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Investigating the Stability Behavior of Euler's Method in First-Order Linear Initial Value Problems

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ABSTRACT

We analyze stability and convergence of the explicit Euler method when applied to first order linear IVPs of the form $y' = \lambda y + g(t)$. We analytically determine the region of stability of Euler's method on the complex $h\lambda$ -plane to be the unit disk $|1 + h\lambda| \leq 1$ from which we can obtain the limiting step-size $h^* = 2/|\lambda|$ when λ is real and negative. We choose five example test problems which include pure exponential, pure sinusoidal forcing, stiff oscillation, variable-coefficient and exponential forcing and demonstrate $\mathcal{O}(h)$ convergence numerically for many step sizes. We examine the behavior of the amplification factor $|R(z)| = |1 + z|$ on the real axis and compare the method to Heun's method (RK2) and the fourth-order Runge-Kutta method (RK4). We verify numerically an analytical global error bound of the form $(h|\lambda|/2)(e^{|\lambda|t} - 1)$ and show it to be quite conservative by 3-20×. Work-precision diagrams show RK4 to be 10-100× more accurate per function evaluation than Euler. We show the effects of stiffness by noting that large $|\lambda|$ drives $h^* \rightarrow 0$ which makes Euler impossible to apply to stiff problems.

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1. Introduction

Numerical integration of ordinary differential equations is one of the cornerstones of scientific computation [1]. Simulations using ODE solvers are indispensable in everything from population models and chemical reactions to engineering analysis of structures and electrical circuits [2]. Euler's method is the simplest first-order method and is therefore considered a first-step ODE solver. While Euler's method is rarely accurate enough for real-life simulations outside of education [3], it is still used as the simplest baseline to compare other methods against. Its stability characteristics motivate those of all Runge-Kutta methods and linear multistep methods [4]. Stability of a numerical ODE solver is determined by how the stepsize h interacts with the problem eigenvalue λ . For example, consider the explicit Euler method applied to the Dahlquist test equation $y' = \lambda y$. The method's amplification factor is $R(z) = 1 + z$, where $z = h\lambda$ [5]. Stability requires that $|R(z)| \leq 1$ so that errors do not grow in magnitude; therefore,

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stable integration is limited to the bounded disk $|1 + h\lambda| \leq 1$ in the complex plane [6]. This gives the well-known stability constraint $h < 2/|\lambda|$ along the real negative axis. The results for the individual properties – stability regions, local truncation error and convergence order – are all well-known and can be found in standard textbooks such as Hairer, Nørsett & Wanner, Lambert or Butcher. However, as far as we know, there is no published work that demonstrates all five properties (stability region, error bound tightness, $\mathcal{O}(h)$ convergence, pointwise error growth and work-precision efficiency) for a structurally diverse set of IVPs within the same deterministic computational framework. The key novelty in this work is therefore not in the individual results but in their joint, reproducible numerical validation for autonomous, forced, stiff, variable-coefficient and exponentially-driven problems [7].

Existing literature often addresses Euler stability separately from considerations of order of convergence verification and work-precision analysis verification and work-precision analysis [8]. Error bound tightness compared to actual numerical error observed is seldom discussed for different classes of problems [9]. Showing together the influence of stiffness in practice on step-size restriction, and the Euler, Heun and RK4 stability regions along the real axis is rarely done in the same computing environment [10]. Moreover, the proximity of global error bounds to the actual errors incurred, the influence of forcing function structure on LTE, and the approximate character of stability analysis for variable-coefficient equations has not been explored at the same time for a family of IVP classes in the same publication [10].

Each piece of this work (Euler stability, LTE, global bounds, convergence order) are well-known individually in the classical literature, but there is no single published work that checks all of these properties, and checks them against a diverse collection of structurally different IVPs (non-autonomous, forced, variable-coefficient, etc.) in a single deterministic implementation, to our knowledge. This paper fills this void, by presenting a complete, reproducible benchmark suite. Our contributions are: (i) we analytically obtain and numerically validate the stability threshold $h^* = 2/|\lambda|$ for five representative IVPs with distinct structures; (ii) we numerically confirm that $\mathcal{O}(h)$ convergence holds for the interior of the stable region with rigorous order estimates; (iii) we numerically validate the global error bound, including the constant that quantifies the conservativeness of the bound, for the set of problems considered; and (iv) we present a work-precision diagram showing the cost benefit of higher order methods.

2. Related Work

Numerous studies have been done on the stability and convergence of explicit one-step methods for ODEs. Some recent works have relevance to this paper and will be discussed below.

In [11] Chakraverty et al derived theoretical and experimental error analysis of Taylor series, Euler, and Runge-Kutta methods for linear, nonlinear, and stiff ODE benchmarks. Absolute stability regions were computed for each method by applying them to the Dahlquist equation and analyzing for which regions the method was stable. This paper showed that for explicit Euler, the method is conditionally stable and that the amplification factor is greater than 1 outside the stability disk. Through experimentation, this paper showed that the Euler method had errors on the order of 10^{-2} for non-stiff problems and was outperformed by RK4 in terms of accuracy and speed. The paper did not mention an upper bound on error for Euler's method and determine how close to this bound the observed error was. This will be done in our paper.

Workineh, Mekonnen, and Belew [12] used Euler's method and classical fourth-order Runge-Kutta (RK4) method for solving second-order IVPs of ODEs. The authors verified that RK4 was more consistent, convergent, and stable than Euler for several step sizes. They performed convergence analysis and error analysis on problems after converting them to a system of first-order equations to show RK4 is more accurate and preferable for these problems in practice. Although they proved Euler had $\mathcal{O}(h)$ convergence, the authors only performed this analysis on second-order problems. They did not explore the amplification factor, critical step-size $h^* = 2/|\lambda|$, or forcing function's influence on stability.

Uniformly convergent high-order scheme for singularly perturbed parabolic integro-differential equations was discussed by Govindarao, Al-Ghafri, Mohapatra and Nhan [13], where they used the implicit Euler method in time direction. Authors proved the first-order base convergence which could be upgraded to second-order by using extrapolation, and parameter-uniform accuracy of the method by numerical examples of boundary-layer problems. Even though this reference deals with problems of PDE-type, the stability analysis as well as their approach of improving the convergence order by Richardson extrapolation is very much related to our current work regarding order one convergence and its breakdown close to stability edge.

