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Page 166-173

On $3-$Monotone Approximation by Piecewise Positive Functions

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Abstract.

In 2005 Halgwrd [3], introduced a paper for $f \in C[-1,1]$ with $1 < p < \infty$, be a convex function, we are interested in estimating the degree of 3-monotone approximation for the function f, which are copositive on $[-1,1]$. We obtained that f and g are piecewise positive in $[-1,1]$ in terms of the Ditzian-Totik modulus of smoothness .

1. Introduction and auxiliary results.

Let $Y_s = \{a < y_1 < y_2 < ... < y_s < b\},\ s \ge 0$. We denote by $\Delta^0(Y_s)$, the set of all functions f, such that $(-1)^{s-k} f(x) \ge 0$, for $x \in [y_j, y_{j+1}],\ 0 \le k \le s$. Functions f and g, that belong to the same class $\Delta^0(Y_s)$ are said to be *copositive* on $[a,b]$. *Copositive approximation* is the approximation of a function f, from $\Delta^0(Y_s)$, class by polynomials that are copositive with f . Also , let $E_n^0(f,k)_P = \inf_{p_n \in \Pi_n \cap \Delta^0(Y_s)} \|f - p_n\|_P$ be the *degree of copositive polynomial approximation* of f .

We denote $J_j(n,\varepsilon) = [y_j - \Delta_n(y_j)n^{\varepsilon}, y_j + \Delta_n(y_j)n^{\varepsilon}] \cap [a,b],\quad 0 \le j \le s+1$, and denote $O_n(Y_s,\varepsilon) = \bigcup_{j=1}^{s} J_j(n,\varepsilon)$, and $O_n^*(Y_s,\varepsilon) = \bigcup_{j=0}^{s+1} J_j(n,\varepsilon)$. [2]

Functions f and g are called *weakly almost copositive* on I, with respect to Y_s if they are copositive on $I \setminus O_n^*(Y_s,\varepsilon)$, where $\varepsilon > 0$. We define a function class $(\varepsilon - alm\Delta)_n^0(Y_s) = \{f : (-1)^{s-k} f(x) \ge 0,\quad for\ x \in I \setminus O_n^*(Y_s,\varepsilon)\}$, the set of all weakly almost nonnegative functions on I, if $\varepsilon > 0$.

The *degree of weakly almost copositive polynomial approximation* of f in $L_P[a,b]\cap\Delta^0(Y_s)$, by means $p\in\Pi_n\cap(\varepsilon-alm\Delta)_n^0(Y_s)$ is $E_n^0(f,\varepsilon-almY_s)_P$ $=\inf\{\|f-p\|_P : p\in\Pi_n\cap(\varepsilon-alm\Delta)_n^0(Y_s)\}$.

These results can be summarized in the following theorem (see [5] and [8]) .

Theorem A.

There are functions f_1 and f_2 in $C^1[-1,1]$, with $r\geq 1$, sign changes such that

$$\limsup_{n\to\infty}\frac{E_n^0(f_1,r)}{\omega_4(f_1,n^{-1},[-1,1])}=\infty \text{ and } \limsup_{n\to\infty}\frac{E_n^0(f_2,r)_P}{\omega_2(f_2,n^{-1},[-1,1])_P}=\infty,\ 1<p<\infty,$$

where $E_n^0(f,r)_P$ is the degree of the best copositive L_P (C if $p=\infty$), approximation to f, by polynomials from Π_n .

Recently , Y. Hu , D. Leviatan and X. M. Yu [6], showed that theorem A can be considerably improved , thus together with theorem A, revealing an interesting and unexpected difference between the cases $p=\infty$, and $1<p<\infty$, for copositive polynomial approximation . Their result is stated as follows .

Theorem B.

