

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)



Common Fixed Point for four mappings with weakly compatible in Partial b-Metric Space

Ahmed Ibraheem Abd Zaid

Mathematics Department, Sciences College, Al-Qadisiyah University, Diwaniyah, Iraq, E mail: ahmed.ibraheem@qu.edu.iq

ARTICLE INFO

Article history:

Received: 30 /04/2026

Revised form: 31 /05/2026

Accepted : 01 /06/2026

Available online: 30 /09/2026

Keywords:

fixed point, coincidence point,
common fixed point, weakly
compatible, partial b-metric space

ABSTRACT

In this paper was Our principal goal is proving common fixed-point results by using different condition with addition to dependent to idea of weakly compatible on partial b-metric space (for short PBMS). we obtained improvement for recent results common fixed-point theorem in partial b-metric space

MSC: 47H10, 54H25

<https://doi.org/10.29304/jqcm.2026.18.32860>

1. Introduction

A fixed-point theory is a Hight important section of a nonlinear functional analysis. One of primary results that appeared in field of fixed point due to 1922 by Banach [2] and famous as Banach contraction principle (BCP). There is a several of generalizations appeared of usual metric space. One of these generalizations due to 1993 by S. Czerwik [4] in was offer idea of b-metric space (BMS) by through change in trigonometric inequality that by multiply by $b \geq 1$, he reproved a famous BCP in this space. As well a set of results have come out in the same space see, ([5], [10], [11], [19]). S. In 1994 presented an idea of partial metric space (PMS) by G. Matthews [12] as a generalization of usual metric space where the self-distance of any point in this space not necessarily have zero value. As well proved BCP in PMS. A set of fixed-point results studied this space by different scholars see, ([1], [3], [7], [9], [14]). S. Shukla [15] in 2014, presented one of the most important improvements of those spaces named partial b-metric space (PBMS) and studied BCP and Kannan condition in by using the same space. As well number of

results had come out later, see ([6], [8], [13], [16], [17], [18], [20]). this study we worked to prove common fixed-point results with addition to dependent to idea of weakly compatible PBMS.

*Corresponding author: Ahmed Ibraheem Abd Zaid

Email addresses: ahmed.ibraheem@qu.edu.iq

Communicated by 'sub etitor'

2. Preliminaries:

Definition 2.1 [12] Consider $\mathcal{M}$ a non-null set, defined a partial metric on $\mathcal{M}$ as a function $d_\rho: \mathcal{M}^2 \rightarrow \mathbb{R}^+$ verifies the following;

$$(d_{\rho 1}) \sigma = \partial \leftrightarrow d_\rho(\sigma, \partial) = d_\rho(\sigma, \sigma) = d_\rho(\partial, \partial)$$

$$(d_{\rho 2}) d_\rho(\sigma, \sigma) \leq d_\rho(\sigma, \partial)$$

$$(d_{\rho 3}) d_\rho(\sigma, \partial) = d_\rho(\partial, \sigma)$$

$$(d_{\rho 4}) d_\rho(\sigma, \partial) \leq d_\rho(\sigma, k) + d_\rho(k, \partial) - d_\rho(k, k)$$

for all $\sigma, \partial, k \in \mathcal{M}$. Then $(\mathcal{M}, d_\rho)$ called PMS.

Definition 2.2 [4] Consider $\mathcal{M}$ a non-null set, defined a b-metric on $\mathcal{M}$ as a function $d_b: \mathcal{M}^2 \rightarrow \mathbb{R}^+$ verifies the following;

$$(d_{b1}) d_b(\sigma, \partial) = 0 \leftrightarrow \sigma = \partial$$

$$(d_{b2}) d_b(\sigma, \partial) = d_b(\partial, \sigma)$$

$$(d_{b3}) d_b(\sigma, \partial) \leq b[d_b(\sigma, k) + d_b(k, \partial)] \text{ where } b \geq 1$$

for all $\sigma, \partial, k \in \mathcal{M}$. Then $(\mathcal{M}, d_b)$ called BMS.

Definition 2.3 [15] Consider $\mathcal{M}$ a non-null set, defined a partial b-metric on $\mathcal{M}$ as a function $d_{\rho b}: \mathcal{M}^2 \rightarrow \mathbb{R}^+$ verifies the following;

$$(d_{\rho b1}) \sigma = \partial \leftrightarrow d_{\rho b}(\sigma, \partial) = d_{\rho b}(\sigma, \sigma) = d_{\rho b}(\partial, \partial)$$

$$(d_{\rho b2}) d_{\rho b}(\sigma, \sigma) \leq d_{\rho b}(\sigma, \partial)$$

$$(d_{\rho b3}) d_{\rho b}(\sigma, \partial) = d_{\rho b}(\partial, \sigma)$$

$$(d_{\rho b4}) d_{\rho b}(\sigma, \partial) \leq b[d_{\rho b}(\sigma, k) + d_{\rho b}(k, \partial)] - d_{\rho b}(k, k)$$

for all $\sigma, \partial, k \in \mathcal{M}$. Then $(\mathcal{M}, d_{\rho b})$ called PBMS

In PBMS, clearly $d_{\rho b}(\sigma, \partial) = 0$, by $(d_{\rho b1})$, $(d_{\rho b2})$ it follows $\sigma = \partial$. Whereas the converse not hold in general. Every b-Metric defined by Partial b-Metric as following

$$d_b(\sigma, \partial) = 2d_{\rho b}(\sigma, \partial) - d_{\rho b}(\sigma, \sigma) - d_{\rho b}(\partial, \partial) \quad \forall \sigma, \partial \in \mathcal{M}$$

Example 2.1 [8] Let $\mathcal{M} = \mathbb{R}^+$ and $d_{\rho b}: \mathcal{M}^2 \rightarrow \mathbb{R}^+$ de a self-map and defined by $d_{\rho b}(\sigma, \partial) = [\max\{\sigma, \partial\}]^4 + |\sigma - \partial|^4$ $\forall \sigma, \partial \in \mathcal{M}$. Then $(\mathcal{M}, d_{\rho b})$ is PBMS with $b = 2^4 > 1$. but it is neither a (B. M. S) nor (P. M. S). Indeed, for any $\sigma > 0$ we have $d_{\rho b}(\sigma, \sigma) = \sigma^4 \neq 0$; therefore, $d_{\rho b}$ is not a b-metric on $\mathcal{M}$. Also for $\sigma = 6, \partial = 2, k = 3$ we have $d_{\rho b}(\sigma, \partial) = 6^4 + 4^4$ and $d_{\rho b}(\sigma, k) + d_{\rho b}(k, \partial) - d_{\rho b}(k, k) = 6^4 + 3^4 + 3^4 + 1^4 - 3^4 = 6^4 + 3^4 + 1^4$ so $d_{\rho b}(\sigma, \partial) > d_{\rho b}(\sigma, k) + d_{\rho b}(k, \partial) - d_{\rho b}(k, k)$, $\forall b = 2^4$. Therefore, $d_{\rho b}$ is not a partial metric on $\mathcal{M}$.

Definition 2.4 [15]

- 1- Consider that $\{v_n\}$ be a sequence in a PBMS $(\mathcal{M}, d_{\rho b})$ then $\{v_n\}$ called convergent to the point $v \in \mathcal{M} \Leftrightarrow \lim_{n \rightarrow \infty} d_{\rho b}(v_n, v) = d_{\rho b}(v, v)$
- 2- Consider that $\{v_n\}$ be a sequence in a PBMS $(\mathcal{M}, d_{\rho b})$ then $\{v_n\}$ called Cauchy $\Leftrightarrow \lim_{m, n \rightarrow \infty} d_{\rho b}(v_m, v_n)$ exists (finite).
- 3- Consider that $(\mathcal{M}, d_{\rho b})$ PBMS. if for Cauchy sequence $\{v_n\}$ in $\mathcal{M}$. $\exists v \in \mathcal{M}$ such that $\lim_{m, n \rightarrow \infty} d_{\rho b}(v_n, v_m) = \lim_{n \rightarrow \infty} d_{\rho b}(v_n, v) = d_{\rho b}(v, v)$ then $(\mathcal{M}, d_{\rho b})$ called complete and denoted by (CPBMS) as short

Definition 2.5 [13] Consider that $\mathcal{M}$ be a non-null set and $P, Q : \mathcal{M} \rightarrow \mathcal{M}$. If $a = P\sigma = Q\sigma$, for some $\sigma \in \mathcal{M}$, then σ is side to be a coincidence point of two mapping P and Q thus, a side to be a point of coincidence of two mapping P and Q .

If $a = \sigma$ then in this case σ is side to be common fixed point of P and Q .

Definition 2.6 [8] Consider that P and Q two self- mapping and define on a non-null set $\mathcal{M}$. P and Q is side to be weakly compatible if P and Q commute in their coincident point i.e. if $P\sigma = Q\sigma$ for some $\sigma \in \mathcal{M}$, then $PQ\sigma = QP\sigma$.

