

Application of Statistical Method in Mathematical Models

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ABSTRACT

In this paper, we define two mathematical models, the first of which is a repeated measurements model with two random effects and two fixed effects, while the second has one random effect and three fixed effects. Analysis of variance for each model investigated and applied to some data, and variance components estimated by using the maximum likelihood method.

MSC..

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1. Introduction

By using a mathematical model, we can deal with any problem related to all sciences [1]. Mathematical models describe data for any study in numerical factors[2]. The repeated measurements model can be used to determine the effect of any factor[3]. In 2021 al-mouel and al-kanan estimate the parameters of the repeated measurements

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model[4]. In 2021, al-hassan and al-mouel identify the analytic form of the likelihood-ratio test[5]. In 2021 al-mouel and kori obtained the estimation corresponding to the expected mean square rate[6]. In 2023 saleh, kadir, and Jamil studied a comparison between some methods[7]. In 2023, langenberg, janczyk, koob, kliegl, and mayer, studied the paired t-test[8]. In 2024 al-kanan study two estimation methods [9]. The mathematical model contains the following: model equation, independent variables, parameterization conditions[10]. A mathematical model may be an equation or a set of equations[11]. The analysis of variance depends on partitioning the total variation sources[12]. The first step to developing a model is about identifying your equation[13]. The objective of this paper is to define two mathematical models, the first of which is a repeated measurements model with two random effects and two fixed effects. The second is a repeated measurements model with one random effect and three fixed effects. Investigate the analysis of variance for each model, and apply it to some data.

2. FIRST MODEL

Consider the following repeated measurements model:

$$\psi_{ijklm} = \mu + \rho_i + \tau_{i(j)} + (\gamma)_{ijk} + (\delta)_{ijkl} + \varepsilon_{ijklm}, \quad (2.1)$$

$$i = 1, \dots, x, j = 1, \dots, y,$$

$$k = 1, \dots, p, l = 1, \dots, q,$$

$$m = 1, \dots, v,$$

where

$$\gamma_{ijk} \sim i.i.d N(0, \sigma_\gamma^2),$$

$$\delta_{ijkl} \sim i.i.d N(0, \sigma_\delta^2),$$

$$\varepsilon_{ijklm} \sim i.i.d N(0, \sigma_\varepsilon^2),$$

$$\sum_{i=1}^x \rho_i = 0, \sum_{i=1}^x \tau_{i(j)} = 0, \text{ for each } i,$$

The expression of the statistics SS_ρ , SS_τ , SS_γ , SS_δ , and SS_ε are computed as follows:

$$SS_\rho = \sum_{i=1}^x \sum_{j=1}^y \sum_{k=1}^p \sum_{l=1}^q \sum_{m=1}^v (\bar{\psi}_{i\dots} - \bar{\psi}_{\dots})^2, MS_\rho = \frac{SS_\rho}{x-1}, \quad (2.2)$$

$$SS_\tau = \sum_{i=1}^x \sum_{j=1}^y \sum_{k=1}^p \sum_{l=1}^q \sum_{m=1}^v (\bar{\psi}_{ij\dots} - \bar{\psi}_{i\dots})^2, MS_\tau = \frac{SS_\tau}{x(y-1)}, \quad (2.3)$$

$$SS_\gamma = \sum_{i=1}^x \sum_{j=1}^y \sum_{k=1}^p \sum_{l=1}^q \sum_{m=1}^v (\bar{\psi}_{ijk\dots} - \bar{\psi}_{ij\dots})^2, MS_\gamma = \frac{SS_\gamma}{xy(p-1)}, \quad (2.4)$$

$$SS_\delta = \sum_{i=1}^x \sum_{j=1}^y \sum_{k=1}^p \sum_{l=1}^q \sum_{m=1}^v (\bar{\psi}_{ijkl} - \bar{\psi}_{ijk\dots})^2, MS_\delta = \frac{SS_\delta}{xyp(q-1)}, \quad (2.5)$$

$$SS_{\varepsilon} = \sum_{i=1}^x \sum_{j=1}^y \sum_{k=1}^p \sum_{l=1}^q \sum_{m=1}^v (\psi_{ijklm} - \bar{\psi}_{ijkl.})^2, MS_{\varepsilon} = \frac{SS_{\varepsilon}}{xypq(v-1)}, \quad (2.6)$$