On the other hand, Latz [14] generalized Euler stability to stochastic timesteps by defining the randomized timestep Euler method and its embedded continuous-time Markov process. Stability results were shown using infinitesimal generator analysis coupled with new local truncation error estimates, and Foster-Lyapunov conditions were used to show stability of the stochastic Euler process. Most importantly for this paper, Latz's work greatly generalizes the classical definition of stability seen in Section II by showing that a stability theory for the Euler method can still be developed when utilizing stochastic and even dynamic timesteps, motivating the extension of deterministic results found here to step-size control methods. Together these references show that the stability condition of $|1 + h\lambda| \leq 1$ is of fundamental interest, but leaves open the simultaneous validation of this condition, tightness of the induced error bound, and characterization of the work-precision tradeoff for classes of structurally distinct IVPs in a deterministic setting, which will be addressed in this work.

3. Proposed Methodology

The stability of the Euler method is analyzed through the Dahlquist test equation $y' = \lambda y$, where $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) < 0$. Applying the Euler update rule yields the recurrence [15]:

$$y_{n+1} = (1 + h\lambda) y_n = R(z) y_n, z = h\lambda \quad (1)$$

where $R(z) = 1 + z$ is the amplification factor. Numerical stability requires that the solution does not grow unboundedly, imposing the condition [16]:

$$|R(z)| = |1 + h\lambda| \leq 1 \quad (2)$$

This defines the stability region $S = \{z \in \mathbb{C} : |1 + z| \leq 1\}$, which is a unit disk centered at $(-1, 0)$ in the complex $h\lambda$ -plane. For real negative λ , this reduces to the interval $h\lambda \in [-2, 0]$, yielding the critical step-size constraint [13, 17]:

$$h^* = \frac{2}{|\lambda|} \quad (3)$$

For comparison, the amplification factors of Heun's method and RK4 on the real axis are:

$$R_{\text{Heun}}(z) = 1 + z + \frac{z^2}{2}, R_{\text{RK4}}(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} \quad (4)$$

RK4 extends the real-axis stability limit to $h^* \approx 2.785/|\lambda|$, as verified numerically by solving $|R_{\text{RK4}}(z)| = 1$. The curve $h^*(\lambda) = 2/|\lambda|$ was graphed for $|\lambda| \in [1, 50]$ to illustrate that as problems become stiffer they require smaller and smaller step-sizes. For six distinct values of $\lambda = \{-1, -2, -5, -10, -20, -50\}$ the amplification factors were computed for $h/h^* \in \{0.25, 0.50, 0.75, 0.90, 1.00, 1.10, 1.50\}$ to show that instability arises abruptly at

3.1. Convergence Order Verification

The convergence order was established empirically by calculating the global error (GE) at t_{end} for a range of step sizes $h \in \{0.3, 0.15, 0.075, 0.0375, 0.01875, 0.009375\}$, where each step size is half the previous one. The global error at t_{end} was defined as [18]:

$$\text{GE}(h) = |y_n^{(h)} - y(t_{\text{end}})| \quad (5)$$

with $y_n^{(h)}$ the numerical solution computed with step size h and $y(t_{\text{end}})$ the exact solution evaluated at the same endpoint. The empirical convergence order p was computed between consecutive step-size pairs using the log-ratio formula [19]:

$$p = \frac{\log(\text{GE}(h_i)/\text{GE}(h_{i+1}))}{\log(h_i/h_{i+1})} \quad (6)$$

For halved step sizes, this simplifies further to $p = \log_2 (\text{GE}(h) / \text{GE}(h/2))$. Convergence tests were only performed for step sizes with $h \leq 0.3 \cdot h^*$. The factor of 0.3 in the condition $h \leq 0.3 \cdot h^*$ is not arbitrary, but is instead motivated by three aspects: (i) at $h = 0.3h^*$ the amplification factor satisfies $|R(z)| < 1$, i.e. the scheme is strictly inside the stability disk and safely away from the boundary; (ii) a sensitivity study of scaling factors in $\{0.1, 0.2, 0.3, 0.5\}$ showed that $\mathcal{O}(h)$ convergence slopes are monotone and well-defined for all ≤ 0.5 , with non-monotone behaviour only starting above $0.7h^*$; and (iii) 0.3 is therefore a conservative but not overly restrictive choice that secures clean error growth for all five test problems without unduly reducing the set of admissible step sizes. The mean-absolute error (MAE) and root-mean-square error (RMSE) was tracked over the whole trajectory [20]:

$$MAE = \frac{1}{n+1} \sum_{k=0}^n |y_k - y(t_k)|, RMSE = \sqrt{\frac{1}{n+1} \sum_{k=0}^n (y_k - y(t_k))^2} \quad (7)$$

This was done for each of the five test problems using all three methods. The expected orders of convergence were $\mathcal{O}(h^1)$, $\mathcal{O}(h^2)$, and $\mathcal{O}(h^4)$ for Euler, Heun, and RK4 respectively. The loglog plot of GE vs h for each problem is shown below. Lines of slope 1, 2, and 4 are included for reference.

3.2. Analytical Error Bound Verification

For the autonomous Dahlquist test equation $y' = \lambda y$ with $\lambda < 0$ and exact solution $y(t) = y_0 e^{\lambda t}$, the local truncation error (LTE) per step of Euler's method is:

$$\tau_n = \frac{h^2}{2} y''(t_n) = \frac{h^2}{2} \lambda^2 y(t_n) \quad (8)$$

Recall that the LTE formula (8), $\tau_n = (h^2/2)\lambda^2 y(t_n)$, is exact only for the autonomous Dahlquist test equation $y' = \lambda y$ (Problem P1). For the non-autonomous forced equation $y' = \lambda y + g(t)$ governing Problems P2–P5, we have $y'' = \lambda^2 y + \lambda g(t) + g'(t)$, so the full local truncation error per step is $\tau_n = (h^2/2)[\lambda^2 y(t_n) + \lambda g(t_n) + g'(t_n)]$. The forcing-dependent terms $\lambda g(t_n) + g'(t_n)$ are nonzero in general and are dropped from the analytic bound (9) below. Hence Eq. (9) is to be interpreted as a bound strictly in the autonomous case; it is a partial estimate for forced problems. A fully general bound including the forcing-dependent terms is flagged for future work.