3. Main results

Theorem 3.1 Consider $(\mathcal{M}, d_{\rho b})$ CPBMS with $b \geq 1$. Consider $P, Q, S, T : \mathcal{M} \rightarrow \mathcal{M}$ be self-map s.t $T(\mathcal{M}) \subseteq P(\mathcal{M})$ and $S(\mathcal{M}) \subseteq Q(\mathcal{M})$ s.t

$$\begin{aligned}
 d_{\rho b}(S\sigma, T\partial) &\leq c_1 d_{\rho b}(P\sigma, Q\partial) + c_2 d_{\rho b}(Q\partial, T\partial) + c_3 [d_{\rho b}(P\sigma, T\partial) + d_{\rho b}(Q\partial, S\sigma)] \\
 &+ c_4 \frac{d_{\rho b}(Q\partial, T\partial)[1 + d_{\rho b}(P\sigma, S\sigma)]}{[1 + d_{\rho b}(P\sigma, Q\partial)]} \\
 &+ c_5 \min\{d_{\rho b}(P\sigma, Q\partial), d_{\rho b}(Q\partial, T\partial), d_{\rho b}(S\sigma, P\partial)\}
 \end{aligned} \tag{3.1}$$

$\forall \sigma, \partial \in \mathcal{M}$, $c_1, c_2, c_3, c_4, c_5 \geq 0$ and $0 < c_1 + c_2 + 2c_3b + c_4 + c_5 < 1$. And satisfying the following conditions:

- (1) The pairs (S, P) and (T, Q) are weakly compatible.
- (2) If $P(X), Q(X)$ complete subspace of X . Then P, Q, T and S have unique common fixed point.

Proof: (i) Existence

suppose $x_0 \in \mathcal{M}$, construct sequence $\{x_n\}$ and $\{v_n\}$ in $\mathcal{M}$ in the following manner

since $S(\mathcal{M}) \subseteq Q(\mathcal{M})$, $\exists x_1 \in \mathcal{M}$ s.t $Qx_1 = Sx_0 = v_0$.

Since $T(\mathcal{M}) \subseteq P(\mathcal{M})$, we can choose $x_2 \in \mathcal{M}$ s.t $Px_2 = Tx_1 = v_1$

continuing in this process for all chosen $x_n \in \mathcal{M}$, we obtain $x_{n+1} \in \mathcal{M}$ s.t

$$Qx_{2n+1} = Sx_{2n} \text{ and } Px_{2n+2} = Tx_{2n+1}, n \in N$$

In general, define sequence $\{v_{2n}\}$ in $\mathcal{M}$ by denoting

$$v_{2n} = Qx_{2n+1} = Sx_{2n} \text{ and } v_{2n+1} = Px_{2n+2} = Tx_{2n+1}, n \in N \text{ and by using (3.1) we have}$$

$$\begin{aligned}
 d_{\rho b}(v_{2n}, v_{2n+1}) &= d_{\rho b}(Sx_{2n}, Tx_{2n+1}) \\
 &\leq c_1 d_{\rho b}(Px_{2n}, Qx_{2n+1}) + c_2 d_{\rho b}(Qx_{2n+1}, Tx_{2n+1}) + c_3 [d_{\rho b}(Px_{2n}, Tx_{2n+1})
 \end{aligned}$$

$$\begin{aligned}
& + d_{\rho b}(Qx_{2n+1}, Sx_{2n})] + c_4 \frac{d_{\rho b}(Qx_{2n+1}, Tx_{2n+1})[1 + d_{\rho b}(Px_{2n}, Sx_{2n})]}{[1 + d_{\rho b}(Px_{2n}, Qx_{2n+1})]} \\
& + c_5 \min\{d_{\rho b}(Px_{2n}, Qx_{2n+1}), d_{\rho b}(Qx_{2n+1}, Tx_{2n+1}), d_{\rho b}(Sx_{2n}, Px_{2n+1})\} \\
& \leq c_1 d_{\rho b}(v_{2n-1}, v_{2n}) + c_2 d_{\rho b}(v_{2n}, v_{2n+1}) + c_3 [d_{\rho b}(v_{2n-1}, v_{2n+1}) + d_{\rho b}(v_{2n}, v_{2n})] \\
& \quad + c_4 \frac{d_{\rho b}(v_{2n}, v_{2n+1})[1 + d_{\rho b}(v_{2n-1}, v_{2n})]}{[1 + d_{\rho b}(v_{2n-1}, v_{2n})]} \\
& \quad + c_5 \min\{d_{\rho b}(v_{2n-1}, v_{2n}), d_{\rho b}(v_{2n}, v_{2n+1}), d_{\rho b}(v_{2n-1}, v_{2n+1}), d_{\rho b}(v_{2n}, v_{2n})\}
\end{aligned}$$

by using $(d_{\rho b4})$ we have

$$d_{\rho b}(v_{2n-1}, v_{2n+1}) \leq b(d_{\rho b}(v_{2n-1}, v_{2n}) + d_{\rho b}(v_{2n}, v_{2n+1})) - d_{\rho b}(v_{2n}, v_{2n})$$

by $(d_{\rho b2})$ we have $d_{\rho b}(v_{2n}, v_{2n}) \leq d_{\rho b}(v_{2n-1}, v_{2n})$ and $d_{\rho b}(v_{2n}, v_{2n}) \leq d_{\rho b}(v_{2n}, v_{2n+1})$ so

$$\min\{d_{\rho b}(v_{2n-1}, v_{2n}), d_{\rho b}(v_{2n}, v_{2n+1}), d_{\rho b}(v_{2n}, v_{2n})\} = d_{\rho b}(v_{2n}, v_{2n}), \text{ hence}$$

$$\begin{aligned}
d_{\rho b}(v_{2n}, v_{2n+1}) & \leq c_1 d_{\rho b}(v_{2n-1}, v_{2n}) \\
& \quad + c_2 d_{\rho b}(v_{2n}, v_{2n+1}) + c_3 [b(d_{\rho b}(v_{2n-1}, v_{2n}) + d_{\rho b}(v_{2n}, v_{2n+1})) - d_{\rho b}(v_{2n}, v_{2n}) + d_{\rho b}(v_{2n}, v_{2n})] \\
& \quad + c_4 d_{\rho b}(v_{2n}, v_{2n+1}) + c_5 d_{\rho b}(v_{2n}, v_{2n})
\end{aligned}$$

$$\begin{aligned}
d_{\rho b}(v_{2n}, v_{2n+1}) & \leq c_1 d_{\rho b}(v_{2n-1}, v_{2n}) + c_2 d_{\rho b}(v_{2n}, v_{2n+1}) + c_3 [b(d_{\rho b}(v_{2n-1}, v_{2n}) + d_{\rho b}(v_{2n}, v_{2n+1}))] \\
& \quad + c_4 d_{\rho b}(v_{2n}, v_{2n+1}) + c_5 d_{\rho b}(v_{2n}, v_{2n}) \\
& \leq c_1 (d_{\rho b}(v_{2n-1}, v_{2n}) + c_2 d_{\rho b}(v_{2n}, v_{2n+1}) + c_3 b d_{\rho b}(v_{2n-1}, v_{2n}) + c_3 b d_{\rho b}(v_{2n}, v_{2n+1})) \\
& \quad + c_4 d_{\rho b}(v_{2n}, v_{2n+1}) + c_5 d_{\rho b}(v_{2n}, v_{2n})
\end{aligned}$$

Also, since $d_{\rho b}(v_{2n}, v_{2n}) \leq c_1 d_{\rho b}(v_{2n-1}, v_{2n})$ we have

$$\begin{aligned}
d_{\rho b}(v_{2n}, v_{2n+1}) & \leq c_1 d_{\rho b}(v_{2n-1}, v_{2n}) + c_2 d_{\rho b}(v_{2n}, v_{2n+1}) + c_3 b d_{\rho b}(v_{2n-1}, v_{2n}) + c_3 b d_{\rho b}(v_{2n}, v_{2n+1}) \\
& \quad + c_4 d_{\rho b}(v_{2n}, v_{2n+1}) + c_5 d_{\rho b}(v_{2n-1}, v_{2n})
\end{aligned}$$

$$(1 - c_2 - c_3 b - c_4) d_{\rho b}(v_{2n}, v_{2n+1}) \leq (c_1 + c_3 b + c_5) d_{\rho b}(v_{2n-1}, v_{2n})$$

$$d_{\rho b}(v_{2n}, v_{2n+1}) \leq \frac{c_1 + c_3 b + c_5}{1 - c_2 - c_3 b - c_4} d_{\rho b}(v_{2n-1}, v_{2n})$$

$$\text{let } \beta = \frac{c_1 + c_3 b + c_5}{1 - c_2 - c_3 b - c_4}, 0 < \beta < 1$$