Theorem 2.1. The expected value of the sum of squares for each factor obtained as follows:

$$\begin{aligned} E(MS_{\rho}) &= E\left(\frac{SS_{\rho}}{x-1}\right) \\ &= \sum_{i=1}^x \sum_{j=1}^y \sum_{k=1}^p \sum_{l=1}^q \sum_{m=1}^v \frac{E(\bar{\psi}_{i....} - \bar{\psi}_{....})^2}{x-1} \\ &= \sum_{i=1}^x \sum_{j=1}^y \sum_{k=1}^p \sum_{l=1}^q \sum_{m=1}^v \frac{\rho_i^2 + E(\bar{y}_{i..} - \bar{y}_{...})^2 + E(\bar{\delta}_{i...} - \bar{\delta}_{....})^2 + E(\bar{\varepsilon}_{i....} - \bar{\varepsilon}_{.....})^2}{x-1} \\ &= ypqv \sum_{i=1}^x \left(\rho_i^2 + \left(\frac{\sigma_y^2}{yp} - \frac{\sigma_y^2}{xyp} \right) + \left(\frac{\sigma_{\delta}^2}{ypq} - \frac{\sigma_{\delta}^2}{xypq} \right) + \left(\frac{\sigma_{\varepsilon}^2}{ypqv} - \frac{\sigma_{\varepsilon}^2}{xypqv} \right) \right) / (x-1) \\ &= ypqv \sum_{i=1}^x \frac{\rho_i^2}{(x-1)} + qv\sigma_y^2 + v\sigma_{\delta}^2 + \sigma_{\varepsilon}^2, \end{aligned}$$

$$\begin{aligned} E(MS_{\tau}) &= E\left(\frac{SS_{\tau}}{y-1}\right) \\ &= pqv \sum_{i=1}^x \sum_{j=1}^y \frac{E(\bar{\psi}_{ij...} - \bar{\psi}_{i....})^2}{y-1} \\ &= pqv \sum_{i=1}^x \sum_{j=1}^y \frac{E(\tau_{ij} - \bar{\tau}_{i.})^2 + E(\bar{y}_{ij.} - \bar{y}_{i..})^2 + E(\bar{\delta}_{ij..} - \bar{\delta}_{i...})^2 + E(\bar{\varepsilon}_{ij...} - \bar{\varepsilon}_{i....})^2}{y-1} \\ &= pqv \sum_{i=1}^x \sum_{j=1}^y \frac{\tau_{ij}^2}{(y-1)} + \frac{\sigma_y^2}{p} + \frac{\sigma_{\delta}^2}{pq} + \frac{\sigma_{\varepsilon}^2}{pqv} \\ &= pqv \sum_{i=1}^x \sum_{j=1}^y \frac{\tau_{ij}^2}{(y-1)} + qv\sigma_y^2 + v\sigma_{\delta}^2 + \sigma_{\varepsilon}^2, \end{aligned}$$

$$\begin{aligned} E(MS_{\gamma}) &= \frac{E(SS_{\gamma})}{xy(p-1)} \\ &= \sum_{i=1}^x \sum_{j=1}^y \sum_{k=1}^p \sum_{l=1}^q \sum_{m=1}^v \frac{(\bar{\psi}_{ijk..} - \bar{\psi}_{ij...})^2}{xy(p-1)} \end{aligned}$$

$$= qv \sum_{i=1}^x \sum_{j=1}^y \sum_{k=1}^p ((\gamma_{ijk} - \bar{\gamma}_{ij.})^2 + (\bar{\delta}_{ijk.} - \bar{\delta}_{ij..})^2 + (\bar{\varepsilon}_{ijk..} - \bar{\varepsilon}_{ij...})^2) / (xy(p-1))$$

$$= qv \sum_{i=1}^x \sum_{j=1}^y \sum_{k=1}^p \frac{qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2}{xypqv}$$

$$= qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2,$$

$$\hat{\sigma}_{\gamma,ANOVA}^2 = \frac{MS_\gamma - MS_\delta}{qv},$$

$$E(MS_\delta) = \frac{E(SS_\delta)}{xyp(q-1)}$$

$$= \sum_{i=1}^x \sum_{j=1}^y \sum_{k=1}^p \sum_{l=1}^q \sum_{m=1}^v \frac{E(\bar{\psi}_{ijkl.} - \bar{\psi}_{ijk..})^2}{xyp(q-1)}$$

$$= \sum_{i=1}^x \sum_{j=1}^y \sum_{k=1}^p \sum_{l=1}^q \sum_{m=1}^v \frac{E(\delta_{ijkl} - \bar{\delta}_{ijk.})^2 + E(\bar{\varepsilon}_{ijkl.} - \bar{\varepsilon}_{ijk..})^2}{xyp(q-1)}$$

$$= \sum_{i=1}^x \sum_{j=1}^y \sum_{k=1}^p \sum_{l=1}^q \sum_{m=1}^v \frac{v\sigma_\delta^2 + \sigma_\varepsilon^2}{xypqv}$$