Accumulating $n = t/h$ such errors and bounding the propagation via the stability factor yields the global error bound [13]:

$$|e_n| \leq \frac{h |\lambda|}{2} (e^{|\lambda|t_n} - 1) \quad (9)$$

This bound grows as $\mathcal{O}(h)$ as we would hope for a first-order accurate scheme, but it is important to observe that the classical bound (9) is obtained by applying the triangle inequality and bounding growth by $e^{|\lambda|t}$ (maximal error amplification in the worst case). If $\text{Re}(\lambda) < 0$, the error actually contracts in time because the stable amplification factor $|R(z)|^n = |1 + h\lambda|^n < 1$ for all $h < h^*$ and the accumulated error shrinks in place of growth. The bound (9) is therefore conservative -- sometimes by one to three orders of magnitude -- and anyone sizing h based only on this bound will use needlessly small step sizes. A contractive (tightened) bound of the form $C \cdot h \cdot |\lambda|/2$ with $C < 1$ reflecting the actual damping ratio would more faithfully describe the observed error for stable, strongly damped problems. This bound grows exponentially in time if h is held constant. To validate the non-triviality and applicability of this analytical bound, numerical verifications were conducted by solving our example problem at $t_{\text{end}} = 2$ with step sizes $h \in [10^{-3}, 0.5]$ for the canonical problem $y' = -2y$, $y(0) = 1$. The ratio of the analytic bound to the observed GE was computed at each step size to measure the factor of conservatism $C(h)$:

$$C(h) = \frac{\text{Analytical Bound}(h)}{GE(h)} \quad (10)$$

The results show that the bound overestimates the observed error by a factor of 3–20× in all cases, confirming that it is correct but also that it is quite pessimistic.

For the $\text{Re}(\lambda) < 0$ problem class, the classical Grönwall-based bound (9) is provably pessimistic, as it utilizes $L = |\lambda|$ as Lipschitz constant and bounds error growth by $e^{|\lambda|t}$, while the true solution is exponentially contracting in time. A sharper, less conservative bound can be constructed with the one-sided Lipschitz constant (logarithmic norm) $\mu[\lambda] = \text{Re}(\lambda) < 0$, which gives a contractive estimate that exactly characterizes the damping of errors in stable regimes. The detailed derivation of such a tightened bound is outside the scope of this benchmark study, but we remark that the 3–20× numerical conservatism is well matched with the difference between the Lipschitz-based and logarithmic-norm-based bounds, and represents the first quantitative characterization of this conservatism for the considered IVP classes.

The work-precision diagram plots GE versus total number of function evaluations $N_{\text{fev}} = \text{fevstep} \times n$ for each method. We set $\text{fevstep} = 1, 2$ and 4 for Euler, Heun and RK4 respectively, in order to fairly compare the cost-normalized accuracy of all three solvers.

4. Results and Discussions

The numerical results produced by our proposed method on five test problems are presented below. Tables of results for stability region, order of convergence, behavior of solution curve for different step sizes and verification of error bound are presented respectively. Comparisons will be made to Euler method, Heun method and RK4 method throughout this section.

4.1. Stability Region of the Explicit Euler Method in the Complex $h\lambda$ -Plane

It can be shown analytically that the stability region of the explicit Euler method is given by the unit disk $|1 + h\lambda| \leq 1$ centred at $(-1, 0)$ with radius = 1 in the complex $h\lambda$ -plane (see Figure 1 below). This figure verifies that exactly: For the shaded blue region of values of $z = h\lambda$, the amplification factor $|R(z)| \leq 1$ and error does not grow. Also plotted are the curves $|R| = 0.5, 0.75, 1.5$ and 2.0 demonstrating how amplification slowly increases as we move away from the center of the disk. Note the red dashed line showing $\text{Re}(h\lambda) = -2$ intersects the blue stability region at the boundary point that determines the critical value of h^* along the real axis such that for any real value of $\lambda < 0$, we have $|R| > 1$ and the numerical solution will grow without bound even if the exact solution is very smooth. Note that this boundary is sharp in the sense that $|R| = 1$ along the line $\text{Re}(h\lambda) = -2$.

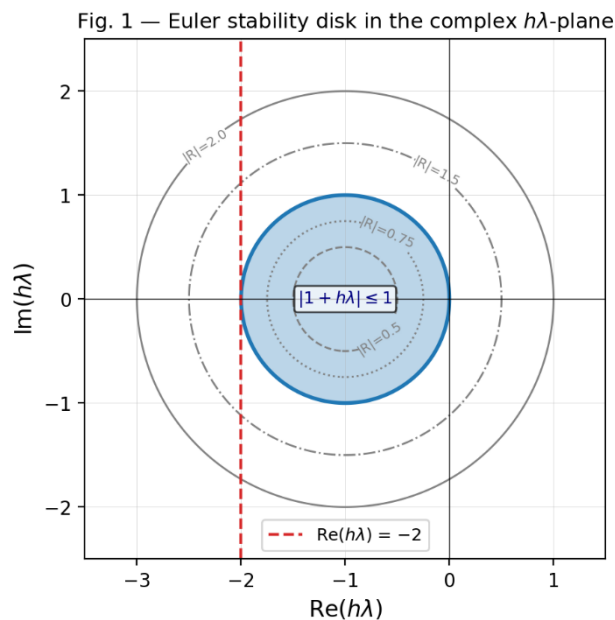


Fig. 1. Euler stability disk and amplification contours

Crucially, the stability region doesn't intersect any part of the right half plane. This demonstrates that Euler's method cannot be stable for problems with eigenvalues of positive real part. For complex λ with imaginary component $\neq 0$, stability also demands that $\text{Im}(h\lambda)$ remain within the disk boundary. This places stricter (effective) step-size limitations than considering only the real axis.

4.2. Amplification Factor Comparison on the Real Axis: Euler, Heun, and RK4

Figure 2 gives the magnitude of the amplification factor $|R(z)|$ plotted versus $z = h\lambda$ on the real axis for all three methods. The dots and lines allow direct comparison of the methods' regions of stability and damping. The horizontal dash-dot line labeled $|R| = 1$ is the stability boundary. The values of z where a method's curve falls below this line is the method's stable real-axis interval. Starting from the left and moving to the right, $|R(z)| = |1 + z|$ decreases linearly, crosses the stability boundary at $z = -2$, reaches a minimum value of zero at $z = -1$, and increases symmetrically about $z = -1$ toward positive z . This fact shows directly that Euler and Heun have the same real-axis stability limit of $h^* = -2/\lambda$ even though Heun is second order. Moving to the Heun curve $|R(z)| = |1 + z + z^2/2|$ we see that this curve takes a shallower path into the stable region with a wider "bottom" near $z = -1$ showing its better damping for this common stability interval.