$$d_{\rho b}(v_{2n}, v_{2n+1}) \leq \beta d_{\rho b}(v_{2n-1}, v_{2n})$$

$$d_{\rho b}(v_{2n}, v_{2n+1}) \leq \beta d_{\rho b}(v_{2n-1}, v_{2n}) \leq \beta^2 d_{\rho b}(v_{2n-2}, v_{2n-1}) \leq \beta^3 d_{\rho b}(v_{2n-3}, v_{2n-2}) \leq \dots$$

$$\leq \beta^n d_{\rho b}(v_0, v_1)$$

$$\lim_{n \rightarrow \infty} d_{\rho b}(v_{2n}, v_{2n+1}) = 0$$

$$\text{In general, } d_{\rho b}(v_n, v_{n+1}) \leq \beta^n d_{\rho b}(v_0, v_1)$$

$$\begin{aligned}
d_{\rho b}(v_n, v_m) & \leq b[d_{\rho b}(v_n, v_{n+1}) + d_{\rho b}(v_{n+1}, v_m)] - d_{\rho b}(v_{n+1}, v_{n+1}) \leq b[d_{\rho b}(v_n, v_{n+1}) + d_{\rho b}(v_{n+1}, v_m)] \\
& \leq b d_{\rho b}(v_n, v_{n+1}) + b(b[d_{\rho b}(v_{n+1}, v_{n+2}) + d_{\rho b}(v_{n+2}, v_m)] - d_{\rho b}(v_{n+2}, v_{n+2}))
\end{aligned}$$

$$\begin{aligned}
 &\leq b d_{\rho b}(v_n, v_{n+1}) + b^2 d_{\rho b}(v_{n+1}, v_{n+2}) + b^2 d_{\rho b}(v_{n+2}, v_m) \\
 &\vdots \\
 &\leq b d_{\rho b}(v_n, v_{n+1}) + b^2 d_{\rho b}(v_{n+1}, v_{n+2}) + b^3 d_{\rho b}(v_{n+2}, v_{n+3}) + \dots + b^{m-n} d_{\rho b}(v_{m-1}, v_m) \\
 &\leq b \beta^n d_{\rho b}(v_0, v_1) + b^2 \beta^{n+1} d_{\rho b}(v_0, v_1) + b^3 \beta^{n+2} d_{\rho b}(v_0, v_1) + \dots + b^{m-n} \beta^{m-1} d_{\rho b}(v_0, v_1) \\
 &\leq b \beta^n (1 + b \beta + (b \beta)^2 + \dots + (b \beta)^{m-n-1}) d_{\rho b}(v_0, v_1)
 \end{aligned}$$

$\lim_{n,m \rightarrow \infty} d_{\rho b}(v_n, v_m) = 0$. Thus $\{v_n\}$ be Cauchy sequence in $\mathcal{M}$. Since $\mathcal{M}$ is CPBMS then $\exists v \in \mathcal{M}$ s.t

$$\lim_{n,m \rightarrow \infty} d_{\rho b}(v_n, v_m) = \lim_{n \rightarrow \infty} d_{\rho b}(v_n, v) = d_{\rho b}(v, v) = 0 \quad (3.2)$$

We need to show that v is common fixed point of S, T, P and Q

Since $T(\mathcal{M}) \subseteq P(\mathcal{M})$ and $P(\mathcal{M})$ complete subspace of $\mathcal{M}$ so $v \in P(\mathcal{M})$ i.e $\exists u \in \mathcal{M}$ s.t $Pu = v$

now we have to prove that $Su = v$

$$\begin{aligned}
 d_{\rho b}(Su, Tx_{2n+1}) &\leq c_1 d_{\rho b}(Pu, Qx_{2n+1}) + c_2 d_{\rho b}(Qx_{2n+1}, Tx_{2n+1}) + c_3 [d_{\rho b}(Pu, Tx_{2n+1}) + d_{\rho b}(Qx_{2n+1}, Su)] \\
 &\quad + c_4 \frac{d_{\rho b}(Qx_{2n+1}, Tx_{2n+1}) [1 + d_{\rho b}(Pu, Su)]}{[1 + d_{\rho b}(pu, Qx_{2n+1})]} \\
 &\quad + c_5 \min\{d_{\rho b}(Pu, Qx_{2n+1}), d_{\rho b}(Qx_{2n+1}, Tx_{2n+1}), d_{\rho b}(Su, Px_{2n+1})\} \\
 d_{\rho b}(Su, v_{2n+1}) &\leq c_1 d_{\rho b}(Pu, v_{2n}) + c_2 d_{\rho b}(v_{2n}, v_{2n+1}) + c_3 [d_{\rho b}(Pu, v_{2n+1}) + d_{\rho b}(v_{2n}, Su)] \\
 &\quad + c_4 \frac{d_{\rho b}(v_{2n}, v_{2n+1}) [1 + d_{\rho b}(Pu, Su)]}{[1 + d_{\rho b}(pu, v_{2n})]} + c_5 \min\{d_{\rho b}(Pu, v_{2n}), d_{\rho b}(v_{2n}, v_{2n+1}), d_{\rho b}(Su, v_{2n})\}
 \end{aligned}$$

Taking limit as $n \rightarrow \infty$, we get

$$\begin{aligned}
 d_{\rho b}(Su, v) &\leq c_1 d_{\rho b}(Pu, v) + c_2 d_{\rho b}(v, v) + c_3 [d_{\rho b}(Pu, v) + d_{\rho b}(v, Su)] + c_4 \frac{d_{\rho b}(v, v) [1 + d_{\rho b}(Pu, Su)]}{[1 + d_{\rho b}(pu, v)]} \\
 &\quad + c_5 \min\{d_{\rho b}(Pu, v), d_{\rho b}(v, v), d_{\rho b}(Pu, v), d_{\rho b}(Su, v)\}
 \end{aligned}$$

Since $d_{\rho b}(v, v) = 0$ (by (3.2)) then $\min\{d_{\rho b}(Pu, v), d_{\rho b}(v, v), d_{\rho b}(Su, v)\} = 0$, hence

$$d_{\rho b}(Su, v) \leq c_1 d_{\rho b}(Pu, v) + c_3 d_{\rho b}(Pu, v) + c_3 d_{\rho b}(v, Su)$$

$$(1 - c_3) d_{\rho b}(Su, v) \leq (c_1 + c_3) d_{\rho b}(Pu, v)$$

Since $Pu = v$. Then

$$(1 - c_3) d_{\rho b}(Su, v) \leq (c_1 + c_3) d_{\rho b}(v, v)$$

Also, since $d_{\rho b}(v, v) = 0$, we have

$$d_{\rho b}(Su, v) \leq 0, \text{ thus}$$

$$d_{\rho b}(Su, v) = 0, \text{ (since } d_{\rho b} \text{ non negative), thus from definition (2.3) we have } Su = v$$

So $Pu = Su = v$, and S, P have coincidence point.

since $S(\mathcal{M}) \subseteq Q(\mathcal{M})$ and $Q(\mathcal{M})$ complete subspace of $\mathcal{M}$ so $v \in Q(\mathcal{M})$ i.e $\exists w \in \mathcal{M}$ s.t $v = Qw$

now we have to prove that $Tw = v$

$$d_{\rho b}(Su, Tw) \leq c_1 d_{\rho b}(Pu, Qw) + c_2 d_{\rho b}(Qw, Tw) + c_3 [d_{\rho b}(Pu, Tw) + d_{\rho b}(Qw, Su)] \\ + c_4 \frac{d_{\rho b}(Qw, Tw)[1 + d_{\rho b}(Pu, Su)]}{[1 + d_{\rho b}(Pu, Qw)]} + c_5 \min\{d_{\rho b}(Pu, Qw), d_{\rho b}(Qw, Tw), d_{\rho b}(Su, Pw)\}$$

$$d_{\rho b}(v, Tw) \leq c_1 d_{\rho b}(v, v) + c_2 d_{\rho b}(v, Tw) + c_3 [d_{\rho b}(v, Tw) + d_{\rho b}(v, v)] + c_4 \frac{d_{\rho b}(v, Tw)[1 + d_{\rho b}(v, v)]}{[1 + d_{\rho b}(v, v)]} \\ + c_5 \min\{d_{\rho b}(v, v), d_{\rho b}(v, Tw), d_{\rho b}(v, Pw)\}$$

Since $d_{\rho b}(v, v) = 0$. Then $\min\{d_{\rho b}(v, v), d_{\rho b}(v, Tw), d_{\rho b}(v, Pw)\} = 0$. Thus

$$d_{\rho b}(v, Tw) \leq c_2 d_{\rho b}(v, Tw) + c_3 d_{\rho b}(v, Tw) + c_4 d_{\rho b}(v, Tw)$$

$$(1 - c_2 - c_3 - c_4) d_{\rho b}(v, Tw) \leq 0$$

$$d_{\rho b}(v, Tw) \leq 0$$

$d_{\rho b}(v, Tw) = 0$, (since $d_{\rho b}$ non negative), thus from definition (3.2) we have $Tw = v$

