



Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
 ISSN:2521-3504(online) ISSN:2074-0204(print)



A Rigorous Treatment of Supra Feebly Open Sets in Supra Topological Spaces

Marwa Qasim Atshan

¹ Department of Mathematics, College of Education, University of AL-Qadisiyah, Diwaniyah-Iraq . marwa.atshan@qu.edu.iq

ARTICLE INFO

Article history:

Received: 23 /04/2026

Rrevised form: 15 /05/2026

Accepted : 17 /05/2026

Available online: 30 /06/2026

Keywords:

Supra Topology, feebly open set,
supra feebly open set, supra open.

ABSTRACT

This work presents a systematic revision and rigorous extension of the classical theory on supra feebly-open sets within supra topological spaces. We formally define a supra feebly-open (**s.f.o.**) subset A by the presence of a supra open set $U \in \mathcal{W}$ such that $U \subseteq A \subseteq \text{scl}_{\text{si}}^{\mathcal{W}}(U)$ and we rigorously establish the equivalent formulation $A \subseteq \text{cl}^{\mathcal{W}}(\text{Int}^{\mathcal{W}}(A))$. The manuscript explores the fundamental properties of these sets, presents the supra feebly interior and supra feebly closure operators, and meticulously charts their inclusion relations with supra α -open, supra semi-open, supra pre-open, supra b-open, and supra β -open sets. This study enhances logical consistency, standardizes notation and puts the results in context of the overall generalized weak openness. The theoretical advantages provide strong useful foundation for future topological applications in the areas of digital topology, rough set theory and fuzzy/neutrosophic supra structures. Mathematical consistency, logical coherence, and formal scholarly presentation have been meticulously ensured throughout the presentation to provide future topological uses. The principal results of this work establish that: (i) every supra feebly-open set is supra semi-open; (ii) the class of supra feebly-open sets is closed under arbitrary unions but not under finite intersections; (iii) the supra feebly interior and closure operators satisfy standard monotonicity and duality properties; and (iv) precise inclusion relations with supra alpha-open, supra pre-open, supra b-open, and supra beta-open sets are rigorously characterized.

MSC..

<https://doi.org/10.29304/jqcm.2026.18.32840>

1. Introduction

Supra topological spaces were introduced by Mashhour et al.in 1983 [1]. They provide a natural generalization of conventional topological spaces. Supra topologies offer a more flexible framework suitable for simulating systems where conventional topological axioms are too limiting, such systems arise in rough set theory, digital topology, and theoretical computer science. Various generalized classes of supra open sets have been studied, including supra α -open sets [10], supra pre-open sets [8], supra b-open sets [9], supra semi-open sets [12], and supra β -open related notions [11]. In classical topology feebly open sets represent a weak-form open set initially studied in ordinary topology spaces [3], [6].

The feebly-open sets were introduced and studied by various authors, including Maheshwari and Jain [3] and Noiri [6] in classical topology. A subset A of a topological space (X, τ) is called feebly-open if there exists an open set $U \in \tau$ such that $U \subseteq A \subseteq \text{Cl}(U)$. it is well known that, in ordinary topological spaces, the class of feebly-open sets coincides

*Corresponding author: Marwa Qasim Atshan

Email addresses:

Communicated by 'sub etitor'

with the class of semi-open sets introduced by Levine [14]. This equivalence should however be re-examined in the supra topological context, as the lack of the finite intersection axiom can result in different behaviors.

The main aim of this manuscript is to generalize the classical concept of feebly openness to the supra topological context, thus filling the gaps that exist between supra generalized open classes. We formally introduce in supra feebly-open sets, establish their basic properties, introduce corresponding interior and closure operators, and precisely chart out their inclusion-relations with known supra generalized open sets. This work provides a rigorous foundation for later investigations in supra continuity, supra separation axioms, and formal applications in the fields of data science and physics [7].

Specifically, Throughout this work, (X, W) denotes a supra topological space, where X is a nonempty set and $W \subseteq P(X)$ is a supra topology on X .

2. Preliminaries and Fundamental Definitions

In this section, we present the definitions concept of feebly open sets, feebly closed sets, and some properties of these supra topological, with their fundamental properties including the feebly interior and feebly closure of any operators. Also, we investigate the union and intersection of any finite feebly open sets in the present section. All definitions are aligned with established literature [3], [8]–[10].

Definition 2.1 (Supra topology) [1].

A subfamily $\mathcal{W}$ of the power set $\mathcal{P}(\mathcal{S})$ of nonempty set $\mathcal{S}$ is called a supra topology on $\mathcal{S}$, if it $\mathcal{W}$ satisfies the following conditions :

1- $\emptyset, \mathcal{S} \in \mathcal{W}$,

2- $\mathcal{W}$ is closed under arbitrary unions, $\{\mathcal{W}_i : i \in I\} \in \mathcal{W}$ for any arbitrary collection $\{\mathcal{W}_i : i \in I\} \subseteq \mathcal{W}$.

The pair $(\mathcal{S}, \mathcal{W})$ is called a supra topological space (briefly, sup. topo. space). The member of $\mathcal{W}$ are called the supra open set. The family of all supra open set is denoted by $\xi_o(\mathcal{S})$ or simply $\mathcal{W}$.

Definitions 2.2 (Supra interior and supra closure) [8], [9] .

Let $(\mathcal{S}, \mathcal{W})$ be a supra topological space and $\mathcal{R} \subseteq \mathcal{S}$

(i) The symbol $cl^{\mathcal{W}}(\mathcal{R})$ means the supra closure and defined as $cl^{\mathcal{W}}(\mathcal{R}) = \{B \subseteq \mathcal{S} : \mathcal{S} \setminus B \in \mathcal{W} \text{ and } \mathcal{R} \subseteq B\}$.