$$= v\sigma_\delta^2 + \sigma_\varepsilon^2,$$

$$\sigma_{\delta,ANOVA}^2 = \frac{MS_\delta - MS_\varepsilon}{v},$$

$$E(MS_\varepsilon) = \frac{E(SS_\varepsilon)}{xypq(v-1)}$$

$$= \sum_{i=1}^x \sum_{j=1}^y \sum_{k=1}^p \sum_{l=1}^q \sum_{m=1}^v \frac{E(\psi_{ijklm} - \bar{\psi}_{ijkl.})^2}{xypq(v-1)}$$

$$= \sum_{i=1}^x \sum_{j=1}^y \sum_{k=1}^p \sum_{l=1}^q \sum_{m=1}^v \frac{E(\varepsilon_{ijklm} - \bar{\varepsilon}_{ijkl.})^2}{xypq(v-1)}$$

$$= \sum_{i=1}^x \sum_{j=1}^y \sum_{k=1}^p \sum_{l=1}^q \sum_{m=1}^v \frac{\sigma_\varepsilon^2}{xypqv}$$

$$= \sigma_\varepsilon^2,$$

$$\hat{\sigma}_{\varepsilon,ANOVA}^2 = MS_\varepsilon.$$

3. Maximum likelihood method

The parameters σ_δ^2 , σ_γ^2 , and σ_ε^2 are estimated by using the maximum likelihood method, where the likelihood function is as follows:

$$L = \left(\exp \left(\frac{-1}{2} \left(\frac{SS_\rho}{qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2} + \frac{SS_\tau}{qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2} + \frac{SS_\gamma}{qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2} + \frac{SS_\delta}{v\sigma_\delta^2 + \sigma_\varepsilon^2} + \frac{SS_\varepsilon}{\sigma_\varepsilon^2} + \frac{xy\bar{p}qv(\bar{\Psi}_{\dots} - \mu)^2}{\sigma_\varepsilon^2 + qv\sigma_\gamma^2 + v\sigma_\delta^2} \right) \right) \right. \\ \left. / ((2\pi)^{\frac{1}{2}xy\bar{p}qv} (\sigma_\varepsilon^2)^{\frac{1}{2}\lambda_\varepsilon} (qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2)^{\frac{1}{2}\lambda_\rho} (qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2)^{\frac{1}{2}\lambda_\tau} (qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2)^{\frac{1}{2}\lambda_\gamma} (v\sigma_\delta^2 + \sigma_\varepsilon^2)^{\frac{1}{2}\lambda_\delta} (\sigma_\varepsilon^2 + qv\sigma_\gamma^2 + v\sigma_\delta^2)^{\frac{1}{2}}) \right)$$

$$\ln(L) = \frac{-1}{2} \left(xy\bar{p}qv \ln(2\pi) + \lambda_\varepsilon \ln(\sigma_\varepsilon^2) + \lambda_\rho \ln(qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2) + \lambda_\tau \ln(qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2) \right. \\ \left. + \lambda_\gamma \ln(qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2) + \lambda_\delta \ln(v\sigma_\delta^2 + \sigma_\varepsilon^2) + \ln(\sigma_\varepsilon^2 + qv\sigma_\gamma^2 + v\sigma_\delta^2) + \frac{SS_\rho}{qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2} \right. \\ \left. + \frac{SS_\tau}{qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2} + \frac{SS_\gamma}{qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2} + \frac{SS_\delta}{v\sigma_\delta^2 + \sigma_\varepsilon^2} + \frac{SS_\varepsilon}{\sigma_\varepsilon^2} + \frac{xy\bar{p}qv(\bar{\Psi}_{\dots} - \mu)^2}{\sigma_\varepsilon^2 + qv\sigma_\gamma^2 + v\sigma_\delta^2} \right),$$

then, we found that

$$\frac{\partial \ln(L)}{\partial \sigma_\varepsilon^2} = \frac{\lambda_\varepsilon}{(\sigma_\varepsilon^2)} - \frac{SS_\varepsilon}{(\sigma_\varepsilon^2)^2},$$

$$\hat{\sigma}_\varepsilon^2 = MS_\varepsilon, \quad (3.1)$$

$$\frac{\partial \ln(L)}{\partial \sigma_\delta^2} = \frac{\lambda_\delta}{(v\sigma_\delta^2 + \sigma_\varepsilon^2)} - \frac{SS_\delta}{(v\sigma_\delta^2 + \sigma_\varepsilon^2)^2},$$