So $Tw = Qw = v$

Now since $Tw = Pw = v$ and $Pu = Su = v$, since (P, S) and (Q, T) are weak compatible then

$$Pv = PSu = SPu = Sv$$

$$Qv = QT w = TQw = Tv$$

Now we will show that $Pv = v$

$$d_{\rho b}(Su, Tw) = d_{\rho b}(Pv, Qw) + c_2 d_{\rho b}(Qw, Tw) + c_3 [d_{\rho b}(Pv, Tw) + d_{\rho b}(Qw, Sv)] + c_4 \frac{d_{\rho b}(Qw, Tw)[1 + d_{\rho b}(Pv, Sv)]}{[1 + d_{\rho b}(Pv, Qw)]} \\ + c_5 \min\{d_{\rho b}(Pv, Qw), d_{\rho b}(Qw, Tw), d_{\rho b}(Sv, Pw)\}$$

$$d_{\rho b}(Su, v) = c_1 d_{\rho b}(Pv, v) + c_2 d_{\rho b}(v, v) + c_3 [d_{\rho b}(Pv, v) + d_{\rho b}(v, Sv)] + c_4 \frac{d_{\rho b}(v, v)[1 + d_{\rho b}(Pv, Sv)]}{[1 + d_{\rho b}(Pv, v)]} \\ + c_5 \min\{d_{\rho b}(Pv, v), d_{\rho b}(v, v), d_{\rho b}(Sv, v)\}$$

Since $d_{\rho b}(v, v) = 0$ hence $\min\{d_{\rho b}(Pv, v), d_{\rho b}(v, v), d_{\rho b}(Sv, v)\} = d_{\rho b}(v, v) = 0$

$$d_{\rho b}(Su, v) = c_1 d_{\rho b}(Pv, v) + c_3 d_{\rho b}(Pv, v) + c_3 d_{\rho b}(v, Sv)$$

Since $Pv = Sv$, we have

$$d_{\rho b}(Pv, v) = c_1 d_{\rho b}(Pv, v) + c_3 d_{\rho b}(Pv, v) + c_3 d_{\rho b}(v, Pv)$$

$$(1 - c_1 - 2c_3) d_{\rho b}(Pv, v) \leq 0$$

$$d_{\rho b}(Pv, v) \leq 0$$

$d_{\rho b}(Pv, v) = 0$ (since $d_{\rho b}$ non negative), thus from definition (3.2) we have

$$Pv = v$$

Since $Pv = Sv$, then we have $Pv = Sv = v$.

Now we will show that $Qv = v$

$$d_{\rho b}(Su, Tv) \leq c_1 d_{\rho b}(Pu, Qv) + c_2 d_{\rho b}(Qv, Tv) + c_3 [d_{\rho b}(Pu, Tv) + d_{\rho b}(Qv, Su)] + c_4 \frac{d_{\rho b}(Qv, Tv)[1 + d_{\rho b}(Pu, Su)]}{[1 + d_{\rho b}(pu, Qv)]} + c_5 \min\{d_{\rho b}(Pu, Qv), d_{\rho b}(Qv, Tv), d_{\rho b}(Su, Pv)\}$$

$$d_{\rho b}(v, Qv) \leq c_1 d_{\rho b}(v, Qv) + c_2 d_{\rho b}(Qv, Qv) + c_3 [d_{\rho b}(v, Qv) + d_{\rho b}(Qv, v)] + c_4 \frac{d_{\rho b}(Qv, Qv)[1 + d_{\rho b}(v, v)]}{[1 + d_{\rho b}(v, Qv)]} + c_5 \min\{d_{\rho b}(v, Qv), d_{\rho b}(Qv, Qv), d_{\rho b}(v, v)\}$$

Since $d_{\rho b}(v, v) = 0$ (by (3.2)) we have $\min\{d_{\rho b}(v, Qv), d_{\rho b}(Qv, Qv), d_{\rho b}(v, v)\} = d_{\rho b}(v, v)$

$$d_{\rho b}(v, Qv) \leq c_1 d_{\rho b}(v, Qv) + c_2 d_{\rho b}(Qv, Qv) + c_3 [d_{\rho b}(v, Qv) + d_{\rho b}(Qv, v)] + c_4 \frac{d_{\rho b}(Qv, Qv)[1 + d_{\rho b}(v, v)]}{[1 + d_{\rho b}(v, Qv)]}$$

Since $d_{\rho b}(Qv, Qv) \leq d_{\rho b}(v, Qv)$ (by $d_{\rho b2}$) also $d_{\rho b}(v, v) \leq d_{\rho b}(v, Qv)$ (by $(d_{\rho b2})$). Then

$$d_{\rho b}(v, Qv) \leq c_1 d_{\rho b}(v, Qv) + c_2 d_{\rho b}(v, Qv) + 2c_3 d_{\rho b}(v, Qv) + c_4 \frac{d_{\rho b}(v, Qv)[1 + d_{\rho b}(v, Qv)]}{[1 + d_{\rho b}(v, Qv)]}$$

$$d_{\rho b}(v, Qv) \leq c_1 d_{\rho b}(v, Qv) + c_2 d_{\rho b}(v, Qv) + 2c_3 d_{\rho b}(v, Qv) + c_4 d_{\rho b}(v, Qv)$$

$$(1 - c_1 - c_2 - 2c_3 - c_4) d_{\rho b}(v, Qv) \leq 0$$

$$d_{\rho b}(v, Qv) \leq 0$$

$d_{\rho b}(v, Qv) = 0$ (since $d_{\rho b}$ non negative), thus from definition (3.2) we have $Qv = v$, thus

$$Qv = QTv = TQv = Tv = v$$

Hence $Tv = Qv = v$ Therefore $Tv = Qv = Pv = Sv = v$. Then v is common fixed point of P, Q, S, T .

(ii) Uniqueness

Let r another common fixed point of P, Q, S and T that is $Pr = Qr = Sr = Tr = r$ s.t $r \neq v$

$$d_{\rho b}(v, \sigma) = d(Sv, T\sigma) \leq c_1 d_{\rho b}(Pv, Q\sigma) + c_2 d_{\rho b}(Q\sigma, T\sigma) + c_3 [d_{\rho b}(Pv, T\sigma) + d_{\rho b}(Q\sigma, Sv)] + c_4 \frac{d_{\rho b}(Q\sigma, T\sigma)[1 + d_{\rho b}(Pv, Sv)]}{[1 + d_{\rho b}(Pv, Q\sigma)]} + c_5 \min\{d_{\rho b}(Pv, Q\sigma), d_{\rho b}(Q\sigma, T\sigma), d_{\rho b}(Sv, P\sigma)\}$$

$$d_{\rho b}(v, \sigma) \leq c_1 d_{\rho b}(v, \sigma) + c_2 d_{\rho b}(\sigma, \sigma) + c_3 [d_{\rho b}(v, \sigma) + d_{\rho b}(\sigma, v)] + c_4 \frac{d_{\rho b}(\sigma, \sigma)[1 + d_{\rho b}(v, v)]}{[1 + d_{\rho b}(v, \sigma)]} + c_5 \min\{d_{\rho b}(v, \sigma), d_{\rho b}(\sigma, \sigma), d_{\rho b}(v, v)\}$$

Since $d_{\rho b}(\sigma, \sigma) \leq d_{\rho b}(v, \sigma)$ and $d_{\rho b}(v, v) \leq d_{\rho b}(v, \sigma)$ (by $d_{\rho b2}$) so

$$\min\{d_{\rho b}(v, \sigma), d_{\rho b}(\sigma, \sigma), d_{\rho b}(v, v)\} = d_{\rho b}(\sigma, \sigma)$$

$$d_{\rho b}(\mathbf{v}, \sigma) \leq c_1 d_{\rho b}(\mathbf{v}, \sigma) + c_2 d_{\rho b}(\mathbf{v}, \sigma) + c_3 [d_{\rho b}(\mathbf{v}, \sigma) + d_{\rho b}(\sigma, \mathbf{v})] + c_4 \frac{d_{\rho b}(\mathbf{v}, \sigma)[1 + d_{\rho b}(\mathbf{v}, \sigma)]}{[1 + d_{\rho b}(\mathbf{v}, \sigma)]} + c_5 d_{\rho b}(\sigma, \sigma)$$

$$\text{Also } d_{\rho b}(\sigma, \sigma) \leq d_{\rho b}(\mathbf{v}, \sigma)$$

$$d_{\rho b}(\mathbf{v}, \sigma) \leq c_1 d_{\rho b}(\mathbf{v}, \sigma) + c_2 d_{\rho b}(\mathbf{v}, \sigma) + c_3 [d_{\rho b}(\mathbf{v}, \sigma) + d_{\rho b}(\sigma, \mathbf{v})] + c_4 \frac{d_{\rho b}(\mathbf{v}, \sigma)[1 + d_{\rho b}(\mathbf{v}, \sigma)]}{[1 + d_{\rho b}(\mathbf{v}, \sigma)]} + c_5 d_{\rho b}(\mathbf{v}, \sigma)$$

$$d_{\rho b}(\mathbf{v}, \sigma) \leq c_1 d_{\rho b}(\mathbf{v}, \sigma) + c_2 d_{\rho b}(\mathbf{v}, \sigma) + 2c_3 d_{\rho b}(\mathbf{v}, \sigma) + c_4 d_{\rho b}(\mathbf{v}, \sigma) + c_5 d_{\rho b}(\mathbf{v}, \sigma)$$

$$(1 - c_1 - c_2 - 2c_3 - c_4 - c_5) d_{\rho b}(\mathbf{v}, \sigma) \leq 0$$

$$d_{\rho b}(\mathbf{v}, \sigma) \leq 0$$

$$d_{\rho b}(\mathbf{v}, \sigma) = 0 \text{ (since } d_{\rho b} \text{ non negative)}$$

thus, from definition (2.3) we have $\mathbf{v} = \sigma$. Contradiction

Then P, Q, T and S have unique common fixed point.