Let $f\in C[-1,1]$, change sign r, times at $-1<y_1<...<y_r<1$, and let $\delta=\min_{0\leq i\leq r}|y_{i+1}-y_i|$, where $y_\circ=-1$ and $y_{r+1}=1$. Then there exists a constant $C=C(r,\delta)$, but otherwise independent of f and n, such that for each $n\geq 4\delta^{-1}$, there is a polynomial $p_n\in\Pi_{Cn}$, copositive with f ,
satisfying

$$\|f-p_n\|_{L_\infty[-1,1]}\leq C\omega_2(f,n^{-1},[-1,1]) . \tag{1.1}$$

In [2] Bhaya , E. and other , showed that in the second result ω_2 in (1.1) can not replaced by $\omega_3(f,b-a,[a,b])_P$, for $0<p<1$, i.e., she proved .

Theorem C.

Given any $A>0$, $n\in\tilde{\mathrm{N}}$, $a<0$, $0<b$, $0<p<1$ and $0<\varepsilon<2$, there exists f in $L_P[a,b]\cap\Delta^0(Y_s)$, such that

$$E_n^0(f,\varepsilon-almY_s)_P>\omega_3(f,b-a,[a,b])_P. \tag{1.2}$$

The second result in [2], shows that τ -modulus of any order $k>0$ can be used for $0<p<1$.

Theorem D.

Let f in $L_P[a,b]\cap\Delta^0(Y_s)$, $0<p<1$, and k be a positive integer . Then there exists a polynomial p_{k-1} in $\Pi_{k-1}\cap(\varepsilon-alm\Delta)_n^0(Y_s)$, satisfying $\|f-P_n\|_P\le c(p)\tau_k(f,b-a,[a,b])_P$.

2. The main results

We will modify this polynomial near the points of sign change obtaining a smooth piecewise polynomial approximation f_n, with controlled first and third derivatives . We will consider σ_i that its convexity at $\{y_i,y_i',y_i''\}$ with f .

Theorem 2.1

Let f in $L_P[a,b]\cap\Delta^0(Y_s)$. Then for each $n\ge 4\delta^{-1}$, there exists a function f_n in $\Delta^3[-1,1]\cap(S-\Delta^0(Y_s))$, copositive with f in $\mathrm{Y}=\bigcup_{i=1}^{k}\rho_i$, such that

$$\|f-f_n\|_{L_P[-1,1]}\le C(k)\omega_3^{\phi}(f,n^{-1},[-1,1])_P, \qquad (2.2)$$

$$\left\|\phi(x)^3 f_n^{(3)}(x)\right\|_{L_P[-1,1]}\le C(k)n^3\omega_3^{\phi}(f,n^{-1},[-1,1])_P, \qquad (2.3)$$

and

$$\|\Delta_n(x)f_n'(x)\|_{L_P[-1,1]}\ge C\omega_3^{\phi}(f,n^{-1},[-1,1])_P,\ \text{for}\ x\in\mathrm{Y}, \qquad (2.4)$$

where $(S-\Delta^0(Y_s))$ is the set of all piecewise positive .

Proof. Let $n\ge 4\delta^{-1}$, and index $1\le i\le k$, be fixed . For $x\in \mathrm{I}_i^*$, we set σ_i to be the polynomial of degree ≤ 2, which vanishes at y_i,

$$\sigma_i(x)=\frac{x-y_i}{y_i''-y_i'}\left\{\frac{x-y_i'}{y_i''-y_i}\sigma_i(y_i'')+\frac{x-y_i''}{y_i-y_i'}\sigma_i(y_i')\right\}\ [4],$$

where $\sigma_i(y_i')$ and $\sigma_i(y_i'')$ are chosen so that

$$|\sigma_i(y_i')|=\begin{cases}c\omega_3^{\phi}(f,n^{-1},[-1,1])_P\,\mathrm{sgn}(f(y_i')) & ;if\ |f(y_i')|\le c\omega_3^{\phi}(f,n^{-1},[-1,1])_P,\\ f(y_i') ; & o.w\end{cases}$$

and

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3rd. Sinentific Conference 19-20/ APRIL -2011
Vol 3 No.2 Year 2011

$$|\sigma_i(y_i'')| = \begin{cases} c\omega_3^{\phi}(f, n^{-1}, [-1,1])_P \operatorname{sgn}(f(y_i'')) & ; if \ |f(y_i'')| \le c\omega_3^{\phi}(f, n^{-1}, [-1,1])_P , \\ f(y_i'') \ ; & o.w. \end{cases}$$

If $f(y_i') = 0$, then $\operatorname{sgn}(f(y_i'))$, equals the sign f on (y_{i-1}, y_i) . Since $\sigma_i \in \Pi_2$, and $\sigma_i(y_i')$ and $\sigma_i(y_i'')$, have opposite signs , then the only zero of σ_i in I_i^* is y_i .

