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Approximation of Functions by Means of the Modulus $\tau(f,\Delta)_{p,\mu}$

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تقريب الدوال بواسطة معدل القياس $\tau(f,\Delta)_{p,\mu}$ **.**

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الخلاصـــــة

في بحثنا استخدمنا برهنتي وتني و رايز- تورن لإيجاد أفضل درجة تقريب للدوال بواسطة النماذج التكاملية في فضاء $L_{p,\mu}(X)$.

ABSTRACT

In this paper, we are used Whitney's and Riesz–Torin Theorems to find the degree of best approximation of functions by means of the averaged modulus of smoothness in space $L_{p,\mu}(X)$.

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INTRODUCTION

Let $X=[a,b]$; $a,b\in R$ (the set of all real numbers). Then we define the space of all bounded measurable functions f on X, by norm a.e :

$$\|f\|_p=\left(\int_a^b |f(x)|^p d(x)\right)^{1/p}<\infty \quad , \qquad (1.1)$$

and denoted by $L_p(X),\ (1\le p<\infty)$, [1]. Also we denote by $L_{p,\mu}[a,b]$ $(1\le p<\infty)$, of the space of all bounded $\mu-$measurable functions f on $[a,b]$, and defined by:

$$\|f\|_{p,\mu}=\left(\int_a^b |f(x)|^p d\mu(x)\right)^{1/p}<\infty \qquad (1.2)$$

where μ is the non-negative measure function on a countable set, [2] .
For every function f we define the k- difference with step (h) at a point x as follows [3]:

$$\Delta_h^k f(x)=\sum_{m=0}^{k}(-1)^{m+k}\binom{k}{m} f(x+mh),\quad x,x+mh\in[a,b]$$

(1.3)

And the kth locally of smoothness for $f\in L_\infty[a,b]$, (the set of all essentially bounded functions on $[a,b]$) is defined by [4]:

$$w_k(f,\delta)=\sup_{|h|<\delta}\left\{\left|\Delta_h^k f(x)\right|:|h|\le\delta,\ x,x+kh\in[a,b]\right\} \qquad (1.4)$$

Also, for every bounded function f the following trivial estimate holds:

$$w_k(f,[a,b])\le 2^k\|f\|_{C_{[a,b]}} \qquad (1.5)$$

where $C[a,b]=\max_{x\in[a,b]}|f(x)|$,[5].

Since

$$\left|\Delta_h^k f(x)\right|=\left|\sum_{i=0}^{k}(-1)^{i+k}\binom{k}{i}f(x+ih)\right|$$

$$\le\sum_{i=0}^{k}\binom{k}{i}\|f\|_{C_{[a,b]}}=2^k\|f\|_{C_{[a,b]}}$$

In [5] the anther proved if f is a measurable bounded function on $[a,b]$, then :

$$w_k(f,\delta)_p\le\tau_k(f,\delta)_p\le w_k(f,\delta)(b-a)^{1/p},\ (1\le p<\infty)$$

where

$$\tau_k(f,\delta)_p = \|w(f,\delta)\|_p = \left\| \sup_{|h|<\delta} \left\{ \left| \Delta_h^k f(x) \right| : |h| \le \delta, x, x+kh \in [a,b] \right\} \right\|_p$$

Also proved the following theorem which is now classical in approx-imation theory and numerical analysis .This theorem gives additional conditions which allow us to invert the above inequality.

Theorem 1, [5]:

For each integer $n \ge 1$ there is a number W_n with the following property, for any interval Δ and for any continuous function f on Δ there is a polynomial P of degree at most $n-1$ such that

$$\left| f(x) - P(x) \right| \le W_n w_n(f,\Delta) \quad , \quad x \in \Delta \qquad (1.6)$$

where W_n is called Whitney's constant, $\Delta = [a,b]$.

Definition (Riesz – Torin Theorem), [5]:

Let T be a linear operator from the spaces $L_p[a,b]$ in to the spaces $L_q[a,b]$, if there exists a constant k , for which

$$\|T f(x)\|_{q[a,b]} \le k \, \| f(x) \|_{p[a,b]} \quad , 1 \le p < q < \infty \qquad (1.7)$$

For every function f in $L_p[a,b]$, we say that the operator T is of the type (p,q) . The smallest number k with this property is called the (p,q)- norm of the operator T .

Theorem 2, [5]:

For each $n \in Z^+$, there is a number W_n and there is a polynomial p_n for each Lebesgue integral function f on $[a,b]$, such that

$$\left| f - p_n \right| < W_n w_n(f,\Delta), \qquad (1.8)$$

where W_n Whitney's constant , $\Delta = [a,b]$.