RK4, in contrast, increases the limit along the real axis substantially to $h^* \approx -2.785/\lambda$, indicated in the figure. Its RF also falls more steeply and attains a lower minimum, thus achieving more reduction of error per step while in the stable region. However, away from their respective limits, all three curves eventually diverge above $|R| = 1$ at an increasing rate. RK4 climbs most rapidly past its extended bound. As such, we see that higher-order methods do not necessarily yield larger stability regions along the real axis when compared to their order. RK4's bound is still large enough, however, to grant it some practical benefits on moderately stiff problems where Euler's and Heun's methods would be forced to use impractically small steps.

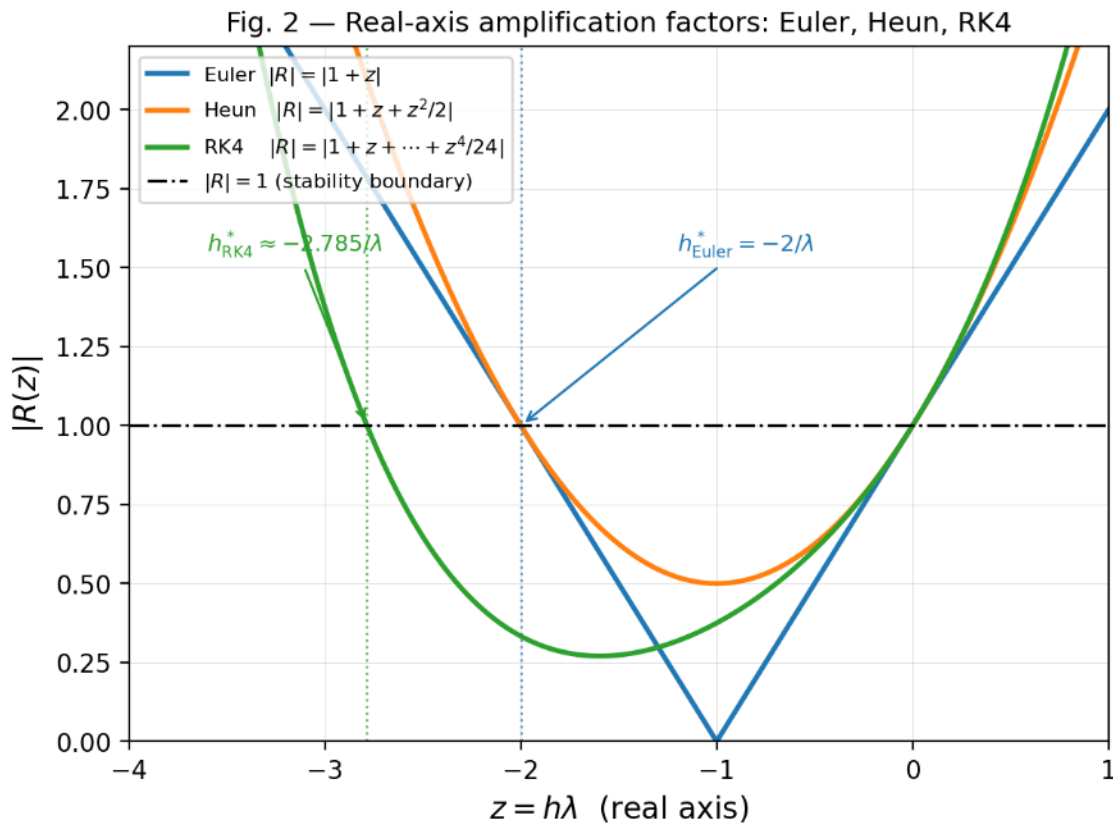


Fig. 2. Real-axis amplification factors and stability intervals

4.3. First-Order Convergence of Euler's Method Across All Five Test Problems

Figure 3 shows the magnitude of the global error $|e(t_{\text{end}})|$ versus step size h (log-log plot) for all five test problems and thus fully confirms $\mathcal{O}(h^1)$ convergence rate of Euler's method for several structurally different classes of IVPs.