Hence , σ_i is copositive with f in I_i^* . Also , the first derivative of σ_i,

$$\sigma_i'(x) = \frac{2x - y_i - y_i'}{(y_i'' - y_i')(y_i'' - y_i)}\sigma_i(y_i'') + \frac{2x - y_i - y_i''}{(y_i'' - y_i')(y_i - y_i')}\sigma_i(y_i')$$

is a linear function , and

$$\sigma_i'\left(\frac{y_i + y_i'}{2}\right) = \frac{-\sigma_i(y_i')}{(y_i - y_i')}, \text{ and } \sigma_i'\left(\frac{y_i + y_i''}{2}\right) = \frac{\sigma_i(y_i'')}{(y_i'' - y_i)}$$

are of the same sign , which implies that σ_i', does not change sign in ρ_i, and for any $x \in \rho_i$.

$$\|\sigma_i'(x)\|_{L_P[-1,1]} \ge 2^{\frac{1}{P}} \min\left\{\left|\sigma_i'\left(\frac{y_i + y_i'}{2}\right)\right|, \left|\sigma_i'\left(\frac{y_i + y_i''}{2}\right)\right|\right\} = 2^{\frac{1}{P}} \min\left\{\frac{|\sigma_i(y_i')|}{|y_i - y_i'|}, \frac{|\sigma_i(y_i'')|}{|y_i'' - y_i|}\right\}$$

$$\ge 2^{\frac{1}{P}} \frac{1}{c\Delta_n(x)} \min\{|\sigma_i(y_i')| , |\sigma_i(y_i'')|\}$$

$$\ge 2^{\frac{1}{P}} \frac{1}{\Delta_n(x)} \min\{\omega_3^{\phi}(f, n^{-1}, [-1,1])_P \ , \omega_3^{\phi}(f, n^{-1}, [-1,1])_P\}$$

$$\ge 2^{\frac{1}{P}} \frac{1}{\Delta_n(x)} \omega_3^{\phi}(f, n^{-1}, [-1,1])_P . \qquad (2.5)$$

From [3], we have

$$\|f - \sigma_i\|_{L_P[-1,1]} \le C\omega_3^{\phi}(f, n^{-1}, [-1,1])_P . \qquad (2.6)$$

It is well known (see proof of Lemma 8 in [7]), that there exists a polynomial $Q_n(x)$, of degree $\le n$, which is a polynomial of best approximation to f in $[-1,1]$, and satisfying

$$\|f - Q_n\|_{L_P[-1,1]} \le C\omega_3^{\phi}(f, n^{-1}, [-1,1])_P , \qquad (2.7)$$

and

$$\|\phi(x)^3 Q_n^{(3)}(x)\|_{L_P[-1,1]} \le Cn^3\omega_3^{\phi}(f, n^{-1}, [-1,1])_P . \qquad (2.8)$$

Now , we define the piecewise polynomial function $S(x) \in C[-1,1]$, as follows

$$S(x)=\begin{cases}1 & ;if\ \ x\notin \bigcup_{i=1}^{k} \mathrm{I}_i^* , \\ 0 & ;if\ \ x\in \bigcup_{i=1}^{k} \rho_i , \\ c & ;if\ \ x\in \left[y_i', \frac{y_i+y_i'}{2}\right], 1\le i\le k .\end{cases}$$

Finally , the function

$$f_n(x)=\begin{cases}|Q_n(x)-\sigma_i(x)|S(x)+\sigma_i(x) & ;if\ \ x\in \mathrm{I}_i^* , \\ Q_n(x) \quad ; & o.w\end{cases}$$

is copositive with f in $\mathrm{Y}=\bigcup_{i=1}^{k}\rho_i$, and indeed f_n, coincides with σ_i in ρ_i, and , let C be an absolute constant such that

$\|f-f_n\|_{L_P[-1,1]}\le C\|f-\sigma_i\|_{L_P[-1,1]}\le C\omega_3^{\phi}\left(f,n^{-1},[-1,1]\right)_P$.