(ii) For $\mathcal{R} \subseteq \mathcal{S}$, the supra interior of a set $\mathcal{R}$ is denoted by $int^{\mathcal{W}}(\mathcal{R})$ and is defined as $int^{\mathcal{W}}(\mathcal{R}) = \{U \in \mathcal{W} : U \subseteq \mathcal{R}\}$.

(iii) Let $(\mathcal{S}, \mathcal{J})$ be a topological space. A supra topology $\mathcal{W}$ on $\mathcal{S}$ is said to be associated with $\mathcal{J}$, if $\mathcal{J} \subseteq \mathcal{W}$.

Definition 2.3 (Generalized supra open sets) [8]–[12]

Let $(\mathcal{S}, \mathcal{W})$ be a supra topological space. A subset $\mathcal{R}$ of $\mathcal{S}$ is said to be:

(a) Supra semi-open $su_{s.o}$ [12] if $\mathcal{R} \subseteq cl^{\mathcal{W}}(int^{\mathcal{W}}(\mathcal{R}))$.

(b) Supra b-open su_{b-o} [9] if $\mathcal{R} \subseteq cl^{\mathcal{W}}(int^{\mathcal{W}}(\mathcal{R})) \cup int^{\mathcal{W}}(cl^{\mathcal{W}}(\mathcal{R}))$.

(c) Supra α -open $su_{\alpha-o}$ [10] if $\mathcal{R} \subseteq int^{\mathcal{W}}(cl^{\mathcal{W}}(int^{\mathcal{W}}(\mathcal{R})))$.

(d) Supra pre-open $su_{p.o}$ [8] if $\mathcal{R} \subseteq int^{\mathcal{W}}(cl^{\mathcal{W}}(\mathcal{R}))$.

(e) Supra β -open $su_{\beta-o}$ [11] if $\mathcal{R} \subseteq cl^{\mathcal{W}}(int^{\mathcal{W}}(cl^{\mathcal{W}}(\mathcal{R})))$.

The families of all supra semi-open, supra b -open, supra α -open, supra pre-open, and supra β -open sets are denoted by $SU_{s.o}$, $SU_{b.o}$, $SU_{\alpha.o}$, $SU_{p.o}$, and $SU_{\beta.o}$ respectively.

Definition 2.4 (Classical feebly open set) [3],[6].

A set $\mathcal{R}$ in a topological space $\mathcal{S}$ is called **feebly-open** (or $\mathcal{F}$ -open) if there exists an open set U such that $U \subseteq \mathcal{R} \subseteq cl_{si}(U)$, which cl_{si} denotes the **semi-closure** operator. Equivalently, $\mathcal{R}$ is $\mathcal{F}$ -open if and only if $\mathcal{R} \subseteq cl_{si}(int(\mathcal{R}))$, the $\mathcal{F}$ -closed is the complement of the $\mathcal{F}$ -open set.

Remark 2.1 [6]

Let $(\mathcal{S}, \mathcal{W})$ be a supra topological space. If $\mathcal{R} \subseteq \mathcal{S}$ is an open set, then $\mathcal{R}$ is $\mathcal{F}$ -open.

The converse does not generally hold, as illustrated below:

Example 2.1

If $\mathcal{S} = \{\tau, q, p\}$ and $\mathcal{J} = \{\emptyset, \mathcal{S}, \{\tau\}\}$ be a topology on $\mathcal{S}$ let $\mathcal{A} = \{\tau, q\} \subseteq \mathcal{S}$ is $\mathcal{F}$ -open because

$int(\mathcal{A}) = \{\tau\}$ and $cl_{si}(\{\tau\}) = \mathcal{S}$, Thus meaning $\mathcal{A} \subseteq cl_{si}(int(\mathcal{A}))$, So $\mathcal{A}$ is $\mathcal{F}$ -open. However, $\mathcal{A} \notin \mathcal{J}$. so $\mathcal{A}$ is not open.

Definition 2.5 (Supra feebly-open and supra feebly-closed sets [3]).

Suppose that $(\mathcal{S}, \mathcal{W})$ is a supra topological space. A subset $\mathcal{A}$ of $\mathcal{S}$ is called **supra feebly open** (briefly, $su_{\mathcal{F}_e, o}$ or $SUFeO$) if there exists supra open sets $U \in \mathcal{W}$ such that $U \subseteq \mathcal{A} \subseteq cl_{si}^{\mathcal{W}}(U)$, or equivalently if $\mathcal{A} \subseteq cl_{si}^{\mathcal{W}}(int^{\mathcal{W}}(\mathcal{A}))$.

The complement of supra feebly open of $\mathcal{A}$ is called **supra feebly-closed** (briefly, $su_{\mathcal{F}_e, c}$); equivalently, if there exist a supra closed set $\mathcal{B}$ of $\mathcal{S}$ such that $int_{si}^{\mathcal{W}}(\mathcal{B}) \subseteq \mathcal{A} \subseteq \mathcal{B}$.

Example 2.2

Suppose that $\mathcal{S} = \{x, y, z\}$ and $\mathcal{W} = \{\emptyset, \mathcal{S}, \{x, y\}, \{y, z\}\}$, it is clear that $\mathcal{A} = \{y, z\} \subseteq \mathcal{S}$ is $su_{\mathcal{F}_e, o}$, since there exist $\mathcal{U} = \{y, z\}$ such that $cl_{si}^{\mathcal{W}}(\mathcal{U}) = \mathcal{S}$. Satisfying $U \subseteq \mathcal{A} \subseteq cl_{si}^{\mathcal{W}}(U)$

Definition 2.6 (Supra feebly interior and supra feebly closure).

