys (infected preys are weakened and hence become easier to predate) according to Holling type-Ⅱ functional response. Further, the mode of disease transmission within the prey population follows the simple law of mass action.
The qualitative dynamical behavior of the proposed model is investigated analytically. Numerical integration is used to investigate the global dynamical behavior of the model system (1). The objective is to explore the possibility of chaotic behavior. Extensive numerical simulations are carried out for various values of control parameters and for different sets of initial conditions. It has been shown that, the system (1) is very sensitive to the parameters $(r, \lambda, c, E_1, E_2, \text{and } E_3)$ and has different types of interesting attracting sets including periodic, periodic doubling, chaos, and periodic halving. Moreover, depending on the simulation results, the following conclusion can be drawn:

1. For small value of intrinsic growth rate of the susceptible prey population $(r < 16)$ the system (1) has a periodic attracting set. However, increasing the growth rate slightly increases the possibility of occurrence of chaotic dynamics. In fact, if the intrinsic growth rate of susceptible prey increases further, the number

of the infected prey increases and hence the predator population still survives and periodic dynamics is observed.

2. The situation is different in case of varying the infection rate (c) keeping other parameters fixed as in Eq. (16) with $r = 18$. In this case, for small value of $c \cong 0.03$ there is small number of infected prey and hence the system undergo periodic dynamic due to the rarity in the sole food of predator. As the infection rate increases slightly $(0.03 < c < 0.065)$ the number of infected prey start increases, consequently the number of predator species will be increase, and hence chaos is observed. Finally, increases the value of infection rate further $(0.065 < c)$ causes decreasing in the number of susceptible prey and then decreases the infected prey due to the effect of the predation. Accordingly the system approaches to the extinction.

3. Similar conclusions can be drawn, in case of increases the natural death rate of infected prey, as those in case of increasing of infection rate.

4. Obviously, the chaotic behavior of the system can be avoided and the system returns to periodic dynamic by increasing the harvest rates, up to specific values. However, increasing the harvest rates further will causes extinction of the system.

Finally, according to the above observation to control the chaotic behavior of the system (1), and hence control the disease, the value of intrinsic growth rate of the susceptible prey should not be very high.

REFERENCES

[1] D. Mukherjee, Uniform persistence in a generalized prey-predator system with parasite infection, *Biosystems*, **47,** pp.149-155, 1998.

[2] Y. Xiao and L. Chen, Modeling and analysis of a predator-prey model with disease in the prey, *Mathematical Biosciences*, **171,** pp.59-82, 2001.

[3] J. Chattopadhyay and O. Arino, A predator-prey model with disease in the prey, *Nonlinear Analysis*, **36,** pp.747-766, 1999.

[4] J. Chattopadhyay and S. Pal, Viral infection on phytoplankton zooplankton system-a mathematical model, *Ecological Modeling*, **151,** pp.15-28, 2002.

[5] D. Mukherjee, Persistence in a prey-predator system with disease in the prey, *J. of Biological systems*, **11,** pp.101-112, 2003.

[6] C.W. Clark, Mathematical Bioeconomics: The Optimal Management of Renewable Resources, Wiley, New York, USA, 1990.

[7] G. Dai and M. Tang, Coexistence region and global dynamics of a harvested predator prey system, *Siam. J. of Applied Mathematical*, **58,** pp.193-210, 1998.

[8] W.P. Cropper and D. Jr. DiResta, Simulation of a Biscayne Bay, Florida commercial sponge population: effects of harvesting after Hurricane Andrew, *Ecological Modeling*, **118,** pp.1-15, 1999.

Journal of Al-Qadisiyah for Computer Science and Mathematics
3nd. Sinentific Conference 19-20/ APRIL -2011
Vol 3 No.2 Year 2011

[9] J. Chattopadhyay, R.R. Sarkar, and G. Ghosal, Removal of infected prey prevent limit cycle oscillations in an infected prey-predator system-a mathematical study, *Ecological modeling*, **156,** pp. 13-121, 2002.
[10] J.H. Hale, Ordinary differential equation, New York, Wiley-Interscience, 1969.
[11] S. Wiggins, Introduction to applied non-linear dynamical system and chaos, spring-verlag, New York, Inc., 1990.
[12] H.I. Freedman and P. Waltman, Persistence in models of three interaction predator-prey populations, *Mathematical Biosciences*, **68,** pp.213-231, 1984.

الفوضى في نوذج حصاد الفريسة-المفترس مع مرض معدن في الفريسة

رائد كامل ناجي ***هبة عبد الله ابراهيم***

قسم الرياضيات – كلية العلوم – جامعة بغداد – بغداد - العراق

المستخلص:

في هذا البحث بحثنا نموذج حصاد الفريسة-المفترس مع مرض معدن في الفريسة. افترضنا ان المفترس يتغذى على الفريسة المصابة فقط اعتمادا على دالة هولنك من النوع الثاني. وجـود، وحدانية و حـــدود الحل للنموذج المقترح بحثـت. تحليـــلات الاستقرارية المحليـة للنموذج نوقشت. الشروط الضرورية والكافية للاصرار في النموذج وجدت. واخيرا الديناميكية الشاملة للنموذج درست تحليليا وعدديـــا. لاحظنـــا بأن النموذج قيد الدراسة يمتلك انـــواع مختلفة من السلوك الديناميكي بضمنها الفوضى.