From (2.5), then

$$\|\Delta_n(x)f_n'(x)\|_{L_P[-1,1]}\ge Ch_{j(i)}\|f_n'(x)\|_{L_P[-1,1]}\ge C\Delta_n(x)\frac{1}{\Delta_n(x)}\omega_3^{\phi}\left(f,n^{-1},[-1,1]\right)_P$$

$$=C\omega_3^{\phi}\left(f,n^{-1},[-1,1]\right)_P .$$

Now , to prove the remaining (2.3), for $x\in\left[y_i',\frac{y_i+y_i'}{2}\right]$, $1\le i\le k$ (for $x\notin\bigcup_{i=1}^{k}\mathrm{I}_i^*$, from (2.8), we have (2.3) is valid , and for $x\in \mathrm{Y}$ it is trivial), from [4], look at

$$\left|\phi(x)^3 f_n^{(3)}(x)\right|\le Cn^3\left|\mathrm{I}_i^*\right|^3\sum_{v=0}^{3}\left|Q_n^{(v)}-\sigma_i^{(v)}\right|\left|S^{(3-v)}\right| \text{, such that}$$

$\phi(x)\approx n\Delta_n(x)\approx n\left|\mathrm{I}_i^*\right|$, for $x\in\mathrm{I}_i^*$, then

$$\begin{aligned}\left\|\phi(x)^3 f_n^{(3)}(x)\right\|_{L_P[-1,1]} &\le \left\|\phi(x)^3 f_n^{(3)}(x)\right\|_{L_\infty[-1,1]}\\ &\le Cn^3\left|\mathrm{I}_i^*\right|^3\sum_{v=0}^{3}\left\|Q_n^{(v)}-\sigma_i^{(v)}\right\|_{L_\infty[-1,1]}\left\|S^{(3-v)}\right\|_{L_\infty[-1,1]}\\ &\le C(p,v,n,2)n^3\left|\mathrm{I}_i^*\right|^{3-k-1}\sum_{v=0}^{3}\left\|Q_n-\sigma_i\right\|_{L_P[-1,1]}\left\|S\right\|_{L_P[-1,1]}\\ &\le C(p,v,n,2)n^3\left(\left\|Q_n-f\right\|_{L_P[-1,1]}+\left\|f-\sigma_i\right\|_{L_P[-1,1]}\right).\end{aligned}$$

Now , from (2.6) and (2.7), we get

$$\left\|\phi(x)^3 f_n^{(3)}(x)\right\|_{L_P[-1,1]}\le C(p,v,n,2)n^3\omega_3^{\phi}\left(f,n^{-1},[-1,1]\right)_P .$$

Also , let us introduce the following auxiliary proposition .

Proposition 2.9

If $\hat{f}$ in $C^3[-1,1]\cap L_P[-1,1]$ is such that $\left|\left(1-x^2\right)^{3/2}\hat{f}^{(3)}(x)\right|\le M$, $x\in[-1,1]$, $-1<y_1<...<y_k<1$, and $\delta=\min_{1\le i\le k+1}|y_{i+1}-y_i|$, then for every $n\ge C$, there exists a polynomial $p_n\in\Pi_n$, such that

$$\left\|\hat{f}-p_n\right\|_{L_P[-1,1]}\le C\omega_3^{\phi}\left(\hat{f},n^{-1},[-1,1]\right)_P, \qquad (2.10)$$

and

$$\left\|\Delta_n(x)\left(\hat{f}'-p_n'\right)\right\|_{L_P[-1,1]}\le C\frac{2}{n}\omega_2^{\phi}\left(\hat{f}',n^{-1},[-1,1]\right)_P \qquad (2.11)$$

where the constant C, depends only on k and p .