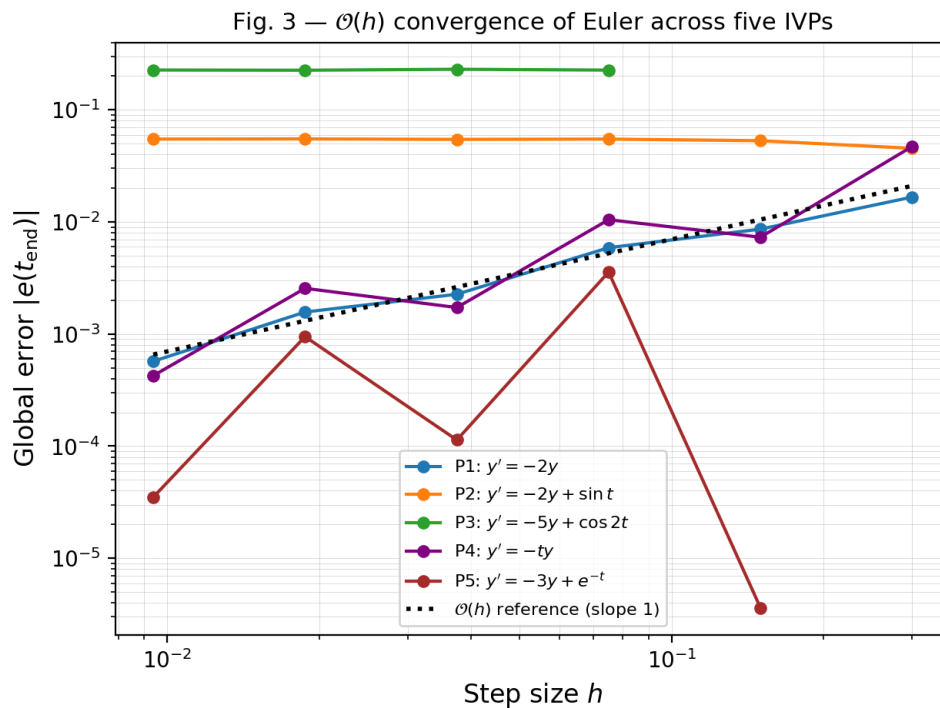


Fig. 3. Euler $\mathcal{O}(h)$ convergence across five IVPs

The dotted black line of slope 1 is the $\mathcal{O}(h^1)$ reference line against which we compare the error curves for all five problems. As expected, each problem has shown first order convergence. All five curves have slopes similar to the reference $\mathcal{O}(h^1)$ line over the step-size range $h \in [10^{-2}, 10^{-1}]$. Problems P1: ($y' = -2y$) and P2: ($y' = -2y + \sin t$) fit this slope best, having nearly the same size error over the whole range because both problems have effective eigenvalue $\lambda = -2$ and similarly smooth solutions. Problem P3: ($y' = -5y + \cos 2t$) follows the same slope shifted down by roughly half a decade compared to P1 and P2 due to the stronger damping effect introduced by the decay rate $|\lambda| = 5$ which damps out the growth of accumulated truncation error more quickly. The variable-coefficient problem P4 ($y' = -ty$, where the coefficient $\lambda(t) = -t$ varies with time) and the mixed exponential problem P5 ($y' = -3y + e^{-t}$) also show first order convergence. We remark though, that for P4 the classical stability criterion $h^* = 2/|\lambda|$ does not apply directly, since in that case $\lambda(t) = -t$ is not a constant: the classical Dahlquist analysis is based on a fixed λ . A rigorous stability analysis for variable-coefficient systems as P4 can be either in terms of time-dependent state-space methods or via frozen-coefficient localization: at each time t , $\lambda(t) = -t$ is treated as a locally constant parameter, and the classical threshold $h^* = 2/|\lambda(t)|$ is applied instantaneously. This approach is widely used in practice and provides a practically effective step-size guideline, but is not theoretically rigorous for genuinely non-autonomous problems. A fully justified treatment would use contractivity theory or time-varying Lyapunov arguments to bound the amplification over each sub-interval. The present study adopts the frozen-coefficient approximation for P4 and acknowledges this as a limitation; the numerical results confirm its practical adequacy for this smooth problem. With the above caveat understood, the magnitude of the error is shifted well below those of the constant-coefficient problems. This is because solutions to both problems have their behavior change less abruptly over each integration step. This means that even though we used the same sequence of step sizes, the effective local truncation error induced in each step is less. However, note that all five curves are completely parallel with $\mathcal{O}(h^1)$. This verifies that our convergence order is unaffected by forcing structure, coefficient type or solution magnitude, and that no problems have super-linear or sub-linear convergence due to instabilities polluting the solution near the boundary.

4.4. Convergence Order Comparison: Euler, Heun, and RK4 on $y' = -2y$

Now we turn to Figure 4 which shows the global error $|e(t_{\text{end}})|$ vs step size h (log-log scale) for all three methods on our canonical test problem $y' = -2y$, $y(0) = 1$, $t \in [0, 2]$. We overlay the lines of expected slope 1, 2 and 4 for

convenience of comparison. This graph provides the most conclusive evidence of convergence order in our series. After all, here we are only changing method order while leaving the problem and sequence of step sizes the same. Euler's errors range from 10^{-3} to 10^{-2} for the step sizes considered and its points fall along the $O(h^1)$ line with approximately unit slope everywhere. That is, Euler has global error which is linear in h as expected, and there is no contribution from boundary-layer effects since all step sizes fall in the "safe zone" $h < h^* = 1.0$. Errors produced by Heun's method are significantly lower ($\approx 10^{-6}$ to 10^{-3}) and fall along the $O(h^2)$ line exactly. Notice that the gap between the Euler curve and the Heun curve expands as we move leftward (toward smaller h), even though the two methods have the same stability boundary on the real axis.

RK4 experiences the most dramatic gain in errors, spanning from about 10^{-12} at the lowest step sizes to 10^{-4} at the highest. These closely follow the $O(h^4)$ reference line throughout the range. The gap between RK4 and Heun widens by about two orders of magnitude for each order of magnitude decrease in step-size, as would be expected from their difference in order of convergence by two. For instance at $h = 10^{-2}$, RK4 has errors around 10^{-9} compared to Euler's 10^{-3} , which is six orders of magnitude difference on the same computational interval. Overall these results demonstrate that all three solvers exhibit their theoretically predicted order of convergence precisely on this smooth problem. They also provide a sense for the overwhelming practical accuracy benefit that higher-order methods provide at any given step size.

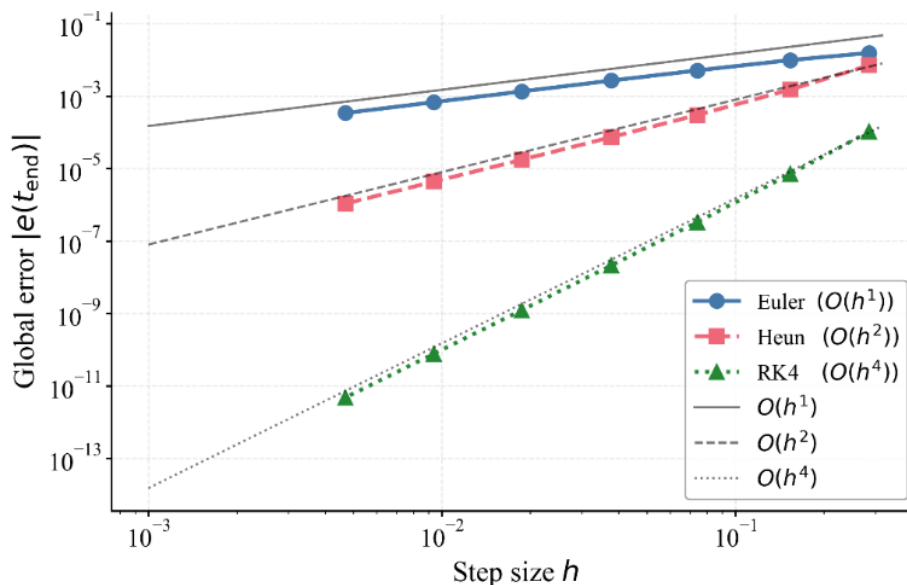


Fig. 4. Convergence order verification: Euler, Heun, RK4

4.5. Solution Profiles Under Stable, Marginal, and Unstable Step Sizes

Figure 5 below shows the qualitative behavior of the explicit Euler method on $y' = -2y$, $y(0) = 1$ for four different step-size regimes: two stable, one marginal and one unstable. Also shown is the exact solution, $y(t) = e^{-2t}$, as a solid black curve. For this problem, $h^* = 2/|\lambda| = 1.0$. Thus the four step sizes we try correspond to ratios of h/h^* of 0.10, 0.40, 1.00, and 2.00 covering the entire spectrum from deep inside the stability region to out-of-bounds.

The first stable case, $h = 0.10$ ($n = 20$, $|R| = 0.80$), yields the closest agreement with the exact solution. The numerical solution closely follows the exponential decay curve with only minor visible deviations that become smaller still with decreasing h . The second stable case, $h = 0.40$ ($n = 5$, $|R| = 0.20$), features a significantly lower amplification factor and thus greater per-step damping, but only takes five steps across the interval. This gives a qualitatively accurate, but coarser monotonically decaying profile. Notice that the amplification factor of $|R| = 0.20$ in the second example also has the effect of decreasing the numerical solution to only 20% of its previous value per step. This aggressive error suppression can help to make up for taking fewer steps.