$$\hat{\sigma}_\delta^2 = \frac{MS_\delta - MS_\varepsilon}{v}, \quad (3.2)$$

$$\frac{\partial \ln(L)}{\partial \sigma_\gamma^2} = \frac{\lambda_\rho}{(qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2)} + \frac{\lambda_\tau}{(qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2)} + \frac{\lambda_\gamma}{(qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2)} - \frac{SS_\rho}{(qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2)^2} \\ - \frac{SS_\tau}{(qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2)^2} - \frac{SS_\gamma}{(qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2)^2},$$

$$\hat{\sigma}_\gamma^2 = \frac{1}{qv} \left(\frac{SS_\rho + SS_\tau + SS_\gamma}{\lambda_\rho + \lambda_\tau + \lambda_\gamma} - MS_\delta \right). \quad (3.3)$$

If $\frac{SS_\delta}{\lambda_\delta} < \frac{SS_\varepsilon}{\lambda_\varepsilon}$, then

$$\frac{\partial \ln(L)}{\partial \sigma_\varepsilon^2} = \frac{\lambda_\varepsilon + \lambda_\delta}{(\sigma_\varepsilon^2)} - \frac{SS_\varepsilon + SS_\delta}{(\sigma_\varepsilon^2)^2},$$

$$\hat{\sigma}_{\varepsilon}^2 = \min\left(\frac{SS_{\varepsilon} + SS_{\delta}}{\lambda_{\varepsilon} + \lambda_{\delta}}, MS_{\varepsilon}\right), \hat{\sigma}_{\delta}^2 = \max\left(0, \frac{MS_{\delta} - MS_{\varepsilon}}{v}\right). \quad (3.4)$$

If $\frac{SS_{\rho}}{\lambda_{\rho}} + \frac{SS_{\tau}}{\lambda_{\tau}} + \frac{SS_{\gamma}}{\lambda_{\gamma}} + \frac{SS_{\delta}}{\lambda_{\delta}} < \frac{SS_{\varepsilon}}{\lambda_{\varepsilon}}$, then

$$\frac{\partial \ln(L)}{\partial \sigma_{\varepsilon}^2} = \frac{\lambda_{\varepsilon} + \lambda_{\delta} + \lambda_{\rho} + \lambda_{\tau} + \lambda_{\gamma}}{(\sigma_{\varepsilon}^2)} - \frac{SS_{\varepsilon} + SS_{\delta} + SS_{\rho} + SS_{\tau} + SS_{\gamma}}{(\sigma_{\varepsilon}^2)^2},$$

$$\hat{\sigma}_{\varepsilon}^2 = \min\left(\frac{SS_{\varepsilon} + SS_{\delta} + SS_{\rho} + SS_{\tau} + SS_{\gamma}}{\lambda_{\varepsilon} + \lambda_{\delta} + \lambda_{\rho} + \lambda_{\tau} + \lambda_{\gamma}}, MS_{\varepsilon}\right), \sigma_{\gamma}^2 = \max\left(0, \frac{1}{qv}\left(\frac{SS_{\rho} + SS_{\tau} + SS_{\gamma}}{\lambda_{\rho} + \lambda_{\tau} + \lambda_{\gamma}} - MS_{\delta}\right)\right). \quad (3.5)$$

We assume that $\lambda_{\rho} = x - 1$, $\lambda_{\tau} = x(y - 1)$, $\lambda_{\gamma} = xy(p - 1)$, $\lambda_{\delta} = xyp(q - 1)$, and $\lambda_{\varepsilon} = xypq(v - 1)$.

Theorem 3.1. The variance component estimator $\hat{\sigma}_{\varepsilon}^2$ when $\frac{SS_{\delta}}{\lambda_{\delta}} < \frac{SS_{\varepsilon}}{\lambda_{\varepsilon}}$ has the following variance and bias equations, where

$$\hat{\sigma}_{\varepsilon}^2 = \min\left(\frac{SS_{\varepsilon} + SS_{\delta}}{\lambda_{\varepsilon} + \lambda_{\delta}}, \frac{SS_{\varepsilon}}{\lambda_{\varepsilon}}\right),$$

$$\frac{SS_{\varepsilon}}{\sigma_{\varepsilon}^2} \sim \chi_{\lambda_{\varepsilon}}^2,$$

$$\frac{SS_{\delta}}{v\sigma_{\delta}^2 + \sigma_{\varepsilon}^2} \sim \chi_{\lambda_{\delta}}^2,$$

$$\left(E\left(\frac{SS_{\varepsilon}}{\lambda_{\varepsilon}}\right) - \sigma_{\varepsilon}^2\right)^2 = \left(\frac{1}{\lambda_{\varepsilon}}(E(SS_{\varepsilon})) - \sigma_{\varepsilon}^2\right)^2$$