Let $\mathcal{A}$ be a subset from $(\mathcal{S}, \mathcal{W})$, We define the following operators:

(1) The supra feebly interior of a set $\mathcal{A}$ is denoted by $int_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A})$, and is defined as:

$$int_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A}) = \{U \subseteq \mathcal{S} \mid U \text{ is s.f.o. and } U \subseteq \mathcal{A}\}.$$

(2) The supra feebly-closure of a set $\mathcal{A}$ is denoted by $cl_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A})$, and is defined as:

$$cl_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A}) = \{F \subseteq \mathcal{S} : F \text{ is s.f.c. and } \mathcal{A} \subseteq F\}.$$

Lemma 2.1

(1) An arbitrary union of $su_{\mathcal{F}_e, o}$ sets is $su_{\mathcal{F}_e, o}$ set. [1]

(2) An arbitrary union of $su_{\mathcal{F}_e, o}$ sets is $su_{\mathcal{F}_e, o}$ set.

(3) An arbitrary intersection of $su_{\mathcal{F}_e, c}$ sets is $su_{\mathcal{F}_e, c}$ set.

Remarks 2.2

(1) A finite intersection of $su_{\mathcal{F}_e, o}$ sets not necessarily $su_{\mathcal{F}_e, o}$.

(2) A finite union of $su_{\mathcal{F}_e, c}$ sets not necessarily $su_{\mathcal{F}_e, c}$.

Example 2.3

Let $\mathcal{S} = \{r, y, m, s\}$, $\mathcal{W} = \{\emptyset, \mathcal{S}, \{r\}, \{r, m\}, \{m, s\}, \{r, m, s\}\}$, Note that $\mathcal{A} = \{r, m\} \subseteq \mathcal{S}$ and $\mathcal{B} = \{m, s\} \subseteq \mathcal{S}$ are both $su_{\mathcal{F}_e.o}$ sets, but $\mathcal{A} \cap \mathcal{B} = \{m\}$ is not $su_{\mathcal{F}_e.o}$ set. Also $Q = \{r, y\} \subseteq \mathcal{S}$ and $\mathcal{T} = \{y, s\} \subseteq \mathcal{S}$ are both $su_{\mathcal{F}_e.c}$ sets, but $Q \cup \mathcal{T} = \{r, y, s\}$ is not $su_{\mathcal{F}_e.c}$ set.

Lemma 2.2

Let $\mathcal{A}$ be a subset from $(\mathcal{S}, \mathcal{W})$, then:

- (1) $\text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A})$ is $su_{\mathcal{F}_e.o}$ and $\text{cl}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A})$ is $su_{\mathcal{F}_e.c}$.
- (2) $\text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A}) \subseteq \mathcal{A}$ and $\mathcal{A} \subseteq \text{cl}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A})$.
- (3) $\mathcal{A} = \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A})$ if and only if $\mathcal{A}$ is $su_{\mathcal{F}_e.o}$ and $\mathcal{A} = \text{cl}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A})$ if and only if $\mathcal{A}$ is $su_{\mathcal{F}_e.c}$.

Proof

Proof. (1)–(3) Follow clearly by Definition 2.6, part (2), and part (3) of Lemma 2.1. Specially, since the arbitrary union of supra feebly open sets is supra feebly open, the union defining the interior is itself feebly open.

Lemma 2.3

Let $\mathcal{A}, \mathcal{B}$ are two subsets of $(\mathcal{S}, \mathcal{W})$, then:

- (1) $\text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{S}/\mathcal{A}) = \mathcal{S} / \text{cl}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A})$.
- (2) $\text{cl}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{S}/\mathcal{A}) = \mathcal{S} / \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A})$.
- (3) $\text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A}) \subseteq \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{B}) \subseteq \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A} \cup \mathcal{B})$.
- (4) $\text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A} \cap \mathcal{B}) \subseteq \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A}) \cap \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{B})$.
- (5) $\text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A}) \cup \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{B}) \subseteq \text{cl}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A} \cup \mathcal{B})$.
- (6) $\text{cl}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A}) \cup \text{cl}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{B}) \subseteq \text{cl}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A} \cup \mathcal{B})$.
- (7) $\text{int}^{\mathcal{W}}(\mathcal{A}) \subseteq \text{cl}^{\mathcal{W}}(\mathcal{A})$ [1].
- (8) $\text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A}) \subseteq \text{cl}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A})$
- (9) $\text{int}_{\text{si}}^{\mathcal{W}}(\mathcal{A}) \subseteq \text{cl}^{\mathcal{W}}(\mathcal{A})$ [12].
- (10) $\text{int}^{\mathcal{W}}(\mathcal{A}) \subseteq \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A})$ and $\text{cl}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A}) \subseteq \text{cl}^{\mathcal{W}}(\mathcal{A})$.

Proof

Part (1) and (2) follow directly from Definition 2.6, and De Morgan's laws.

(3) Let $x \in \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A}) \cup \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{B})$, so either $x \in \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A})$ or $x \in \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{B})$, if $x \in \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A})$, thus by definition (2.6) $x \in \bigcup Q_i$ where Q_i are $su_{\mathcal{F}_e.o}$ for all i and $Q_i \subseteq \mathcal{A}$, so

$x \in Q_i \subseteq \mathcal{A} \subseteq \mathcal{A} \cup \mathcal{B}$, hence $x \in \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A} \cup \mathcal{B})$, or if $x \in \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{B})$ is symmetric, thus $x \in \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A} \cup \mathcal{B})$, hence $\text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A}) \cup \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{B}) \subseteq \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A} \cup \mathcal{B})$.