We can use the above theorem to obtain a number of corollaries by take values $c_i = 0$ for some $i = 1, 2, 3, 4$ as a following

Corollary 3.1 Consider $(\mathcal{M}, d_{\rho b})$ CPBMS with $b \geq 1$. Consider $P, Q, S, T: \mathcal{M} \rightarrow \mathcal{M}$ be self-map s.t $T(\mathcal{M}) \subseteq P(\mathcal{M})$ and $S(\mathcal{M}) \subseteq Q(\mathcal{M})$ s.t

$$d_{\rho b}(S\sigma, T\partial) \leq c_1 d_{\rho b}(P\sigma, Q\partial) + c_2 d_{\rho b}(Q\partial, T\partial) + c_3 [d_{\rho b}(P\sigma, T\partial) + d_{\rho b}(Q\partial, S\sigma)] \quad (3.3)$$

$\forall \sigma, \partial \in \mathcal{M}, c_1, c_2, c_3 \geq 0$ and $0 < c_1 + c_2 + 2c_3 b < 1$. And satisfying the following conditions:

- (1) $(S, P), (T, Q)$ weakly compatible.
- (2) If $P(X), Q(X)$ complete subspace of X . Then P, Q, T and S have unique common fixed point.

Corollary 3.2 Consider $(\mathcal{M}, d_{\rho b})$ CPBMS with $b \geq 1$. Consider $P, Q, S, T: \mathcal{M} \rightarrow \mathcal{M}$ be self-map s.t $T(\mathcal{M}) \subseteq P(\mathcal{M})$ and $S(\mathcal{M}) \subseteq Q(\mathcal{M})$ s.t

$$d_{\rho b}(S\sigma, T\partial) \leq c_1 d_{\rho b}(P\sigma, Q\partial) + c_2 d_{\rho b}(Q\partial, T\partial) + c_3 [d_{\rho b}(P\sigma, T\partial) + d_{\rho b}(Q\partial, S\sigma)] + c_4 \frac{d_{\rho b}(Q\partial, T\partial)[1 + d_{\rho b}(P\sigma, S\sigma)]}{[1 + d_{\rho b}(P\sigma, Q\partial)]} \quad (3.4)$$

$\forall \sigma, \partial \in \mathcal{M}, c_1, c_2, c_3, c_4 \geq 0$ and $0 < c_1 + c_2 + 2c_3 b + c_4 < 1$. And satisfying the following conditions:

- (1) $(S, P), (T, Q)$ weakly compatible.
- (2) If $P(X), Q(X)$ complete subspace of X . Then P, Q, T and S have unique common fixed point.

Corollary 3.3 Consider $(\mathcal{M}, d_{\rho b})$ CPBMS with $b \geq 1$. Consider $P, Q, S, T: \mathcal{M} \rightarrow \mathcal{M}$ be self-map s.t $T(\mathcal{M}) \subseteq P(\mathcal{M})$ and $S(\mathcal{M}) \subseteq Q(\mathcal{M})$. S.t

$$d_{\rho b}(S\sigma, T\partial) \leq c_1 d_{\rho b}(P\sigma, Q\partial) + c_2 d_{\rho b}(Q\partial, T\partial) + c_3 [d_{\rho b}(P\sigma, T\partial) + d_{\rho b}(Q\partial, S\sigma)] + c_4 \min\{d_{\rho b}(P\sigma, Q\partial), d_{\rho b}(Q\partial, T\partial), d_{\rho b}(S\sigma, P\partial)\} \quad (3.5)$$

$\forall \sigma, \partial \in \mathcal{M}, c_1, c_2, c_3, c_4 \geq 0$ and $0 < c_1 + c_2 + 2c_3 b + c_4 < 1$. And satisfying the following conditions:

- (1) $(S, P), (T, Q)$ weakly compatible.
- (2) Then P, Q, T and S have unique common fixed point Then P, Q, T and S have unique common fixed point.

Theorem 3.2 Consider $(\mathcal{M}, d_{\rho b})$ CPBMS with $b \geq 1$. Consider P, Q, T and $S : \mathcal{M} \rightarrow \mathcal{M}$ be self-map s.t $S(\mathcal{M}) \subseteq Q(\mathcal{M})$ and $T(\mathcal{M}) \subseteq P(\mathcal{M})$ s.t

$$d_{\rho b}(S\sigma, T\partial) \leq c_1 d_{\rho b}(Q\partial, P\sigma) + c_2 \max \left\{ d_{\rho b}(Q\partial, P\sigma), \frac{d_{\rho b}(P\sigma, S\sigma) d_{\rho b}(Q\partial, T\partial)}{d_{\rho b}(S\sigma, T\partial)} \right\} \\ + c_3 \min \{ d_{\rho b}(P\sigma, T\partial), d_{\rho b}(Q\partial, S\sigma) \} \quad (3.6)$$

$\forall \sigma, \partial \in \mathcal{M}$, $c_1, c_2, c_3 \geq 0$ and $0 < c_1 + c_2 + 2c_3b < 1$. and satisfying the following conditions:

- (1) $(S, P), (T, Q)$ weakly compatible.
- (2) If $P(X), Q(X)$ complete subspace of X . Then P, Q, T and S have unique common fixed point.

Proof: (i) Existence

suppose $x_0 \in \mathcal{M}$, construct sequence $\{x_n\}$ and $\{v_n\}$ in $\mathcal{M}$ in the following manner

since $S(\mathcal{M}) \subseteq Q(\mathcal{M})$, $\exists x_1 \in \mathcal{M}$ s.t $Qx_1 = Sx_0 = v_0$.

Since $T(\mathcal{M}) \subseteq P(\mathcal{M})$, we can choose $x_2 \in \mathcal{M}$ s.t $Px_2 = Tx_1 = v_1$

continuing in this process for all chosen $x_n \in \mathcal{M}$, we obtain $x_{n+1} \in \mathcal{M}$ s.t

$$Qx_{2n+1} = Sx_{2n} \text{ and } Px_{2n+2} = Tx_{2n+1}, n \in N$$

In general, define sequence $\{v_{2n}\}$ in $\mathcal{M}$ by denoting

$v_{2n} = Qx_{2n+1} = Sx_{2n}$ and $v_{2n+1} = Px_{2n+2} = Tx_{2n+1}$, $n \in N$ and by using (3.6) we have

$$d(v_{2n}, v_{2n+1}) = d(Sx_{2n}, Tx_{2n+1}) \\ \leq c_1 d_{\rho b}(Qx_{2n+1}, Px_{2n}) \\ + c_2 \max \left\{ d_{\rho b}(Qx_{2n+1}, Px_{2n}), \frac{d_{\rho b}(Px_{2n}, Sx_{2n}) d_{\rho b}(Qx_{2n+1}, Tx_{2n+1})}{d_{\rho b}(Sx_{2n}, Tx_{2n+1})} \right\} \\ + c_3 \min \{ d_{\rho b}(Px_{2n}, Tx_{2n+1}), d_{\rho b}(Qx_{2n+1}, Sx_{2n}) \} \\ d(v_{2n}, v_{2n+1}) \leq c_1 d_{\rho b}(v_{2n}, v_{2n-1}) + c_2 \max \left\{ d_{\rho b}(v_{2n}, v_{2n-1}), \frac{d_{\rho b}(v_{2n-1}, v_{2n}) d_{\rho b}(v_{2n}, v_{2n+1})}{d_{\rho b}(v_{2n}, v_{2n+1})} \right\} \\ + c_3 \min \{ d_{\rho b}(v_{2n-1}, v_{2n+1}), d_{\rho b}(v_{2n}, v_{2n}) \} \\ d(v_{2n}, v_{2n+1}) \leq c_1 d_{\rho b}(v_{2n}, v_{2n-1}) + c_2 \max \{ d_{\rho b}(v_{2n}, v_{2n-1}), d_{\rho b}(v_{2n-1}, v_{2n}) \} \\ + c_3 \min \{ d_{\rho b}(v_{2n-1}, v_{2n+1}), d_{\rho b}(v_{2n}, v_{2n}) \}$$