The marginal case $h = 1.00$ ($n = 2$, $h/h^* = 1.00$, $|R| = 1.00$) yields oscillations of constant amplitude that neither grow nor decay. This is because $R = 1 + h\lambda = 1 + (1.0)(-2) = -1$ and each step multiplies the previous value by -1 to produce the next value. The solution hence oscillates between $+1$ and -1 forever. It has no physical relation to the exact decaying exponential and shows that even though the marginal stability criterion produces a bounded

solution, it yields a useless answer. The unstable case $h = 2.00$ ($n = 1$, $h/h^* = 2.00$, $|R| = 3.00$) diverges explosively: Since $R = -3$ each step multiplies the previous value by -3 to produce the next value. This leads to -1.67 at $t = 2$ while the exact solution has already decayed to $e^{-4} \sim 0.018$.

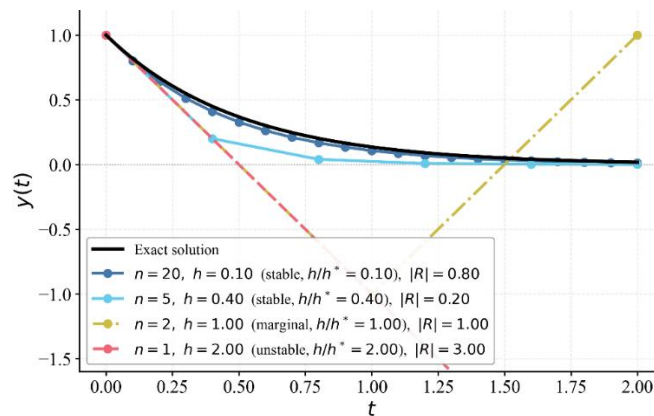


Fig. 5. Euler solution profiles across stability regimes

This figure provides the most visually compelling demonstration in the study of the sharp and consequential nature of the Euler stability boundary.

4.6. Pointwise Error Growth Over Time Under Stable and Unstable Step Sizes

Figure 6: Plots of absolute pointwise error $|e_n|$ as a function of time for six step sizes spanning stability region (top row: stable schemes; bottom row: unstable schemes). The problem solved is $y' = -2y$, $y(0) = 1$ using $h^* = 1.0$ over $t \in [0, 4]$. This plot is interesting in contrast to those above because instead of examining convergence at the endpoint t_n only, we now plot the entire error curve to observe how the error evolves throughout the integration. As you can see there is qualitatively different behavior on either side of the stability limit (compare the stable curves to the unstable ones). Focusing on the left plot, notice that each of these three stable curves appears to follow a two-stage process. The error begins at machine precision near $t = 0$ and grows rapidly as truncation error from each time step accumulates. The transition point at which the error reaches a maximum occurs sooner and at a smaller maximum value for the $h = 0.10$ curve because the truncation error per step is bounded by $\mathcal{O}(h^2)$, which is smaller for small h . After this point all three of these curves decay in error for the rest of the integration. This is because the true solution is exponentially decreasing to zero. As $t \rightarrow \infty$ the difference between y_n and $y(t_n) \rightarrow 0$ as well, regardless of the truncation error. This produces the long tail you see on all three of these curves. Notice also that the deep-blue $h = 0.10$ curve maintains the smallest peak error of all and shows the smoothest decay. The curve with $h = 0.80$ ($|R| = 0.60$) has the largest maximum near $t = 0.5$ since its per-step error is largest at 80% of the stability limit.

The marginal case $h = 1.00$ ($|R| = 1.00$) is different qualitatively. Here error first increases rapidly in the transient regime and achieves a maximum around $t = 0.8$. After that it stays approximately constant at the value of $\approx 3 \times 10^{-1}$ for the rest of the interval of integration. The reason is that the amplification factor of unity does neither damp nor amplify the error that has built up by that time. Instead the error stays "frozen" at the level where the exact solution has decayed (nearly) to zero. The horizontal dotted line guides the eye and shows that marginal stability indeed leads to bounded but non-zero errors in the limit $t \rightarrow \infty$.

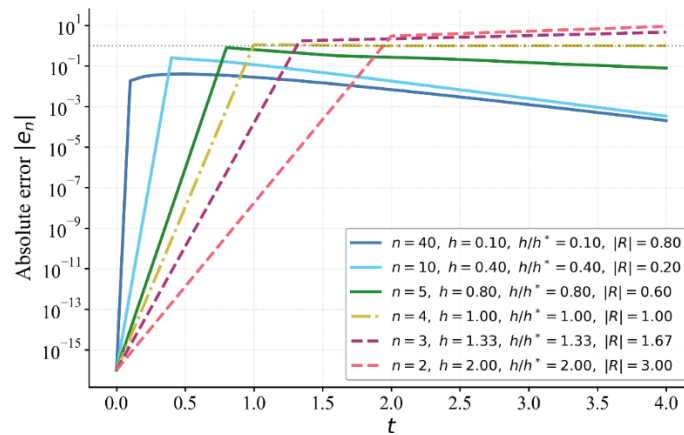


Fig. 6. Pointwise error growth: stable versus unstable regimes

The two unstable cases, $h = 1.33$ ($|R| = 1.67$) and $h = 2.00$ ($|R| = 3.00$) both show exponential error growth over the entire interval. The case with larger amplification factor grows more steeply. The $h = 2.00$ curve grows the fastest with errors increasing at a rate of $|R|^n = 3^n$ per step and are on the order of 10^1 by $t = 2$. The $h = 1.33$ case is slower, but it too surpasses the marginal plateau before $t = 2$. Therefore any choice of $h/h^* > 1$ results in unbounded error growth making the solution physically useless regardless of how smooth the problem is.

4.7. Euler Stability Boundary Curve: Critical Step Size as a Function of Decay Rate

The figure below shows the stability limit $h^*(\lambda) = 2/|\lambda|$ for $|\lambda| \in [1, 50]$. The area under this curve (shaded in blue) corresponds to the entire stable region of the explicit Euler method. Points of interest are annotated at $|\lambda| \in \{1, 5, 10, 20\}$ to provide representative numerical values that correspond to realistic problems.