(4) Similar to the proof of part (3) using the fact that $A \cap B \subseteq A$ and $A \cap B \subseteq B$, which implies the interior of the intersection must be contained in the intersection of the interiors. (5)–(10) Follow from Definitions 2.2, Lemmas 2.1 and 2.2.

Example 2.4

In the setting of Example 2.3, if $\mathcal{A} = \{r, y, s\} \subseteq \mathcal{S}$ and $\mathcal{B} = \{y, m, s\} \subseteq \mathcal{S}$, so $\text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A} \cup \mathcal{B}) = \mathcal{S}$, while $\text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A}) = \{r\}$ and $\text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{B}) = \{m, s\}$. Note that $\text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A} \cup \mathcal{B}) \not\subseteq \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A}) \cup \text{int}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{B}) = \{r, m, s\}$. also $\text{cl}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A}) = \mathcal{S}$ and $\text{cl}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{B}) = \{y, m, s\}$, $\text{cl}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A} \cap \mathcal{B}) = \{y, s\}$, but $\text{cl}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A} \cap \mathcal{B}) \not\subseteq \text{cl}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{A}) \cap \text{cl}_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{B}) = \{y, m, s\}$.

Lemma 2.4

Let $(\mathcal{S}, \mathcal{W})$ be a supra topological space, and let $\mathcal{R}, \mathcal{B} \subseteq \mathcal{S}$, then :

- (1) If $\mathcal{R} \subseteq \mathcal{B}$, then $cl_{si}^{\mathcal{W}}(\mathcal{R}) \subseteq cl_{si}^{\mathcal{W}}(\mathcal{B})$ [12].
- (2) If $\mathcal{R} \subseteq \mathcal{B}$, then $int_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{R}) \subseteq int_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{B})$ and $cl_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{R}) \subseteq cl_{\mathcal{F}_e}^{\mathcal{W}}(\mathcal{B})$.

Proof

(2) Directly by using Definition (2.6), and the monotonicity of the subset relation over unions and intersections.

3. Relations Between Supra Feebly-Open Sets and Other Generalized Supra Open Sets

In this section we studied the relations between our set $su_{\mathcal{F}_e.o}$ and other types of supra collections like $su_{.o}$, $su_{.b-o}$, $su_{.si.o}$, $su_{.b-o}$, $su_{.p.o}$, etc. We introduced Figure 1 summarizes these relations diagrammatically. Table 1 (following Figure 1) provides explicit counterexamples confirming that none of the arrows in Figure 1 can be reversed, thereby showing that all inclusion relations are strict.

Proposition 3.1

Let $(\mathcal{S}, \mathcal{W})$ be a supra topological space. if $\mathcal{A} \subseteq \mathcal{S}$ is $su_{.o}$ set, then $\mathcal{A}$ is $su_{\mathcal{F}_e.o}$.

Proof

Let $\mathcal{A}$ be supra open. Since $\mathcal{A} \in \mathcal{W}$, we have $int^{\mathcal{W}}(\mathcal{A}) = \mathcal{A}$. Setting $U = \mathcal{A}$, we get $U \in \mathcal{W}$ with $U \subseteq \mathcal{A}$ and $\mathcal{A} \subseteq cl_{si}^{\mathcal{W}}(U) = cl_{si}^{\mathcal{W}}(\mathcal{A})$. Hence $\mathcal{A}$ satisfies Definition 2.5, so $\mathcal{A}$ is $su_{\mathcal{F}_e.o}$.

The Statements above are untrue in the opposite direction in general , as demonstrated by the subsequent cases.

Examples 3.1

- (1) Let $\mathcal{S} = \{x, y, m, w\}$, $\mathcal{W} = \{\emptyset, \mathcal{S}, \{x, m\}, \{x, y\}, \{x, y, m\}\}$. Note that $\mathcal{A} = \{x, y, w\} \subseteq \mathcal{S}$ is $su_{\mathcal{F}_e.o}$, but not $su_{.o}$.
- (2) Let $\mathcal{S} = \{x, g, z, t\}$, $\mathcal{W} = \{\emptyset, \mathcal{S}, \{x, t\}, \{x, z, t\}, \{g, z, t\}\}$. Note that $\mathcal{A} = \{x, g, z\} \subseteq \mathcal{S}$ is $su_{.p.o}$ but not $su_{\mathcal{F}_e.o}$.

Proposition 3.2

Let $(\mathcal{S}, \mathcal{W})$ be a sup. topo. space, let $\mathcal{A} \subseteq \mathcal{S}$:

- (1) If $\mathcal{A}$ is $su_{.o}$ then $\mathcal{A}$ is $su_{.si.o}$.
- (2) If $\mathcal{A}$ is $su_{.o}$ then $\mathcal{A}$ is $su_{.p.o}$.
- (3) If $\mathcal{A}$ is $su_{.o}$ then $\mathcal{A}$ is $su_{.b-o}$.
- (4) If $\mathcal{A}$ is $su_{.o}$ then $\mathcal{A}$ is $su_{.b-o}$.

Proof

(1) Let $\mathcal{A}$ be a $su_{.o}$, since $\mathcal{A} \subseteq cl^{\mathcal{W}}(\mathcal{A})$, therefore $\mathcal{A} = int^{\mathcal{W}}(\mathcal{A}) \subseteq cl^{\mathcal{W}}(int^{\mathcal{W}}(\mathcal{A}))$. Thus we get the required results, and analogous formal phrasing for parts (2) – (4) relying on the fact that for any supra open set $\mathcal{A} = int^{\mathcal{W}}(\mathcal{A})$, the successive applications of closure and interior operators preserve the superset relation.