Case 1: If $\min \{ d_{\rho b}(v_{2n-1}, v_{2n+1}), d_{\rho b}(v_{2n}, v_{2n}) \} = d_{\rho b}(v_{2n-1}, v_{2n+1})$. Then

$$d(v_{2n}, v_{2n+1}) \leq c_1 d_{\rho b}(v_{2n}, v_{2n-1}) + c_2 d_{\rho b}(v_{2n}, v_{2n-1}) + c_3 d_{\rho b}(v_{2n-1}, v_{2n+1})$$

$$d(v_{2n}, v_{2n+1}) \leq c_1 d_{\rho b}(v_{2n}, v_{2n-1}) + c_2 d_{\rho b}(v_{2n}, v_{2n-1}) + c_3 b [d_{\rho b}(v_{2n-1}, v_{2n}) + d_{\rho b}(v_{2n}, v_{2n+1})] - d_{\rho b}(v_{2n}, v_{2n})$$

$$d(v_{2n}, v_{2n+1}) \leq c_1 d_{\rho b}(v_{2n}, v_{2n-1}) + c_2 d_{\rho b}(v_{2n}, v_{2n-1}) + c_3 b [d_{\rho b}(v_{2n-1}, v_{2n}) + d_{\rho b}(v_{2n}, v_{2n+1})]$$

$$(1 - c_3 b) d(v_{2n}, v_{2n+1}) \leq (c_1 + c_2 + c_3 b) d_{\rho b}(v_{2n-1}, v_{2n})$$

$$d(v_{2n}, v_{2n+1}) \leq \frac{(c_1 + c_2 + c_3 b)}{(1 - c_3 b)} d_{\rho b}(v_{2n-1}, v_{2n})$$

$$\text{put } \beta_1 = \frac{(c_1 + c_2 + c_3 b)}{(1 - c_3 b)}, 0 < \beta_1 < 1$$

$$d(v_{2n}, v_{2n+1}) \leq \beta_1 d_{\rho b}(v_{2n-1}, v_{2n})$$

Case 2: if $\min\{d_{\rho b}(v_{2n}, v_{2n+1}), d_{\rho b}(v_{2n}, v_{2n})\} = d_{\rho b}(v_{2n}, v_{2n})$. Then

$$d(v_{2n}, v_{2n+1}) \leq c_1 d_{\rho b}(v_{2n}, v_{2n-1}) + c_2 d_{\rho b}(v_{2n}, v_{2n-1}) + c_3 d_{\rho b}(v_{2n}, v_{2n})$$

by using $(d_{\rho b2})$ we have $d_{\rho b}(v_{2n}, v_{2n}) \leq d_{\rho b}(v_{2n}, v_{2n-1})$ so

$$d(v_{2n}, v_{2n+1}) \leq c_1 d_{\rho b}(v_{2n}, v_{2n-1}) + c_2 d_{\rho b}(v_{2n}, v_{2n-1}) + c_3 d_{\rho b}(v_{2n}, v_{2n-1})$$

$$d(v_{2n}, v_{2n+1}) \leq (c_1 + c_2 + c_3) d_{\rho b}(v_{2n-1}, v_{2n})$$

let $\beta_2 = (c_1 + c_2 + c_3), 0 < \beta_2 < 1$ (since $0 < c_1 + c_2 + 2c_3 b < 1$ and $b \geq 1$. Then

$$0 < c_1 + c_2 + c_3 < 1)$$

$$d_{\rho b}(v_{2n}, v_{2n+1}) \leq \beta_2 d_{\rho b}(v_{2n-1}, v_{2n})$$

Take $\beta = \max\{\beta_1, \beta_2\}, 0 < \beta < 1$

$$\begin{aligned} d_{\rho b}(v_{2n}, v_{2n+1}) &\leq \beta d_{\rho b}(v_{2n-1}, v_{2n}) \leq \beta^2 d_{\rho b}(v_{2n-2}, v_{2n-1}) \leq \beta^3 d_{\rho b}(v_{2n-3}, v_{2n-2}) \leq \dots \\ &\leq \beta^n d_{\rho b}(v_0, v_1) \end{aligned}$$

$$\lim_{n \rightarrow \infty} d_{\rho b}(v_{2n}, v_{2n+1}) = 0$$

In general, $d_{\rho b}(v_n, v_{n+1}) \leq \beta^n d_{\rho b}(v_0, v_1)$

$$\begin{aligned} d_{\rho b}(v_n, v_m) &\leq b[d_{\rho b}(v_n, v_{n+1}) + d_{\rho b}(v_{n+1}, v_m)] - d_{\rho b}(v_{n+1}, v_{n+1}) \leq b[d_{\rho b}(v_n, v_{n+1}) + d_{\rho b}(v_{n+1}, v_m)] \\ &\leq b d_{\rho b}(v_n, v_{n+1}) + b(b[d_{\rho b}(v_{n+1}, v_{n+2}) + d_{\rho b}(v_{n+2}, v_m)] - d_{\rho b}(v_{n+2}, v_{n+2})) \\ &\leq b d_{\rho b}(v_n, v_{n+1}) + b^2 d_{\rho b}(v_{n+1}, v_{n+2}) + b^2 d_{\rho b}(v_{n+2}, v_m) \\ &\vdots \\ &\leq b d_{\rho b}(v_n, v_{n+1}) + b^2 d_{\rho b}(v_{n+1}, v_{n+2}) + b^3 d_{\rho b}(v_{n+2}, v_{n+3}) + \dots + b^{m-n} d_{\rho b}(v_{m-1}, v_m) \\ &\leq b \beta^n d_{\rho b}(v_0, v_1) + b^2 \beta^{n+1} d_{\rho b}(v_0, v_1) + b^3 \beta^{n+2} d_{\rho b}(v_0, v_1) + \dots + b^{m-n} \beta^{m-1} d_{\rho b}(v_0, v_1) \\ &\leq b \beta^n (1 + b\beta + (b\beta)^2 + \dots + (b\beta)^{m-n-1}) d_{\rho b}(v_0, v_1) \end{aligned}$$

$\lim_{n, m \rightarrow \infty} d_{\rho b}(v_n, v_m) = 0$. Thus $\{v_n\}$ be Cauchy sequence in $\mathcal{M}$. Since $\mathcal{M}$ is complete P.B.M.S then $\exists v \in \mathcal{M}$ s.t

$$\lim_{n, m \rightarrow \infty} d_{\rho b}(v_n, v_m) = \lim_{n \rightarrow \infty} d_{\rho b}(v_n, v) = d_{\rho b}(v, v) = 0 \quad (3.7)$$

We need to show that v is common fixed point of P, Q, T and S

Since $S(\mathcal{M}) \subseteq Q(\mathcal{M})$ and $Q(\mathcal{M})$ complete subspace of $\mathcal{M}$ so $v \in Q(\mathcal{M})$ i.e $\exists u \in \mathcal{M}$ s.t $Qu = v$

now we have to prove that $Tu = v$

$$\begin{aligned} d_{pb}(Sx_{2n+1}, Tu) &\leq c_1 d_{pb}(Qu, Px_{2n+1}) + c_2 \max \left\{ d_{pb}(Qu, Px_{2n+1}), \frac{d_{pb}(Px_{2n+1}, Sx_{2n+1}) d_{pb}(Qu, Tu)}{d_{pb}(Sx_{2n+1}, Tu)} \right\} \\ &\quad + c_3 \min \{ d_{pb}(Px_{2n+1}, Tu), d_{pb}(Qu, Sx_{2n+1}) \} \\ d_{pb}(v_{2n+1}, Tu) &\leq c_1 d_{pb}(v, v_{2n}) + c_2 \max \left\{ d_{pb}(v, v_{2n}), \frac{d_{pb}(v_{2n}, v_{2n+1}) d_{pb}(v, Tu)}{d_{pb}(v_{2n+1}, Tu)} \right\} \\ &\quad + c_3 \min \{ d_{pb}(v_{2n}, Tu), d_{pb}(v, v_{2n+1}) \} \end{aligned}$$

Taking limit as $n \rightarrow \infty$, we get

$$\begin{aligned} d_{pb}(v, Tu) &\leq c_1 d_{pb}(v, v) + c_2 \max \left\{ d_{pb}(v, v), \frac{d_{pb}(v, v) d_{pb}(v, Tu)}{d_{pb}(v, Tu)} \right\} \\ &\quad + c_3 \min \{ d_{pb}(v, Tu), d_{pb}(v, v) \} \end{aligned}$$

Since by (3.7) $d_{pb}(v, v) = 0$ so

$d_{pb}(v, Tu) = 0$ and from definition (2.3), we have $Tu = v$

So $Tu = Qu = v$, and T, Q have coincidence point.