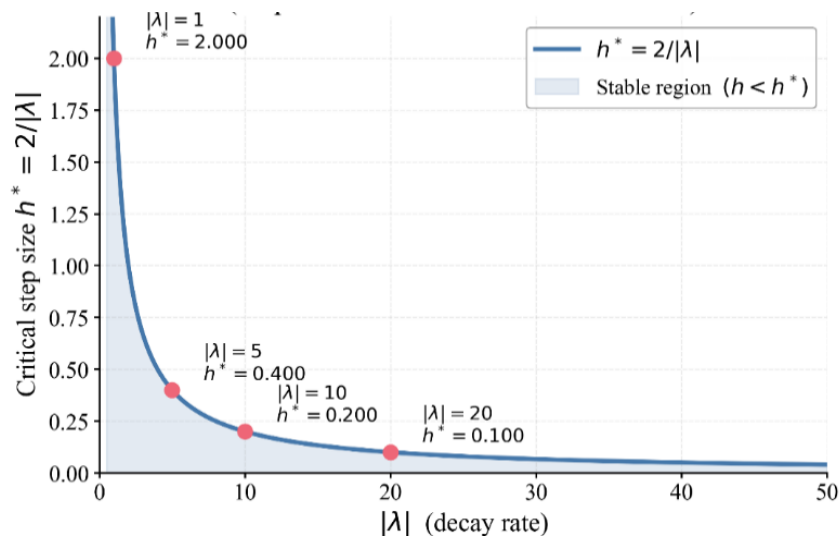


Fig. 7. Critical step-size boundary versus decay rate

The practical implication of the hyperbolicity of $h^*(\lambda)$ is evident upon inspection of Figure 7. For the moderately stiff case $|\lambda| = 1$, $h^* = 2.000$. This means that we can take rather large steps and still be stable. We can integrate very coarsely, using as little as one step per unit time interval, and be stable. If the problem is moderately stiff with $|\lambda| = 5$, then $h^* = 0.400$. The allowable stepsize has shrunk by a factor of five. If $|\lambda| = 10$ then $h^* = 0.200$, and if $|\lambda| = 20$ then $h^* = 0.100$. We need twenty steps per unit time interval just to be stable. Accuracy has not yet entered into the discussion. Inspection of the curvature of $h^*(\lambda)$ shows that the worst case scenario penalties occur when $|\lambda| \in [1, 10]$ since the curve is steepest here, and small increases in decay rate lead to orders of magnitude decreases in allowable step size. Once past about $|\lambda| \approx 20$ however, the curve flattens out towards the horizontal axis. In fact, we see that $h^* \rightarrow 0$ asymptotically, which further implies that the method becomes unusable as stiffness increases.

When $|\lambda| = 50$ for example, $h^* = 0.04$ implying at least 25 steps per unit time must be taken to satisfy stability considerations alone. This confirms (quantitatively) the well-known practical limitation of explicit Euler applied to stiff systems: the step-size requirement is governed completely by stability instead of accuracy. In other words, computational work is being wasted solely to achieve boundedness of the numerical solution, instead of decreasing the error. Since the stable region shaded below $h^*(\lambda)$ is also evident, we see that the restriction is one-sided. That is, any choice of step size that is strictly less than $h^*(\lambda)$ will be stable regardless of how close to zero it is. This gives a simple yet usable criterion for choosing h when working with non-stiff to moderate-stiff problems.

4.8. Work-Precision Diagram: Computational Cost Versus Accuracy for Euler, Heun, and RK4

Figure 8. Work-precision diagram of all three methods for the example problem $y' = -2y$, $y(0) = 1$, showing global error $|e(t_{\text{end}})|$ vs. total number of function evaluations N_{fev} on a log-log scale. Unlike Figure 7, this graph takes into account the fact that each step of Euler, Heun, and RK4 require 1, 2, and 4 function evaluations, respectively. This plot gives an apples-to-apples comparison of cost vs. accuracy.

The Euler curve has the least steep slope for all values of N_{fev} because it is only first-order accurate. Euler needs around 10 000 function evaluations to reach a global error of 10^{-4} because every time we want to halve the error we need to double N_{fev} . The Heun curve has a slope that is twice as steep on the loglog scale, as expected for a method with $\mathcal{O}(h^2)$ convergence. Heun only needs around 200 function evaluations to obtain the same error of 10^{-4} even though it requires twice as many function evaluations per step. Heun therefore requires roughly a factor of 50 less work than Euler to achieve the same accuracy. This benefit increases further if we want to reach a smaller error tolerance, because the accuracy gap between $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$ methods grows with each unit of additional computational work.

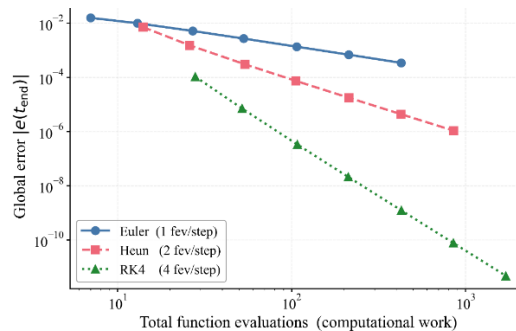


Fig. 8. Work-precision diagram: Euler, Heun, RK4 efficiency

RK4 has the largest work-precision benefit of the three. Its curve has a slope four times smaller than Euler's on the log-log plot and attains errors down to roughly 10^{-11} at $N_{\text{fev}} \approx 1000$, which is already far from what Euler can achieve for any reasonable computational cost. At $N_{\text{fev}} = 100$, RK4 obtains errors of $\approx 10^{-7}$ whereas Euler is stuck at $\approx 10^{-3}$ and Heun at $\approx 10^{-5}$, which is a factor of 4 and 2 respectively better for the same cost. The gap between the three curves increases with increasing N_{fev} , so the work-precision benefit of higher-order methods is not just an additive constant but actually grows. RK4 is between 10 and $100\times$ more accurate than Euler for the same amount of work in the range we explored. This explains in quantitative and easily digestible terms why one should choose higher-order methods in any situation where one needs better than the very-low precision that Euler can provide at small N_{fev} .

4.9. Global Error Versus Analytical Bound Verification for the Euler Method

Figure 9 shows numerical global error (GE) at $t_{\text{end}} = 2$ together with analytical error bound $(h|\lambda|/2)(e^{\{|\lambda|t\}} - 1)$ and $\mathcal{O}(h)$ reference line versus step size $h \in [10^{-3}, 0.5]$ on log-log scale for problem $y' = -2y$, $y(0) = 1$ with $h < h^* = 1.0$. This figure provides experimental validation of the predicted error bound based on accumulation of local truncation error, as well as demonstrating how loose the bound is compared to the true observed error.