The converses of the above statement do not hold the general, as illustrated by the following examples

Examples 3.2

- (1) Let $\mathcal{S} = \{1, 2, 3, 4\}$, $\mathcal{W} = \{\emptyset, \mathcal{S}, \{1, 3\}, \{3, 4\}, \{1, 3, 4\}\}$. Note that $\mathcal{A} = \{1, 2, 3\} \subseteq \mathcal{S}$ is $su_{.si.o}$ but not $su_{.o}$.

(2) Suppose $\mathcal{S} = \{j, i, h, d\}$, $\mathcal{W} = \{\emptyset, \mathcal{S}, \{j, i\}, \{i, h\}, \{j, i, h\}\}$, it is clear that $\mathcal{M} = \{j, i, d\} \subseteq \mathcal{S}$ is $su_{p.o}$ however not su_{o} .

(3) Let $\mathcal{S} = \{\mathcal{L}, \mathcal{N}, q, p\}$, $\mathcal{W} = \{\emptyset, \mathcal{S}, \{\mathcal{L}\}, \{\mathcal{L}, q\}, \{q, p\}, \{\mathcal{L}, q, p\}\}$, we can easily show that $\mathcal{N} = \{\mathcal{L}, \mathcal{N}\} \subset \mathcal{S}$ is su_{b-o} but not su_{o} .

(4) Let $\mathcal{S} = \{1, 2, 3, 4\}$, $\mathcal{W} = \{\emptyset, \mathcal{S}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$, where $\mathcal{K} = \{1, 3, 4\} \subseteq \mathcal{S}$ is $su_{\beta-o}$ however not su_{o} .

Proposition 3.3

Let $(\mathcal{S}, \mathcal{W})$ be a sup. top. space, let $\mathcal{R} \subseteq \mathcal{S}$:

(1) If $\mathcal{R}$ is $su_{\mathcal{F}_{e.o}}$ set, then $\mathcal{R}$ is $su_{si.o}$.

(2) If $\mathcal{R}$ is $su_{\mathcal{F}_{e.o}}$ and su_{o} , therefore $\mathcal{R}$ is su_{b-o} .

(3) If $\mathcal{R}$ is $su_{\mathcal{F}_{e.o}}$ and su_{o} , then $\mathcal{R}$ is $su_{\beta-o}$.

Proof

(1) Let $\mathcal{R}$ be $su_{\mathcal{F}_{e.o}}$, so there exists $\mathcal{H}$ which is su_{o} such that $\mathcal{H} \subseteq \mathcal{R} \subseteq cl_s^{\mathcal{W}}(\mathcal{H})$. By Proposition 3.2 Part (1) we obtain $cl_{si}^{\mathcal{W}}(\mathcal{H}) \subseteq cl^{\mathcal{W}}(\mathcal{H})$. Thus $\mathcal{H} \subseteq \mathcal{R} \subseteq cl^{\mathcal{W}}(\mathcal{H})$, and $\mathcal{H} \subseteq int^{\mathcal{W}}(\mathcal{R})$ implying $\mathcal{R} \subseteq cl^{\mathcal{W}}(int^{\mathcal{W}}(\mathcal{R}))$, which means $\mathcal{R}$ is $su_{si.o}$.

(2) Suppose $\mathcal{R}$ is $su_{\mathcal{F}_{e.o}}$, by Part (1) we conclude that $\mathcal{R} \subseteq cl^{\mathcal{W}}(int^{\mathcal{W}}(\mathcal{R}))$. Also by Proposition 3.1 $\mathcal{R} \subseteq int^{\mathcal{W}}(cl^{\mathcal{W}}(\mathcal{R}))$. Therefore $\mathcal{R} = \mathcal{R} \cup \mathcal{R} \subseteq cl^{\mathcal{W}}(int^{\mathcal{W}}(\mathcal{R})) \cup int^{\mathcal{W}}(cl^{\mathcal{W}}(\mathcal{R}))$. Thus $\mathcal{R}$ is su_{b-o} .

(3) Let $\mathcal{R}$ is $su_{\mathcal{F}_{e.o}}$ and su_{o} so by Proposition 3.2 Part (2) we get $\mathcal{R} \subseteq int^{\mathcal{W}}(cl^{\mathcal{W}}(\mathcal{R}))$, hence $cl^{\mathcal{W}}(\mathcal{R}) \subseteq cl^{\mathcal{W}}(int^{\mathcal{W}}(cl^{\mathcal{W}}(\mathcal{R})))$, since $\mathcal{R} \subseteq cl^{\mathcal{W}}(\mathcal{R})$. Thus $\mathcal{R} \subseteq cl^{\mathcal{W}}(int^{\mathcal{W}}(cl^{\mathcal{W}}(\mathcal{R})))$ is $su_{\beta-o}$.

The Statement above is untrue in its opposite direction in general, as shown the Subsequent cases.

Examples 3.3

(1) Let $\mathcal{S} = \{\ell, m, n\}$, $\mathcal{W} = \{\emptyset, \mathcal{S}, \{\ell\}, \{n\}, \{\ell, n\}\}$. Note that $\mathcal{A} = \{\ell, m\} \subseteq \mathcal{S}$ is $su_{si.o}$ then not $su_{\mathcal{F}_{e.o}}$.