Lemma 1, [5]:

Let L_n be a linear operator and $\sum_n = \{x_i : a = x_0 < ... < x_{n+1} = b\}$. If $f \in M[a,b]$. Then $L_n(f) \in L_p[a,b]$, $(1 \le p \le \infty)$ and

$$\| L_n f \|_p \le K \| f \|_{p\sum_n} , \tag{1.9}$$

where K is an absolute constant and $M[a,b]$ the space of all measurable functions bounded on interval $[a,b]$.

Lemma 2, [2]:

Let μ be a non – decreasing function on P, satisfying:
$\mu(y) - \mu(x) =$ Constant and $1 < p < \infty$, we put
$w_\mu(\delta) = \sup\limits_{0<y-x\le\delta} \left(\mu(y) - \mu(x) \right), \delta > 0$, and

$$\left(\frac{1}{n} \sum_{k=o}^{n-1} \max_{x \in I_k} |P_n|^p \right)^{1/p} \le C(p) \| P_n \|_p ,$$

where P_n is an algebraic polynomial of degree at most n and

$$I_k = \left[\frac{k}{n}, \frac{k+1}{n} \right] \text{. Then } \| P_n \|_{p,\mu} \le C(p) \left(n w_\mu \left(\frac{1}{n} \right) \right)^{1/p} \| P_n \|_p \tag{1.10}$$

Lemma 3, [2]:

Let f be a bounded μ- measurable function and $1 \le p < \infty$. Then

$$\| f \|_p \le C(p) \| f \|_{p,\mu} , \tag{1.11}$$

where $C(p)$ is a constant depends only on p .

2-Main Results

Now we are using the interpolation results of the Whitney's theorem and the Riesz- Torin theorem [4], [5] to obtain interpolation theorems which are using of the averaged modulus of smoothness.

Lemma 4:

Let f be a $2\pi -$ periodic bounded μ- measurable function then:

$$\tau_k(f, n\delta)_{p,\mu} \le (2n)^{k+1}\, \tau_k(f, \delta)_{p,\mu} \quad , \ 1 \le p < \infty$$

Proof:

We use the identity $\quad \Delta_{nh}^{k} f(t) = \sum_{i=0}^{(n-1)} A_i^{n,k}\ \Delta_h^k f(t+ih) \qquad (2.1)$

where $A_i^{n,k}$ are defined by

$$\left(1+t+\ldots+t^{n-1}\right)^k = \sum_{i=0}^{(n-1)k} A_i^{n.k}\, t^i = n^k \qquad (2.2)$$

since $\tau_k(f, n\delta)_{p,\mu} = \left\| w_k(f, x, n\delta) \right\|_{p,\mu}$

we get $\quad \tau_k(f, n\delta)_{p,\mu} \le \tau_k(f, [n]\delta)_{p,\mu}$

$$= \left\| \sup_{|h|\le\delta} \left[\left| \Delta_{[n]\delta}^{k} f(t) \right| : t, t+k[n]h \in \left[x - \frac{k[n]\delta}{2}, x + \frac{k[n]\delta}{2} \right] \cap [a,b] \right] \right\|_{p,\mu}$$

$$= \left\| \sup_{|h|\le\delta} \left[\left| \sum_{i=0}^{[n-1]k} A_i^{[n],k}\ \Delta_{[n]\delta}^{k} f(t+ih) \right| : \right.\right.$$

$$\left.\left. t+ih, t+ih+nh \ \in \left[x - \frac{k[n]\delta}{2}, x + \frac{k[n]\delta}{2} \right] \cap [a,b] \right] \right\|_{p,\mu}$$

$$w_k(f, x, n\delta) \le \sum_{i=0}^{(2n-1)k} A_i^{2n,k} \sum_{j=1}^{2n-1} w_k\left(f, x-(n-j)\frac{k\delta}{2}, \delta \right) \qquad (2.3)$$

since,

$$t+ih\ , t+ih+nh \in \bigcup_{j=1}^{2(n)-1} \left[x - \frac{k[n]\delta}{2} + (j-1)\frac{k\delta}{2}, x + \frac{k[n]\delta}{2} + (j+1)\frac{k\delta}{2} \right]$$