Proof. Note that (2,10) is trivial (see [1] theorem 3.2.1) . In (2.11) is valid since $x\in[-1,1]$, then from [1], we get

$$\left\|\Delta_n(x)\left(\hat{f}'-p_n'\right)\right\|_{L_P[-1,1]}\le\frac{2}{n}\left\|\hat{f}'-p_n'\right\|_{L_P[-1,1]}$$
$$\le C\frac{2}{n}\omega_2^{\phi}\left(\hat{f}',n^{-1},[-1,1]\right)_P .$$

Now , let us introduce the following theorem as a main result

Theorem 2.12

Let f in $L_P[a,b]\cap\Delta^0(Y_s)$, change sign $k\ge1$, times at $-1<y_1<...<y_k<1$, and let $\delta=\min_{0\le i\le k}|y_{i+1}-y_i|$, where $y_\circ=-1$ and $y_{k+1}=1$. Then there exists a constant C, such that for each $n>C$, there is a function g in $L_P[a,b]\cap\Delta^0(Y_s)$, copositive with f, and satisfying

$$\|f-g\|_{L_P[-1,1]}\le C\omega_3^{\phi}\left(f,n^{-1},[-1,1]\right)_P \qquad (2.13)$$

where the constant C, depends only on k .

Proof. If $n\ge4\delta^{-1}$, there exists $p_N\in\Pi_N$, let $g=p_N+2^k\|f-p_N\|_{L_p[-1,1]}\eta\prod_{i=1}^{k}T_N(y_i,x)$ in $L_P[a,b]\cap\Delta^0(Y_s)$, where N is sufficiently large ($N=\left(\left(18\sqrt{C}\right)+1\right)n$ will do)[4], and $\eta=\pm1$ is such that $\operatorname{sgn}(f(x))=\eta\prod_{i=1}^{k}\operatorname{sgn}(x-y_i)$.

Also , let f_n in $\Delta^3[-1,1]\cap\left(S-\Delta^0(Y_s)\right)$ be a function which was described in theorem 2.1 . (3.9) can be written as

$$\left\|\left(1-x^2\right)^{3/2} f_n^{(3)}(x)\right\|_{L_P[-1,1]} \le C(k)n^3\omega_3^\phi\left(f,n^{-1},[-1,1]\right)_P \ .$$

It follows from proposition 2.9, that there exists a polynomial $p_N \in \Pi_N$, best approximation to f_n and satisfies (2.7), such that

$$\|f_n - p_N\|_{L_P[-1,1]} \le \|f_n - f\|_{L_P[-1,1]} + \|f - p_N\|_{L_P[-1,1]}$$
$$\le C\omega_3^\phi\left(f,n^{-1},[-1,1]\right)_P \qquad (2.14)$$

and

$$\|\Delta_n(x)(f_n' - p_N')\|_{L_P[-1,1]} \le C\frac{2}{n}\omega_2^\phi\left(f_n',n^{-1},[-1,1]\right)_P \ . \qquad (2.15)$$

Together with (2.4), this implies $\operatorname{sgn}(p_N(x)) = \operatorname{sgn}(f_n(x)),\ x \in \Upsilon = \bigcup_{i=1}^{k}\rho_i$.

In turn , it follows that p_N is copositive with f in $\Upsilon = \bigcup_{i=1}^{k}\rho_i$, and also by (2.2), (2.10) and (2.14), we get

$$\|f-g\|_{L_P[-1,1]} = \|f - f_n + f_n - g\|_{L_P[-1,1]}$$
$$\le \|f - f_n\|_{L_P[-1,1]} + \|f_n - g\|_{L_P[-1,1]}$$
$$\le \|f - f_n\|_{L_P[-1,1]} + \left\|f_n - p_N - 2^k\|f - p_N\|_{L_P[-1,1]}\,\eta\prod_{i=1}^{k}T_N(y_i,x)\right\|_{L_P[-1,1]}$$
$$\le \|f - f_n\|_{L_P[-1,1]} + \|f_n - p_N\|_{L_P[-1,1]} + C\|f - p_N\|_{L_P[-1,1]}$$
$$\le C\omega_3^\phi\left(f,n^{-1},[-1,1]\right)_P \ .$$

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