(2) Let $\mathcal{S} = \{m, t, r, w\}$, $\mathcal{W} = \{\emptyset, \mathcal{S}, \{m, w\}, \{t, r\}, \{m, r, w\}, \{t, r, w\}\}$. Note that $\mathcal{A} = \{m, t, r\} \subseteq \mathcal{S}$ is $su_{p.o}$ and $su_{\beta-o}$ but not $su_{\mathcal{F}_{e.o}}$.

(3) Suppose $\mathcal{S} = \{m, n, h, w\}$, $\mathcal{W} = \{\emptyset, \mathcal{S}, \{m\}, \{m, n\}, \{n, h\}, \{m, n, h\}\}$. Note that $\mathcal{A} = \{m, h, w\} \subseteq \mathcal{S}$ is su_{b-o} and not $su_{\mathcal{F}_{e.o}}$.

Proposition 3.4

Suppose $(\mathcal{S}, \mathcal{W})$ be a sup. topo. space, let $\mathcal{A} \subseteq \mathcal{S}$:

(1) If $\mathcal{A}$ $su_{\alpha-o}$, then $\mathcal{A}$ $su_{si.o}$ [10].

(2) If $\mathcal{A}$ su_{o} , then $\mathcal{A}$ $su_{\alpha-o}$ [10].

(3) If $\mathcal{A}$ $su_{p.o}$, then $\mathcal{A}$ su_{b-o} [8].

(4) If $\mathcal{A}$ $su_{\alpha-o}$, then $\mathcal{A}$ $su_{p.o}$ [8].

Proposition 3.5

Suppose $(\mathcal{S}, \mathcal{W})$ be a sup. topo. space, let $\mathcal{M} \subseteq \mathcal{S}$, then:

(1) If $\mathcal{M}$ is $su_{\mathcal{F}_{e.o}}$ and su_{o} set, then $\mathcal{M}$ is $su_{\alpha-o}$.

(2) If $\mathcal{M}$ is su_{b-o} , then $\mathcal{M}$ is $su_{\beta-o}$.

(3) If $\mathcal{M}$ is $su_{.si.o}$, then $\mathcal{M}$ is $su_{.\beta-o}$.

Proof

(1) Let $\mathcal{M}$ is $su_{.\mathcal{F}_e.o}$ set, by Proposition 3.3 Part(1) $\mathcal{M} \subseteq cl^w(int^w(\mathcal{M}))$, since $\mathcal{M}$ is supra open $int^w(\mathcal{M}) = int^w(cl^w(int^w(\mathcal{M})))$ and $\mathcal{M}$ is $su_{.o}$. Therefore $\mathcal{M} = int_{\mathcal{F}_e}^w(\mathcal{M}) = int^w(\mathcal{M}) \subseteq int^w(cl^w(int^w(\mathcal{M})))$. Thus $\mathcal{M}$ is $su_{.\alpha-o}$.

(2) Let $\mathcal{M}$ is $su_{.b-o}$, so $\mathcal{M} \subseteq cl^w(int^w(\mathcal{M})) \cup int^w(cl^w(\mathcal{M}))$, therefore either $\mathcal{M} \subseteq cl^w(int^w(\mathcal{M}))$ or $\mathcal{M} \subseteq int^w(cl^w(\mathcal{M}))$.

Case 1: If $\mathcal{M} \subseteq cl^w(int^w(\mathcal{M}))$, then $\mathcal{M} \subseteq cl^w(int^w(\mathcal{M})) \subseteq cl^w(int^w(cl^w(int^w(\mathcal{M}))))$, so $\mathcal{M}$ is $su_{.\beta-o}$.

Case 2: If $\mathcal{M} \subseteq int^w(cl^w(\mathcal{M}))$, then since $int^w(cl^w(\mathcal{M})) \subseteq cl^w(int^w(cl^w(\mathcal{M})))$, we get $\mathcal{M} \subseteq cl^w(int^w(cl^w(\mathcal{M})))$, so $\mathcal{M}$ is $su_{.\beta-o}$.

(3) clearly This is immediate from Proposition 3.4.

The Statement above is untrue in its opposite direction in general. As shown the Subsequent cases.

Example 3.4

Let $\mathcal{S} = \{v, u, t, q\}$, $\mathcal{W} = \{\emptyset, \mathcal{S}, \{v\}, \{v, t\}, \{t, q\}, \{v, t, q\}\}$. Note that:

$\mathcal{A} = \{u, t\} \subset \mathcal{S}$ is $su_{.\beta-o}$ but not $su_{.b-o}$. Also note that $\mathcal{R} = \{v, q\} \subset \mathcal{S}$ is $su_{.\beta-o}$ but not $su_{.si.o}$.

Remark 3.1. In all examples throughout this paper, the collection $\mathcal{W}$ satisfies the conditions of Definition 2.1 (supra topology): it contains $\emptyset$ and $\mathcal{S}$, and is closed under arbitrary unions. The subsets chosen as examples are arbitrary subsets of $\mathcal{S}$; no additional membership condition in $\mathcal{W}$ is imposed beyond what is stated in the relevant definitions. In particular, in Example 3.4, the sets $\mathcal{A}=\{u,t\}$ and $\mathcal{R}=\{v,q\}$ are chosen as arbitrary subsets of $\mathcal{S}=\{v,u,t,q\}$ to illustrate the strict separation of $su_{.\beta}$ -open sets from $su_{.b}$ -open and $su_{.si}$ -open sets, respectively.