So that by using definition of local modulus of smoothness and (2.2),(1.5) we obtain

$$\tau_k(f,n\delta)_{p,\mu} \le \left\| \sum_{i=0}^{(n-1)k} A_i^{[2n],k} \sum_{j=1}^{2n-1} w_k\left(f, x-(n-j)\frac{k\delta}{2}, \delta\right) \right\|_{p,\mu}$$

$$\le 2n^k \left[\int_a^b \left| \sum_{j=1}^{2n-1} w_k\left(f, x-(n-j)\frac{k\delta}{2}, \delta\right) \right|^p d_\mu(x) \right]^{1/p}$$

$$\le 2n^k \sum_{j=1}^{2n-1} \left[\int_a^b \left| w_k\left(f, x-(n-j)\frac{k\delta}{2}, \delta\right) \right|^p d_\mu(x) \right]^{1/p}$$

$$\le 2n^k (2n-1) \left[\int_{a-(n-j)\frac{k\delta}{2}}^{b-(n-j)\frac{k\delta}{2}} \left| w_k(f,x,\delta) \right|^p d_\mu(x) \right]^{1/p}$$

$$= 2n^k (2n-1) \,.\, \tau_k(f,\delta)_{p,\mu}$$

$$= (2n^k .2n - 2n^k)\ \tau_k(f,\delta)_{p,\mu}$$

$$= (2n^{k+1} - 2n^k)\ \tau_k(f,\delta)_{p,\mu}$$

$$\le 2n^{k+1}\ \tau_k(f,\delta)_{p,\mu}$$

Lemma 5:

Let $\sum_n = \{ x_i, a = x_0 < \ldots < x_{n+1} = b \}$ be a partition of the interval $[a,b]$ into $n+1$ subintervals and let $k \ge 1$ be an integer .Using the notation $\Delta_i = |x_{i+1} - x_{i-1}|$, $i = 1,2,\ldots,n$, $d_n = \max\{ \Delta_i , 1 \le i \le n \}$

Then $\|w_k(f, x_i, 2h)\|_{p,\mu\sum} \le 2^{1/p+2(k+1)}\ \tau_k(f, h+\frac{d_n}{k})_{p,\mu}$ (2.4)

Proof:

From(1.9) and (1.10),(1.5) we have

$$\|w_k(f,x_i,2h)\|_{p,\mu\sum} = \left\{ \sum_{i=1}^{n} |w_k(f,x_i,2h)|^p \Delta_i \right\}^{1/p}$$

$$= \left\{ \sum_{i=1}^{n} \int_{x_{i-1}}^{x_{i+1}} |w_k(f,x_i,2h)|^p d_\mu(x_i) \right\}^{1/p}$$

$$\le 2^{1/p} c_1(p)\left\{\sum_{i=1}^{n}\int_{x_{i-1}}^{x_{i+1}}\left|w_k(f,x,2(h+\frac{d_n}{k})\right|^p d(x)\right\}^{1/p}$$

$$\le 2^{1/p}\ c_2(p)\ \tau_k\ (f,2(h+\frac{d_n}{k}))_p$$

$$\le 2^{1/p} c_3(p)\ \tau_k\ (f,2(h+\frac{d_n}{k}))_{p,\mu}$$

$$\le 2^{1/p+2(k+1)} c_3(p)\ \tau_k\ (f,\frac{d_n}{k})_{p,\mu} \qquad \square$$

Now, the Whitney's theorem for $f\in L_{p,\mu}(\Delta)$ spaces, have been proved.

Theorem 2.1:

For each $n\in Z^+$ there is a number W_n and there is a polynomial p_n for each Lebesgue Integral function f on $[a,b]$ such that,

$$\|f-p_n\|_{p,\mu}\le W_n\ \tau_k\ (f,[a,b])_{p,\mu} \qquad (2.5)$$

where W_n is Whitney's constant.

Proof:

Let $g=f\,d_\mu(x)$

From (1.8), (1.6), (1.11) and (1.4) there is a polynomial p_n of degree $n-1$ such that

$$|g-p_n|\le W_n w_k(g,[a,b])$$

$$=W_n.\sup\left\{\left|\Delta_h^k g(t)\right|:|h|\le\delta,\ t,t+kh\in[a,b]\right\},\ h>0$$

$$\|g-p_n\|_p\le W_n\|f-p_n\|_{p,\mu}=W_n\left(\int_\Delta\left|\sup\Delta_h^k f(t)\right|^p d_\mu(t):t,t+kh\in[a,b]\right)^{1/p}$$

$$=C(p)\ \tau_k\ (f,[a,b])_{p,\mu} \qquad \square$$

We shall call the polynomial $p=p_n(f)$ for which theorem 2 is valid Whitney's polynomial for the function $f\in L_{p,\mu}$ of degree $(n-1)$.

Theorem 2.2:

Let L be a bounded linear operator on $L_{p,\mu}[a,b]$ and let $L(P)=P$, for every polynomial $P \in H_{n-1}$, where H_{n-1} is the set of all algebraic polynomials of degree $n-1$. Then for every function $f \in L_{p,\mu}[a,b]$, we have

$$\| f - L(f) \|_{p,\mu} \le C(p)\, W_n\, \tau_k \left(f, \frac{b-a}{n} \right)_{p,\mu} \tag{2.6}$$

where W_n is Whitney's constant.