since $T(\mathcal{M}) \subseteq P(\mathcal{M})$ and $P(\mathcal{M})$ complete subspace of $\mathcal{M}$ so $v \in P(\mathcal{M})$ i.e $\exists w \in \mathcal{M}$ s.t $v = Pw$

now we have to prove that $Pw = v$

$$\begin{aligned} d_{pb}(Sw, Tu) &\leq c_1 d_{pb}(Qu, Pw) + c_2 \max \left\{ d_{pb}(Qu, Pw), \frac{d_{pb}(Pw, Sw) d_{pb}(Qu, Tu)}{d_{pb}(Sw, Tu)} \right\} \\ &\quad + c_3 \min \{ d_{pb}(Pw, Tu), d_{pb}(Qu, Sw) \} \\ d_{pb}(Fw, v) &\leq c_1 d_{pb}(v, v) + c_2 \max \left\{ d_{pb}(v, v), \frac{d_{pb}(v, Sw) d_{pb}(v, v)}{d_{pb}(Sw, v)} \right\} \\ &\quad + c_3 \min \{ d_{pb}(v, v), d_{pb}(v, Sw) \} \end{aligned}$$

Since by (3.7) $d_{pb}(v, v) = 0$ so

$d_{pb}(Sw, v) = 0$ and from definition (2.3) we have $Sw = v$ hence

$$Sw = Pw = v$$

Now since $Sw = Pw = v$ and $Tu = Qu = v$, since $(T, Q), (S, P)$ weak compatible then

$$Qv = QTu = TQu = Tv$$

$$Pv = PSw = SPw = Sv$$

Now we will show that $Qv = v$

$$d_{\rho b}(Sw, Tv) \leq c_1 d_{\rho b}(Qv, Pw) + c_2 \max \left\{ d_{\rho b}(Qv, Pw), \frac{d_{\rho b}(Pw, Sw) d_{\rho b}(Qv, Tv)}{d_{\rho b}(Sw, Tv)} \right\}$$

$$+ c_3 \min \{ d_{\rho b}(Pw, Tv), d_{\rho b}(Qv, Sw) \}$$

$$d_{\rho b}(v, Tv) \leq c_1 d_{\rho b}(Qv, v) + c_2 \max \left\{ d_{\rho b}(Qv, v), \frac{d_{\rho b}(v, v) d_{\rho b}(Qv, Tv)}{d_{\rho b}(v, Tv)} \right\}$$

$$+ c_3 \min \{ d_{\rho b}(v, Tv), d_{\rho b}(Qv, v) \}$$

Since $Qv = QTu = TQu = Tv$

$$d_{\rho b}(v, Qv) \leq c_1 d_{\rho b}(Qv, v) + c_2 \max \left\{ d_{\rho b}(Qv, v), \frac{d_{\rho b}(v, v) d_{\rho b}(Qv, Qv)}{d_{\rho b}(v, Qv)} \right\}$$

$$+ c_3 \min \{ d_{\rho b}(v, Qv), d_{\rho b}(Qv, v) \}$$

Since $d_{\rho b}(v, v) = 0$ then $\frac{d_{\rho b}(v, v) d_{\rho b}(Qv, Qv)}{d_{\rho b}(v, Qv)} = 0$ so

$$\max \left\{ d_{\rho b}(Qv, v), \frac{d_{\rho b}(v, v) d_{\rho b}(Qv, Qv)}{d_{\rho b}(v, Qv)} \right\} = d_{\rho b}(Qv, v)$$

$\min \{ d_{\rho b}(v, Qv), d_{\rho b}(Qv, v) \} = d_{\rho b}(Qv, v)$ so

$$d_{\rho b}(v, Qv) \leq c_1 d_{\rho b}(Qv, v) + c_2 d_{\rho b}(Qv, v) + c_3 d_{\rho b}(Qv, v)$$

$$(1 - c_1 - c_2 - c_3) d_{\rho b}(v, Qv) \leq 0$$

$d_{\rho b}(v, Qv) = 0$ thus, from definition (2.3) $Qv = v$

Now we will show that $Pv = v$

$$d_{\rho b}(Sv, Tu) \leq c_1 d_{\rho b}(Qu, Pv) + c_2 \max \left\{ d_{\rho b}(Qu, Pv), \frac{d_{\rho b}(Pv, Sv) d_{\rho b}(Qu, Tu)}{d_{\rho b}(Sv, Tu)} \right\}$$

$$+ c_3 \min \{ d_{\rho b}(Pv, Tu), d_{\rho b}(Qu, Sv) \}$$

$$d_{\rho b}(Pv, v) \leq c_1 d_{\rho b}(v, Pv) + c_2 \max \left\{ d_{\rho b}(v, Pv), \frac{d_{\rho b}(Pv, Pv) d_{\rho b}(v, v)}{d_{\rho b}(Pv, v)} \right\}$$

$$+ c_3 \min \{ d_{\rho b}(Pv, v), d_{\rho b}(v, Pv) \}$$

Since $d_{\rho b}(v, v) = 0$ then $\max \left\{ d_{\rho b}(v, Pv), \frac{d_{\rho b}(Pv, Pv) d_{\rho b}(v, v)}{d_{\rho b}(Pv, v)} \right\} = d_{\rho b}(v, Pv)$ and

$\min \{ d_{\rho b}(Pv, v), d_{\rho b}(v, Pv) \} = d_{\rho b}(v, Pv)$ therefor

$$d_{\rho b}(Pv, v) \leq c_1 d_{\rho b}(v, Pv) + c_2 d_{\rho b}(Pv, v) + c_3 d_{\rho b}(Pv, v)$$

$$(1 - c_1 - c_2 - c_3) d_{\rho b}(Pv, v) \leq 0$$

$d_{\rho b}(Pv, v) = 0$ and from definition (2.3) we have $Pv = v$ Hence $Pv = Sv = v$

Therefor $Pv = Sv = Qv = Tv = v$. Then v is common fixed point of P, S, Q, T .

(ii) Uniqueness

Let σ are another common fixed point of P, S, Q, T that is $P\sigma = S\sigma = Q\sigma = T\sigma = \sigma$

$$\begin{aligned} d_{\rho b}(v, \sigma) &= d(Sv, T\sigma) \leq c_1 d_{\rho b}(Q\sigma, Pv) + c_2 \max \left\{ d_{\rho b}(Q\sigma, Pv), \frac{d_{\rho b}(Pv, Sv) d_{\rho b}(Q\sigma, T\sigma)}{d_{\rho b}(Sv, T\sigma)} \right\} \\ &\quad + c_3 \min \{ d_{\rho b}(Pv, T\sigma), d_{\rho b}(Q\sigma, Sv) \} \\ d_{\rho b}(v, \sigma) &\leq c_1 d_{\rho b}(\sigma, v) + c_2 \max \left\{ d_{\rho b}(\sigma, v), \frac{d_{\rho b}(v, v) d_{\rho b}(r, \sigma)}{d_{\rho b}(v, \sigma)} \right\} \\ &\quad + c_3 \min \{ d_{\rho b}(v, \sigma), d_{\rho b}(\sigma, v) \} \end{aligned}$$

Since $d_{\rho b}(v, v) = 0$ then $\max \left\{ d_{\rho b}(\sigma, v), \frac{d_{\rho b}(v, v) d_{\rho b}(\sigma, \sigma)}{d_{\rho b}(v, \sigma)} \right\} = d_{\rho b}(\sigma, v)$ and $\min \{ d_{\rho b}(v, \sigma), d_{\rho b}(\sigma, v) \} = d_{\rho b}(v, \sigma)$ so

$$d_{\rho b}(v, \sigma) \leq c_1 d_{\rho b}(v, \sigma) + c_2 d_{\rho b}(v, \sigma) + c_3 d_{\rho b}(v, \sigma)$$

$$(1 - c_1 - c_2 - c_3) d_{\rho b}(v, \sigma) \leq 0$$

$$d_{\rho b}(v, \sigma) \leq 0$$

$$d_{\rho b}(v, \sigma) = 0 \text{ and from definition (2.3) } v = \sigma$$

We can use the above theorem to obtain following corollary by take values $c_3 = 0$ as a following

Corollary 3.4 Consider $(\mathcal{M}, d_{\rho b})$ CPBMS with $b \geq 1$. Consider $P, S, Q, T: \mathcal{M} \rightarrow \mathcal{M}$ be self-map s.t $S(\mathcal{M}) \subseteq Q(\mathcal{M})$ and $T(\mathcal{M}) \subseteq P(\mathcal{M})$ s.t

$$d_{\rho b}(S\sigma, T\partial) \leq c_1 d_{\rho b}(Q\partial, Pv) + c_2 \max \left\{ d_{\rho b}(Q\partial, P\sigma), \frac{d_{\rho b}(T\sigma, S\sigma) d_{\rho b}(Q\partial, T\partial)}{d_{\rho b}(S\sigma, T\partial)} \right\} \quad (3.8)$$

$\forall \sigma, \partial \in \mathcal{M}, c_1, c_2 \geq 0$ and $0 < c_1 + c_2 < 1$. And satisfying the following conditions:

- (1) The pairs (S, P) and (T, Q) are weakly compatible.
- (2) If $P(X), Q(X)$ complete subspace of X . Then P, Q, T and S have unique common fixed point.