Proposition 3.6

Let $(\mathcal{S}, \mathcal{W})$ be a supra topological space, and let $\mathcal{R}, \mathcal{M}$ be two subset of $\mathcal{S}$ Such that $\mathcal{R} \subseteq \mathcal{M}$. If $\mathcal{R}$ is $su_{.\mathcal{F}_e.o}$ set and $\mathcal{M}$ is $su_{.o}$ set, then $\mathcal{R} \cap \mathcal{M}$ is $su_{.\mathcal{F}_e.o}$ Set.

Proof

Since $\mathcal{R}$ is $su_{.\mathcal{F}_e.o}$, we have and $\mathcal{M}$ is $su_{.o}$ set that is $int_{\mathcal{F}_e}^w(\mathcal{R}) = \mathcal{R}$ and $int^w(\mathcal{M}) = \mathcal{M}$ by

Lemma 2.3 Part (10) and Proposition 3.1 $int^w(\mathcal{M}) \subseteq int_{\mathcal{F}_e}^w(\mathcal{M})$, then

$$\mathcal{R} \cap \mathcal{M} = int_{\mathcal{F}_e}^w(\mathcal{R}) \cap int^w(\mathcal{M}) \subseteq int_{\mathcal{F}_e}^w(\mathcal{R}) \cap int_{\mathcal{F}_e}^w(\mathcal{M}) \subseteq int_{\mathcal{F}_e}^w(\mathcal{R} \cap \mathcal{M}) \dots \dots (1),$$

Also by Lemma 2.3 Part (4) and notation 3.1 we have,

$int_{\mathcal{F}_e}^w(\mathcal{R} \cap \mathcal{M}) \subseteq int_{\mathcal{F}_e}^w(\mathcal{R}) \cap int_{\mathcal{F}_e}^w(\mathcal{M}) \subseteq \mathcal{R} \cap \mathcal{M} \dots \dots (2)$, from (1) and (2) we have $\mathcal{R} \cap \mathcal{M} = int_{\mathcal{F}_e}^w(\mathcal{R} \cap \mathcal{M})$. Therefore, which show $\mathcal{R} \cap \mathcal{M}$ is $su_{.Fe.o}$

The relationship between a particular kind of supra -open sets is depicted in the figure 1 below. Arrows indicate logical implication ($A \rightarrow B$) Mean every A is a B .

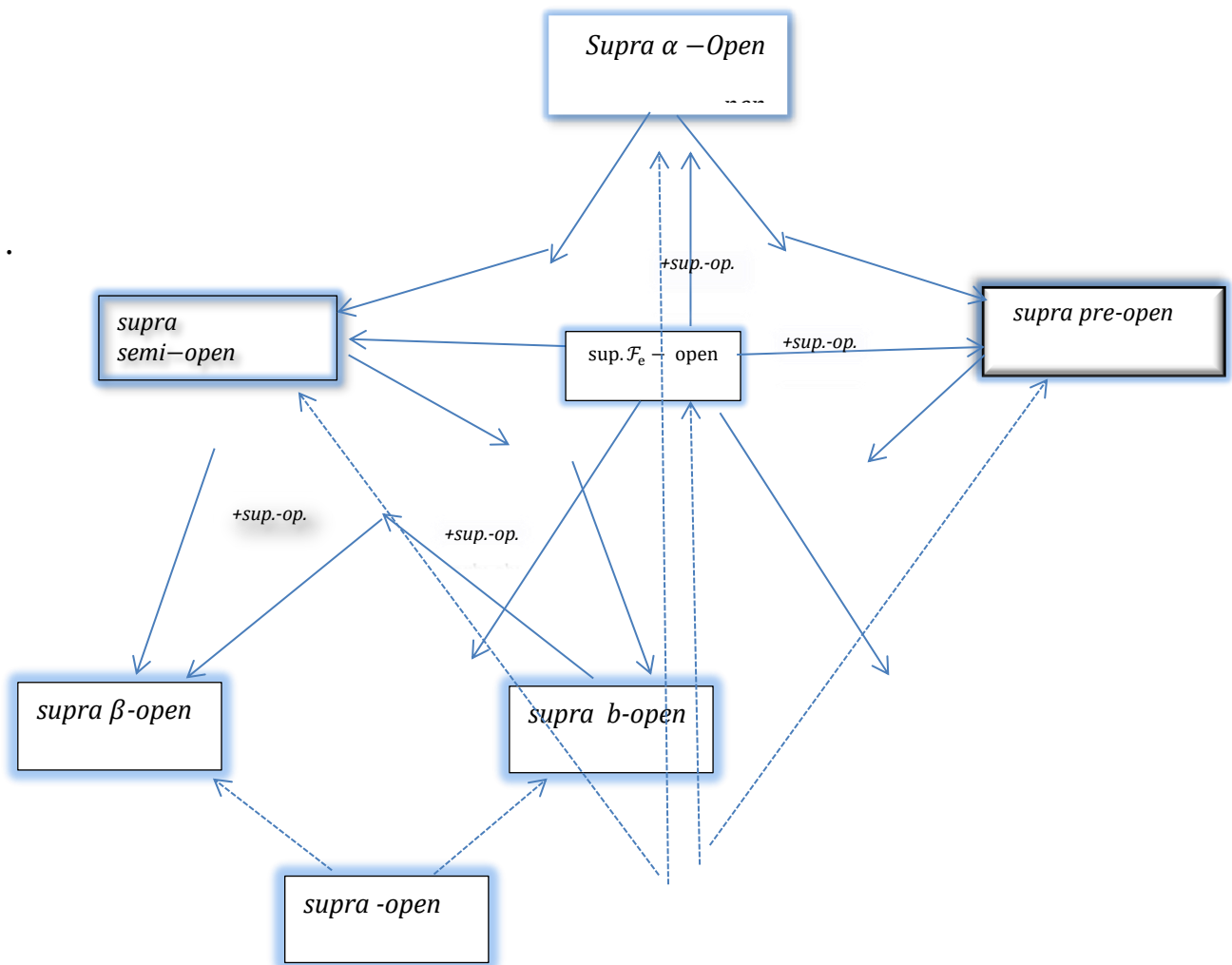


Figure.1. Diagrammatic Representation of Inclusion Relations

To confirm that no arrow in Figure 1 can be reversed, the following table provides explicit counterexamples showing that each implication is strict