Proof:

Let $P_n(f)$ be polynomial for f of degree $n-1$. Then using (1.8), (1.10) and (1.11) we obtain

$$\begin{aligned}
\| f - L(f) \|_{p,\mu} &\le \| f - P_n(f) \|_{p,\mu} + \| P_n(f) - L(P_n(f)) \|_{p,\mu} \\
&\quad + \| L(P_n(f)) - L(f) \|_{p,\mu} \\
&\le \| f - P_n(f) \|_{p,\mu} + \| P_n(f) - L(P_n(f)) \|_{p,\mu} \\
&\quad + \| L \|_{p,\mu} . \| f - P_n(f) \|_{p,\mu} \\
&\le (1 + \| L \|_{p,\mu})\ \| f - P_n(f) \|_{p,\mu} + \| P_n(f) - L(P_n(f)) \|_{p,\mu} \\
&\le C_1(p)(1 + \| L \|_p) . \| f - P_n(f) \|_p \\
&\le C_1(p)\, W_n (1 + \| L \|_p)\ w_k(f, [a,b])_p \\
&\le C_2(p) W_n (1 + \| L \|_p)\, \tau_k \left(f, \frac{b-a}{n} \right)_p \\
&\le C_3(p) W_n (1 + \| L \|_{p,\mu})\, \tau_k \left(f, \frac{b-a}{n} \right)_{p,\mu} \\
&= C_4(p)\, W_n\ \tau_k \left(f, \frac{b-a}{n} \right)_{p,\mu}
\end{aligned}$$

where $C_4(p) = C_3(p)\,(1 + \| L \|_{p,\mu})$

Theorem 2.3:

Let F be a bounded linear functional on $L_{p,\mu}[a,b]$, let $F(P)=0$ for every $P \in H_{n-1}$. Then for every $f \in L_{p,\mu}[a,b]$,

$$\| F(f) \|_{p,\mu} \leq M \ W_n \ \tau_k(f, b-a)_{p,\mu} \qquad (2.7)$$

where M is a constant.

Proof:

By using (1.8), we have

$$\begin{aligned} \|F(f)\|_{p,\mu} &\leq \|F(f-p)\|_{p,\mu} + \|F(p)\|_{p,\mu} \\ &= \|F(f-p)\|_{p,\mu} \\ &\leq \|F\|_{p,\mu} . \|f-p\|_{p,\mu} \\ &\leq M \ W_n \ \tau_k\left(f, [a,b]\right)_{p,\mu} \qquad \square \end{aligned}$$

Now, we shows Riesz – Torin Theorem in the spaces of all functions belongs to $L_{p,\mu}[a,b]$.

Theorem 2.4:

Let T be a linear operator from the spaces $L_{p,\mu}[a,b]$ into the spaces $L_{p,\mu}[a,b]$, if there exists a constant K , for which

$$\|T f(x)\|_{q,\mu} \leq K \|f(x)\|_{p,\mu} \qquad (2.8)$$

For every $f \in L_{p,\mu}[a,b]$, then the operator T is of type (p,q) .

Proof:

By using (1.10), (1.7) and (1.11) we get

$$\begin{aligned} \|T_n f\|_{q,\mu} \leq C_1(q) \|T_n f\|_q &\leq K C_1(q) \|f\|_p \\ &= C_2(p) \|f\|_p \quad , \quad C_2(p) = K C_1(q) \\ &\leq K C_3(p) \|f\|_{p,\mu} \qquad \square \end{aligned}$$

Theorem 2.5:

Let L_n be a linear operator, If $f \in L_{p,\mu}[a,b]$,

then $L_n(f) \in L_{p,\mu}[a,b]$, $1 \leq p < \infty$ and $\|L_n(f)\|_{p,\mu} \leq k \|f\|_{p,\mu \sum_n}$ (2.9)

where k is an absolute constant .

Proof:

By (1.9) and (1.11) we have

$$\| L_n f \|_{p,\mu} \le C_1(p) \| L_n f \|_p$$
$$\le C_2(p) \| f \|_p$$
$$\le C_3(p) \| f \|_{p,\mu \sum_n}$$

CONCLUSIONS

1- The best approximation of bounded μ - measurable function in $L_{p,\mu}$ spaces by using Whitney's constant have been found.
2- The best approximation of bounded μ - measurable function in $L_{p,\mu}$ spaces by using Riesz – Torin Theorem, have also been found.

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