$$= \left(\frac{1}{\lambda_{\varepsilon}}(\lambda_{\varepsilon}\sigma_{\varepsilon}^2) - \sigma_{\varepsilon}^2\right)^2$$

$$= 0,$$

$$\text{var}\left(\frac{SS_{\varepsilon}}{\lambda_{\varepsilon}}\right) = \frac{1}{(\lambda_{\varepsilon})^2} \text{var}(SS_{\varepsilon})$$

$$= \frac{1}{(\lambda_{\varepsilon})^2} 2\lambda_{\varepsilon}(\sigma_{\varepsilon}^2)^2,$$

$$\left(E\left(\frac{SS_{\varepsilon} + SS_{\delta}}{\lambda_{\delta} + \lambda_{\varepsilon}}\right) - \sigma_{\varepsilon}^2\right)^2 = \left(\frac{1}{\lambda_{\delta} + \lambda_{\varepsilon}}(E(SS_{\varepsilon}) + E(SS_{\delta})) - \sigma_{\varepsilon}^2\right)^2$$

$$= \left(\frac{1}{\lambda_{\delta} + \lambda_{\varepsilon}}(\lambda_{\varepsilon}\sigma_{\varepsilon}^2 + \lambda_{\delta}(v\sigma_{\delta}^2 + \sigma_{\varepsilon}^2)) - \sigma_{\varepsilon}^2\right)^2$$

$$= \left(\frac{\lambda_{\delta}v\sigma_{\delta}^2}{\lambda_{\delta} + \lambda_{\varepsilon}}\right)^2,$$

Theorem 3.2. The variance component estimator $\hat{\sigma}_\varepsilon^2$ when $\frac{SS_\rho}{\lambda_\rho} + \frac{SS_\tau}{\lambda_\tau} + \frac{SS_\gamma}{\lambda_\gamma} + \frac{SS_\delta}{\lambda_\delta} < \frac{SS_\varepsilon}{\lambda_\varepsilon}$ has the following variance and bias equations, where

$$\hat{\sigma}_\varepsilon^2 = \min \left(\frac{SS_\varepsilon + SS_\delta + SS_\rho + SS_\tau + SS_\gamma}{\lambda_\varepsilon + \lambda_\delta + \lambda_\rho + \lambda_\tau + \lambda_\gamma}, \frac{SS_\varepsilon}{\lambda_\varepsilon} \right),$$

$$\frac{SS_\varepsilon}{\sigma_\varepsilon^2} \sim \chi_{\lambda_\varepsilon}^2,$$

$$\frac{SS_\delta}{v\sigma_\delta^2 + \sigma_\varepsilon^2} \sim \chi_{\lambda_\delta}^2,$$

$$\frac{SS_\rho}{qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2} \sim \chi_{\lambda_\rho}^2, \text{ where } \sum_{i=1}^x \rho_i = 0,$$

$$\frac{SS_\tau}{qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2} \sim \chi_{\lambda_\tau}^2, \quad \text{where } \sum_{i=1}^x \tau_{i(j)} = 0,$$

$$\frac{SS_\gamma}{qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2} \sim \chi_{\lambda_\gamma}^2.$$

$$\left(E \left(\frac{SS_\varepsilon}{\lambda_\varepsilon} \right) - \sigma_\varepsilon^2 \right)^2 = \left(\frac{1}{\lambda_\varepsilon} (E(SS_\varepsilon)) - \sigma_\varepsilon^2 \right)^2$$

$$= \left(\frac{1}{\lambda_\varepsilon} (\lambda_\varepsilon \sigma_\varepsilon^2) - \sigma_\varepsilon^2 \right)^2$$

$$= 0,$$

$$\text{var} \left(\frac{SS_\varepsilon}{\lambda_\varepsilon} \right) = \frac{1}{(\lambda_\varepsilon)^2} \text{var}(SS_\varepsilon)$$

$$= \frac{1}{(\lambda_\varepsilon)^2} 2\lambda_\varepsilon (\sigma_\varepsilon^2)^2,$$

$$\left(E \left(\frac{SS_\varepsilon + SS_\delta + SS_\rho + SS_\tau + SS_\gamma}{\lambda_\varepsilon + \lambda_\delta + \lambda_\rho + \lambda_\tau + \lambda_\gamma} \right) - \sigma_\varepsilon^2 \right)^2 = \left(\frac{(E(SS_\varepsilon) + E(SS_\delta) + E(SS_\rho) + E(SS_\tau) + E(SS_\gamma)))}{\lambda_\varepsilon + \lambda_\delta + \lambda_\rho + \lambda_\tau + \lambda_\gamma} - \sigma_\varepsilon^2 \right)^2$$