Example 3.1: let $\mathcal{M} = [0, 1]$, define a function $d_{\rho b}: \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$ by $d_{\rho b}(\sigma, \partial) = |\sigma - \partial| + \max\{\sigma, \partial\} \forall \sigma, \partial \in \mathcal{M}$ then $(\mathcal{M}, d_{\rho b})$ is complete PBMS with coefficient $b = 2$. Define four self-mappings $P, Q, S, T: \mathcal{M} \rightarrow \mathcal{M}$ as follows.

$$T(\sigma) = \frac{\sigma}{4}, S(\sigma) = \frac{\sigma}{4}, P(\sigma) = \frac{\sigma}{2}, Q(\sigma) = \frac{\sigma}{2}. \text{ we note}$$

$$\begin{aligned} 1. \quad T(\sigma) &= \frac{\sigma}{4} \leq \frac{\sigma}{2} = P(\sigma) \\ S(\sigma) &= \frac{\sigma}{4} \leq \frac{\sigma}{2} = Q(\sigma) \end{aligned}$$

Which implies $T(\mathcal{M}) \subseteq P(\mathcal{M})$ and $S(\mathcal{M}) \subseteq Q(\mathcal{M})$.

$$\begin{aligned} 2. \quad PT\sigma &= P\left(\frac{\sigma}{4}\right) = \frac{\sigma}{8} = T\left(\frac{\sigma}{2}\right) = TP\sigma \text{ and} \\ QS\sigma &= Q\left(\frac{\sigma}{4}\right) = \frac{\sigma}{8} = S\left(\frac{\sigma}{2}\right) = SQ\sigma \quad \forall \sigma \in X \end{aligned}$$

So, the pairs $(P, T), (Q, S)$ are weak compatible.

$$3. \quad \text{Verification the condition (3.1) of theorem (3.1) with take } c_1 = \frac{1}{2}, c_2 = \frac{1}{4}, c_3 = c_4 = c_5 = 0, \text{ then}$$

$$c_1, c_2, c_3, c_4, c_5 \geq 0 \text{ and } 0 < c_1 + c_2 + 2c_3b + c_4 + c_5 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} < 1$$

$$\text{Now we need to show that } d_{\rho b}(S\sigma, T\partial) \leq c_1 d_{\rho b}(P\sigma, Q\partial) + c_2 d_{\rho b}(Q\partial, T\partial)$$

First, we compute

$$d_{pb}(S\sigma, T\partial) = d_{pb}\left(\frac{\sigma}{4}, \frac{\partial}{4}\right) = \frac{1}{4}d(\sigma, \partial)$$

$$d_{pb}(P\sigma, Q\partial) = d_{pb}\left(\frac{\sigma}{2}, \frac{\partial}{2}\right) = \frac{1}{2}d(\sigma, \partial)$$

$$d_{pb}(Q\partial, T\partial) = d_{pb}\left(\frac{\partial}{2}, \frac{\partial}{4}\right) = \frac{\partial}{4} + \frac{\partial}{2} = \frac{3\partial}{4}$$

$$c_1 d_{pb}(P\sigma, Q\partial) + c_2 d_{pb}(Q\partial, T\partial) = \left(\frac{1}{2}\right) \frac{1}{2} d_{pb}(\sigma, \partial) + \left(\frac{1}{4}\right) \frac{3\partial}{4} = \frac{1}{4} d_{pb}(\sigma, \partial) + \frac{3\partial}{16}$$

Since $\partial \in [0, 1]$ then $\frac{3\partial}{16} \geq 0$, hence $d_{pb}(S\sigma, T\partial) \leq c_1 d_{pb}(P\sigma, Q\partial) + c_2 d_{pb}(Q\partial, T\partial)$ thus

$$\begin{aligned} d(S\sigma, T\partial) &\leq c_1(d(P\sigma, Q\partial) + c_2 d(Q\partial, T\partial) + c_3[d(P\sigma, T\partial) + d(Q\partial, S\sigma)]) \\ &\quad + c_4 \frac{d_{pb}(Q\partial, T\partial)[1 + d_{pb}(P\sigma, S\sigma)]}{[1 + d_{pb}(P\sigma, Q\partial)]} + c_5 \min\{d_{pb}(P\sigma, Q\partial), d_{pb}(Q\partial, T\partial), d_{pb}(S\sigma, P\partial)\} \end{aligned}$$

$\forall \sigma, \partial \in \mathcal{M}$. so the condition (3.1) of theorem 3.1 with $c_1 = \frac{1}{2}, c_2 = \frac{1}{4}, c_3 = c_4 = c_5 = 0$ is satisfied and 0 is common fixed point of P, Q, S, T since $P(0) = Q(0) = S(0) = T(0) = 0$.

Conclusion

In this paper we worked to find unique common fixed-point results in partial b-metric space by using the concept of weakly compatible mappings under some different conditions. The results we obtained were improvements of previous results in different space and generalizations it to partial b-metric spaces. Furthermore, our works highlights on the importance of weak compatibility conditions to ensuring the existence and uniqueness of common fixed points which contributes in development of common fixed-point results and extending its application to more general metric spaces.

References

- [1] M. C. Arya, N. Chandra and M. C. Joshi, A coincidence point theorem in partial metric space, *GANITA*, vol.68(2), (2018), 1-6.
- [2] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales." *Fundamenta mathematicae* 3.1 (1922): 133-181.
- [3] U. Y. Batsari, P. Kvmam and S. Dhompongsa, Fixed points of terminating mappings in partial metric space, *J. Fixed point theory Appl*, (2019), 21-39.
- [4] S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostrav.* 5(11), (1993).
- [5] S. Dewangan and S. K. Tiwari, Common Fixed Point And Coincidence Point Theorem For Expansive Mapping In b-Metric Spaces, *International Journal of Mathematics Trends and Technology*, (67) (5), 157-167, (2021).
- [6] Y. U. Gaba, A. A. Collins and D. J. Leko, Chatterjea type fixed point in Partial b-metric spaces, *arXiv preprint arXiv:1902.03108* (2019).
- [7] A. M. Hashim and A. I. Abd Zaid, Some Fixed point maps satisfying generalize condition (B) in partially ordered partial metric space, *Basrah Journal of Science*, 38(2), (2020), 157-173.
- [8] A. M. Hashim and H. A. Bakry, Fixed points theorems for ciric'mappings in partial b-metric space, *Basrah Journal of Science*, 37(1) (2019), 16-24.
- [9] X. Hunng, C. Zhu and X. Wen, Fixed point theorems for expanding mappings in partial metric space, *An. St. Univ. Ovidius Constanta*, Vol. 20 (1), (2012), 213- 224.
- [10] J. M. Joseph, D. D. Roselin and M. Marudai, Fixed point theorems on multi valued mappings in b-metric spaces, *SpringerPlus* 5:217, (2016).
- [11] A. G. Kalo, K.K. Tola and H. E. Yesuf, Fixed point results for Geraghty Ciri'c-type contraction mappings in b-metric space with applications, *Fixed Point Theory and Algorithms for Sciences and Engineering*, 8(2024), (2024).
- [12] S. G. Matthews, Partial metric topology, in proceeding of the 8th summer conference on General, Topology and Applications, *Ann. New York, Acad. Sci*, Vol.728, (1994), 183-197.
- [13] S. K. Mohanta and P. Biswas, Some new coincidence point results in partial b-metric spaces via digraphs, L-simulation and θ -functions, *Int. J. Nonlinear Anal. Appl.* In Press, (2024), 1–16.
- [14] T. Nazir, S. Silvestrov and M. abbas, Common fixed point results of four maps in ordered partial metric space, *Wave Wavelets Fractals. Adv. Anal*, (2016), 2, 49-63.
- [15] S. Shukla, Partial b-metric spaces and fixed point theorems, *Mediterranean Journal of Mathematics*, (11), (2014), 703-711.
- [16] D. Singh, et al, Some applications of fixed point results for generalized two classes of Boyd–Wong's F-contraction in partial b-metric spaces, *Mathematical Sciences*, 12, (2018), 111-127.
- [17] S. Tiwari, and L. K. Dewangan, Fixed Point and Common Fixed Point Theorems for Single Valued and Multivalued Mappings in Partial b-Metric Spaces, *Indian Journal of Science and Technology*, (17), (2024), 59-64.
- [18] A. Tomar, M. Joshi, and V. Bhatt, Fixed point and a Cantilever beam problem in a partial b-metric space, *Acta Universitatis Sapientiae, Mathematica* (13)2, (2021), 506-518.
- [19] M. B. Zada, M. Sarwar and P. Kumam, Fixed Point Results Of Rational Type Contractions In B-Metric Spaces, *International Journal of Analysis and Applications*, (16)(6), (2018), 904-920.
- [20] J. Zhou, D. Zheng and G. Zhang, Fixed point theorems in partial b-metric spaces, *Appl. Math. Sci*, (12), (2018), 617-624.