Table 1. Counterexamples for Non-Reversibility of Inclusion Relations

Space (S, W)	Set A	A is	A is NOT
$S=\{r,y,m,w\},$ $W=\{\emptyset, S, \{r\}, \{m\}, \{r,y\}, \{r,y,m\}\}$	$A=\{r,y,w\}$	su.Fe.o	su.o (Ex. 3.1.1)
$S=\{l,m,n\}, W=\{\emptyset, S, \{l\}, \{n\}, \{l,n\}\}$	$A=\{l,m\}$	su.si.o	su.Fe.o (Ex. 3.3.1)
$S=\{x,g,z,t\},$ $W=\{\emptyset, S, \{x,t\}, \{x,z,t\}, \{g,z,t\}\}$	$A=\{x,g,z\}$	su.p.o	su.Fe.o (Ex. 3.1.2)
$S=\{m,n,h,w\},$ $W=\{\emptyset, S, \{m\}, \{m,n\}, \{n,h\}, \{m,n,h\}\}$	$A=\{m,h,w\}$	su.b.o	su.Fe.o (Ex. 3.3.3)

Space (S, W)	Set A	A is	A is NOT
$S=\{v,u,t,q\},$ $W=\{\emptyset, S, \{v\}, \{v,t\}, \{t,q\}, \{v,t,q\}\}$	$A=\{u,t\}$	su.β.o	su.b.o (Ex. 3.4)
$S=\{m,t,r,w\},$ $W=\{\emptyset, S, \{m\}, \{w\}, \{t,r\}, \{m,r,w\}, \{t,r,w\}\}$	$A=\{m,t,r\}$	su.p.o, su.β.o	su.Fe.o (Ex. 3.3.2)

4. Conclusion and Future Work

In conclusion, the study of supra feebly topological spaces has introduced a new perspective on topological structures, emphasizing the role of “weak” or “fragile” open sets. This research has explored the fundamental properties of these spaces, including connectedness and disconnectedness, and has laid the groundwork for further investigation. We have given a strict mathematical transition between the classical feebly open sets and the generalized supra topology structure by formally defining the supra feebly interior and closure operators.

The findings of this study contribute to the advancement of mathematical knowledge in generalized topology and have potential implications for various fields, such as computer science and physics, where topological concepts are applied. Particularly in digital topology and rough set theory where classical topological axioms is often too rigid.

Future research directions may include:

- Developing the theory of supra feebly topological spaces
- Exploring applications in specific domains (e.g., data mining and pattern recognition).
- Investigating relationships with other topological structures such as fuzzy supra topologies or neutrosophic supra structures.

This study serves as a foundation for continued exploration and innovation in the field of topology.

References

- [1] A. S. Mashhour, A. A. Allam, F. S. Mahmoud, and F. H. Khedr, “On supra topological spaces,” *Indian Journal of Pure and Applied Mathematics*, vol. 14, no. 4, pp. 502–510, 1983.
- [2] C. Parvathy and S. Deepa, “Supra regular B-closed sets in supra topological spaces,” *Journal of Propulsion Technology*, vol. 45, no. 3, pp. 1446–1452, 2024.
- [3] S. N. Maheshwari and R. Jain, “Some new mappings,” *Mathematica Slovaca*, vol. 24, no. 3, pp. 245–252, 1974.
- [4] L. Thivagar and M. Devi, “Some new class of generalized continuous function,” in *Proc. Int. Conf. Mathematics and Computer Science*, 2011, pp. 431–434.
- [5] M. Baloush and S. Chedzul-Kifli, “On some chaos notions of supra topological spaces,” *Open Access Library Journal*, vol. 10, no. 6, pp. 1–8, 2023.
- [6] T. Noiri, “On feebly continuous mappings,” *Journal of the Korean Mathematical Society*, vol. 13, no. 1, pp. 1–8, 1976.
- [7] O. Njåstad, “On some classes of nearly open sets,” *Pacific Journal of Mathematics*, vol. 15, no. 3, pp. 961–970, 1965.
- [8] O. R. Sayed, “Supra pre-open sets and supra pre-continuity on topological spaces,” *Scientific Studies and Research, Series Mathematics and Informatics*, vol. 20, no. 2, pp. 79–88, 2010.
- [9] O. R. Sayed and T. Noiri, “On supra b-open sets and supra b-continuity on topological spaces,” *European Journal of Pure and Applied Mathematics*, vol. 3, no. 2, pp. 295–302, 2010.
- [10] R. Devi, S. Sampathkumar, and M. Caldas, “On supra α-open sets and supra α-continuous functions,” *General Mathematics*, vol. 16, no. 2, pp. 77–84, 2008.
- [11] S. Dayana Mary and N. Nagaveni, “Decomposition of β-closed sets in supra topological spaces,” *IOSR Journal of Engineering*, vol. 3, no. 1, pp. 70–74, 2013.
- [12] T. M. Al-Shami, “On supra semi-open sets and some applications on topological spaces,” *Journal of Advanced Studies in Topology*, vol. 8, no. 2, pp. 144–153, 2017.