$$= \left(\frac{(\lambda_\varepsilon \sigma_\varepsilon^2 + \lambda_\delta (v\sigma_\delta^2 + \sigma_\varepsilon^2) + \lambda_\rho (qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2) + \lambda_\tau (qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2) + \lambda_\gamma (qv\sigma_\gamma^2 + v\sigma_\delta^2 + \sigma_\varepsilon^2))}{\lambda_\varepsilon + \lambda_\delta + \lambda_\rho + \lambda_\tau + \lambda_\gamma} - \sigma_\varepsilon^2 \right)^2$$

$$= \left(\frac{\lambda_\varepsilon \sigma_\varepsilon^2 + (\lambda_\delta + \lambda_\rho + \lambda_\tau + \lambda_\gamma) v\sigma_\delta^2 + (\lambda_\rho + \lambda_\tau + \lambda_\gamma) qv\sigma_\gamma^2}{\lambda_\varepsilon + \lambda_\delta + \lambda_\rho + \lambda_\tau + \lambda_\gamma} \right)^2.$$

4. APPLICATION SIDE

For the first model, we use two levels of i (i_1, i_2), two levels of j (j_1, j_2), two levels of k (k_1, k_2), and two levels of l (l_1, l_2). We assume that ρ_i is the fixed effect of i levels, $\tau_{i(j)}$ is the fixed effect of the i, j levels, $(\gamma)_{ijk}$ is the random effect of i, j , and k levels, and $(\delta)_{ijkl}$ is the random effect of the i, j, k , and l levels.

Table 1- i_1 levels [15].

i_1							
j_1				j_2			
k_1		k_2		k_1		k_2	
l_1	l_2	l_1	l_2	l_1	l_2	l_1	l_2
12.51	25	11.67	23.66	13.41	10.43	12.56	23.55
18.12	16	18.8	16.06	19.911	22	18.943	19.37

Table 2- i_2 levels [15].

i_2							
j_1				j_2			
k_1		k_2		k_1		k_2	
l_1	l_2	l_1	l_2	l_1	l_2	l_1	l_2
21.32	22.33	25.03	19.66	24.44	18	23.596	19.99
19.501	19.03	21.7	37	20.301	32.26	20.5	29.43

Table 3- ANOVA for the first model

v.o.s	d.f	SS	MS	F table	F statistics
ρ	1	265.0407761250026	265.0407761250026	4.494	16.02350141673759
τ	2	0.711764499999803	0.355882249999902	3.63372	0.02151548082685
γ	4	66.16301125000064	16.54075281250016	3.00692	1.09827586587543
δ	8	120.485231999999909	15.060653999999989	2.5911	0.43790797285991
ε	16	550.2764939999997	34.39228087499998		

The analysis of variance is obtained and determines the F statistic for each factor, since $\frac{SS_{\delta}}{\lambda_{\delta}} < \frac{SS_{\varepsilon}}{\lambda_{\varepsilon}}$, then

$$\hat{\sigma}_{\varepsilon}^2 = \min\left(\frac{SS_{\varepsilon} + SS_{\delta}}{\lambda_{\varepsilon} + \lambda_{\delta}}, MS_{\varepsilon}\right) = 27.948405249999995, \hat{\sigma}_{\delta}^2 = \max\left(0, \frac{MS_{\delta} - MS_{\varepsilon}}{v}\right) = 0, \quad \text{and} \quad \sigma_{\gamma}^2 = \frac{1}{qv} \left(\frac{SS_{\rho} + SS_{\tau} + SS_{\gamma}}{\lambda_{\rho} + \lambda_{\tau} + \lambda_{\gamma}} - MS_{\delta} \right) = 8.08896335267864.$$

5. Second mathematical model

Kori and al-mouel [6,14] study the following model:

$$h_{abc} = \theta + A_b + \pi_{a(b)} + B_c + (AB)_{bc} + \epsilon_{abc}, a = 1, \dots, i, b = 1, \dots, j, c = 1, \dots, k. \quad (10)$$

Table 4- Effects classifies

The factor	The type	The definition	The condition
θ	fixed	Overall mean	$-\infty < \theta < \infty$
A_b	fixed	Fixed effect	$\sum_{b=1}^j A_b = 0$
$\pi_{a(b)}$	random	Random effect	$\pi_{a(b)} \sim N(0, \sigma_\pi^2)$
B_c	fixed	Fixed effect	$\sum_{c=1}^k B_c = 0$
$(AB)_{bc}$	fixed	Fixed effect	$\sum_{b=1}^j (AB)_{bc} = \sum_{c=1}^k (AB)_{bc} = 0$
ϵ_{abc}	random	Random error	$\epsilon_{abc} \sim N(0, \sigma_\epsilon^2)$

For the practical side, we assume that A_b is the fixed effect of b levels, $\pi_{a(b)}$ is the random effect of a, and b levels, B_c is the fixed effect of the c levels, and $(AB)_{bc}$ is the fixed effect of b, and c levels. We use the following data:

Table 5- Second model data [15]

		0 SA mg. L ⁻¹	0.5 SA mg. L ⁻¹	1 SA mg. L ⁻¹
50 Put. Mg. L ⁻¹	c1	12.51	11.76	13.41
	c2	18.12	18.8	19.911
	c3	7.275	9.45	10.43
	c4	25	23.66	22
100 Put. mg. L ⁻¹	c1	21.32	25.03	24.44
	c2	19.501	21.7	20.301
	c3	10.41	14.501	12.61
	c4	22.33	19.66	18

By using the formula below, we found the following ANOVA table:

$$\sum_{a=1}^2 \frac{(h_{a..})^2}{12} = 7483.225757083333, \sum_{b=1}^3 \frac{(h_{.b.})^2}{8} = 7428.828285125001, \sum_{c=1}^4 \frac{(h_{..c})^2}{6} = 7836.807877499999,$$

$$\sum_{a=1}^2 \sum_{b=1}^3 \sum_{c=1}^4 (h_{abc})^2 = 8101.126949, \sum_{a=1}^2 \sum_{b=1}^3 \frac{(h_{ab.})^2}{4} = 7491.612432250001, \sum_{b=1}^3 \sum_{c=1}^4 \frac{(h_{bc})^2}{2} = 7868.5963055,$$

$$\frac{(h_{...})^2}{24} = 7424.703860041667.$$

Table 6- ANOVA table

v.o.s	d.f	SS	MS	F table	F statistics
A	2	4.12442508333334	2.06221254166667	9.55209	0.09853821240388
π	3	62.78414712500035	20.92804904166679	3.86255	1.10961018576134
B	3	412.10401745833224	137.36800581944408	3.86255	7.28328465551218
AB	6	27.66400291666741	4.6106671527779	3.37375	0.24445867962617
ϵ	9	169.7464963750008	18.86072181944453		

Where $SS_A = \sum_{a=1}^2 \sum_{b=1}^3 \sum_{c=1}^4 (\bar{h}_{b.} - \bar{h}_{...})^2$, $SS_{\pi} = \sum_{a=1}^2 \sum_{b=1}^3 \sum_{c=1}^4 (\bar{h}_{ab.} - \bar{h}_{b.})^2$, $SS_B = \sum_{a=1}^2 \sum_{b=1}^3 \sum_{c=1}^4 (\bar{h}_{.c} - \bar{h}_{...})^2$, $SS_{AB} = \sum_{a=1}^2 \sum_{b=1}^3 \sum_{c=1}^4 (\bar{h}_{bc} - \bar{h}_{b.} - \bar{h}_{.c} + \bar{h}_{...})^2$, and $SS_{\epsilon} = \sum_{a=1}^2 \sum_{b=1}^3 \sum_{c=1}^4 (h_{abc} + \bar{h}_{b.} - \bar{h}_{ab.} - \bar{h}_{bc})^2$. since $F[0.95,3,9] < F_B$, then, there is a statistically significant difference.

6. Conclusion

For each model, the analysis of variance is obtained and determines the F statistic for each factor. For the first model, since $F[0.95,1,16] < F_p$, then there is a statistically significant difference for p factor such that $F_p = 16.02350141673759$, $H_0^p: \rho_i = 0, H_1^p: \rho_i \neq 0$, and $F_p = \frac{MS_p}{MS_y}$. We assume that $H_0^{\tau}: \tau_{i(j)} = 0, H_1^{\tau}: \tau_{i(j)} \neq 0, F_{\tau} = \frac{MS_{\tau}}{MS_y}, H_0^{\gamma}: \sigma_{\gamma}^2 = 0, H_1^{\gamma}: \sigma_{\gamma}^2 > 0, F_{\gamma} = \frac{MS_{\gamma}}{MS_{\delta}}$ and $H_0^{\delta}: \sigma_{\delta}^2 = 0, H_1^{\delta}: \sigma_{\delta}^2 > 0, F_{\delta} = \frac{MS_{\delta}}{MS_{\epsilon}}$. We found that $\hat{\sigma}_{\epsilon}^2 = 27.94840524999995$, $\hat{\sigma}_{\delta}^2 = 0$, and $\sigma_{\gamma}^2 = 8.08896335267864$, for the second model, the F table value and F statistics for the second model are as follows: $F_A = \frac{MS_A}{MS_{\pi}} = 0.09853821240388, F[0.95,2,3] = 9.55209, H_0^A: A_b = 0, H_1^A: A_b \neq 0, F_{\pi} = \frac{MS_{\pi}}{MS_{\epsilon}} = 1.10961018576134, F[0.95,3,9] = 3.86255, H_0^{\pi}: \sigma_{\pi}^2 = 0, H_1^{\pi}: \sigma_{\pi}^2 > 0, F_B = \frac{MS_B}{MS_{\epsilon}} = 7.28328465551218, F[0.95,3,9] = 3.86255, H_0^B: B_c = 0, H_1^B: B_c \neq 0, and F_{AB} = \frac{MS_{AB}}{MS_{\epsilon}} = 0.24445867962617, F[0.95,6,9] = 3.37375, H_0^{AB}: (AB)_{bc} = 0, H_1^{AB}: (AB)_{bc} \neq 0$.

References

- [1] C. Eck, H. Garcke, and P. Knabner, Mathematical modeling, 1st ed., Springer Int. Publ. AG, 2017. DOI:10.1007/978-3-319-55161-6
- [2] J. D. Brown, Linear models in matrix form, 1st ed., Springer, 2014. <https://link.springer.com/book/10.1007/978-3-319-11734-8>
- [3] F. H. Al-Kanan, "Confidence intervals in three-way repeated measurements model", Basrah. J. Sci., 42(2024) 176-190. <https://basjsci.edu.iq/index.php/basjsci/ar/article/view/220/194>
- [4] A. H. S. AL-Mouel, F.H. Al-Kanan, "A penalized likelihood estimator for variance components in repeated measurements model", Basrah. J. Sci., 39(2021)192-203. <https://basjsci.edu.iq/index.php/basjsci/ar/article/view/104/84>
- [5] A. A. AL-Hassan, A. H.S. AL-Moul, "Some inference of the repeated measurements model", Int. J. Nonlinear Anal. Appl., 12(2021)569-581. DOI: 10.22075/IJNAA.2021.5109
- [6] H. A. Kori, A. H.S. AL-Mouel, "Expected mean square rate estimation of repeated measurements model", Int. J. Nonlinear Anal. Appl., 12(2021)75-83. <https://doi.org/10.22075/IJNAA.2021.5018.Hh>
- [7] D. M. Saleh, D. H. Kadir, D. I. Jamil, "A comparison between some penalized methods for estimating parameters", Qalaai. Zanistscientific. J., 8(2023)2518-6566. Doi: 10.25212/lfu.qzj.8.1.44
- [8] B. Langenberg, M. Janczyk, V. Koob, R. Kliegl, A. Mayer, "A tutorial on using the paired t test for power calculations in repeated measures anova with interaction", Behav. Res. Methods., 55(2023)2467-2484. <https://link.springer.com/article/10.3758/s13428-022-01902-8>
- [9] F. H. Al-Kanan, "Estimation methods for three-way repeated measurements model", Basrah. Res. Sci., 50(2024)35-45. <https://jou.jobs.edu.iq/index.php/home/article/view/143>
- [10] S. Serovajsky, Mathematical modelling, 1st ed., Chpman and hall/CRC, 2021.

<https://doi.org/10.1201/9781003035602>

[11] D. L. Timmons, C. W. Johnson, S. M. McCook, Fundamentals of algebraic modeling, 5th ed., Cengage Learning, 2013.

https://books.google.iq/books/about/Fundamentals_of_Algebraic_Modeling.html?id=FXsWAAAAQBAJ&redir_esc=y

[12] M. C. Gacula, J. Singh, J. Bi, S. Altan, Statistical methods in food and consumer research, 2nd ed., Elsevier, 2008.

<https://www.researchgate.net/publication>

[13] T. E. Essington, Introduction to quantitative Ecology: mathematical and statistical modelling for beginners, Oxford, 2021.

DOI:10.1093/oso/9780192843470.001.0001

[14] H. I. Mustafa, A. S. Al-mouel, Reduced bias estimation of variance for repeated measurements model, J. Stat. Manag. Syst., 27(2024) 663-671.

DOI:10.47974/JSMS-1084

[15] K. A. Muhsen, E. A. Hantosh, M. A. Darweash, The effect of putrescine and sal salic acid and their interaction on the multiplication of vegetative buds and their characteristics for date palm cultivars al-barhi in vitro, Plant. Cell. Biotechnol. Mol. Biol., 21(2020)112-125.