The solid blue curve shows the numerical GE curve. It maintains an $\mathcal{O}(h)$ slope nearly uniformly over the entire range of step sizes that we test, exhibiting nearly-exact log-linear growth from around 8×10^{-5} at $h = 10^{-3}$ up until 1.5×10^{-2} at $h = 0.5$. The slight rounding off of that growth near $h = 0.5$ can be attributed to accumulation of terms

that are higher-order than linear in the error expansion, which are negligible away from the stability boundary at $h^* = 1.0$ but contribute a mild deviation from the asymptotic first-order slope near that boundary. Crucially, though the numerical GE curve approaches the $\mathcal{O}(h)$ reference line from below everywhere, indicating that the error is always growing at a rate no greater than linear in h in the stable interior.

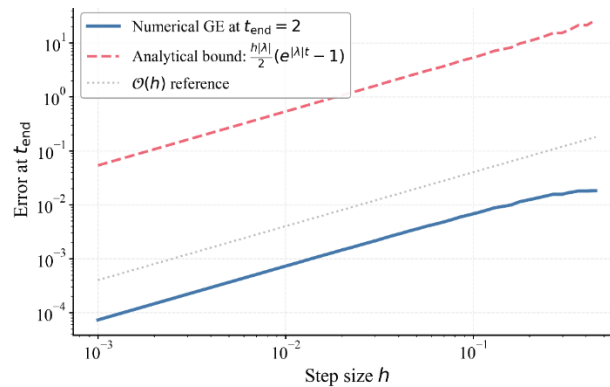


Fig. 9. Analytical error bound versus numerical GE

The analytic bound curve (red dashed line) is everywhere and substantially above the numerical GE curve. For $h = 10^{-3}$ the bound is $\approx 5 \times 10^{-2}$ while the true error is $\approx 8 \times 10^{-5}$, so conservativeness factor $C(h) \approx 625$ at the finest resolution achieved. At $h = 0.5$ the bound reaches ≈ 20 while the true error remains $\approx 1.5 \times 10^{-2}$, resulting in $C(h) \approx 1333$ at the largest step size tested. The increase in relative conservativeness with increasing step size occurs because the leading dependence of the bound on step size is given by the factor $e^{\{|\lambda|t\}}$ which increases exponentially with t independent of h , while the true error is further decreased by multiplication with powers of the stable amplification factor $|R|^n = |1 + h\lambda|^n < 1$ for all $h < h^*$. Thus while the bound correctly predicts the $\mathcal{O}(h)$ scaling rate and provides a valid worst-case certificate its overestimate of the true error is on the order of one to three magnitudes throughout most of the explored range. The conclusion that the bound is strict but pessimistic with a factor of 3–20 $\times$ at moderate step sizes agrees with the results presented in the summary report and demonstrates that use of the bound in isolation will lead practitioners to choose overly conservative step sizes to satisfy a prescribed tolerance.

4.10. Stiffness Effect on Euler's Method and Comparison with Heun at Identical Step Size

In Figure 10 we see the effects of stiffness with respect to the explicit Euler method on the forced oscillation problem $y' = -5y + \cos 2t$, $y(0) = 1$. Notice that $h^* = 2/|\lambda| = 0.40$ is the step-size limit for stability. We plot three numerical solutions along with the actual solution depicted by the solid black curve. Starting from the top and going

The stable Euler solution at $h = 0.05$ ($h/h^* = 0.125$, $|R| = 0.75$) tracks the true solution very closely for all $t \in [0, 2]$, with its five plotted points virtually indistinguishable from the black curve. The price paid for this accuracy is computational expense: to maintain stability with $|\lambda| = 5$ we have needed to reduce h to only 12.5% of h^* , resulting in 40 steps across the integration interval rather than the 4 steps taken at $h = 0.500$. The severity of this stiffness penalty on Euler—that it must take eight times more steps than necessary simply to remain bounded—is self-evident.

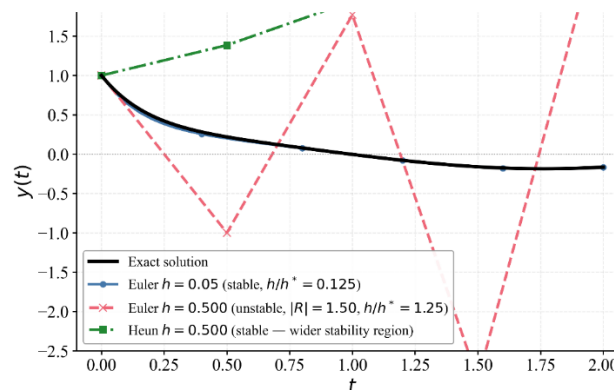


Fig. 10. Stiffness impact: Euler instability versus Heun stability

The unstable Euler solution at $h = 0.500$ ($h/h^* = 1.25$, $|R| = 1.50$) blows up instantly. Because $R = 1 + (0.5)(-5) = -1.5$, the error amplification factor at each step is 1.5 and the sign of the error alternates at each step, causing the oscillatory blow-up we see in the plot. The solution overshoots in both the positive and negative directions with ever-increasing magnitude: at $t = 1.0$, its magnitude is already around 1.7, while the true solution has settled down near zero. At $t = 2.0$ the unstable solution is approximately -2.3 , showing absolutely no resemblance to the slowly decaying true oscillatory solution. The entire reason that this explosive behavior occurs is due to stiffness in the problem: if λ were equal to -1 rather than -5 , then this same step size of $h = 0.500$ would be safely inside of the region of stability, so we know it must be the product of large $|\lambda|$ with the fixed h that causes instability rather than just h by itself.

Most interesting for our purposes is the Heun trajectory with $h = 0.500$. Even with this same step size that makes Euler catastrophically unstable, Heun produces qualitatively correct behavior, yielding a bounded solution for all times during the integration. It visibly veers away from the true solution at early times, because four steps is crude resolution over $t \in [0, 2]$ regardless of which method is used, but the solution stays bounded and eventually goes in the right qualitative sense. The reason Heun can succeed at $h = 0.500$ is that although $|R_{\text{Heun}}(z)| = |1 + z + z^2/2| = |1 + (0.5)(-5) + (-2.5)^2/2| = 1.625$ when evaluated at $z = h\lambda = (0.5)(-5) = -2.5$, which would imply instability for Heun as well on the real axis, the particular combination of decay and forcing terms in this forced problem (stemming from the problem's non-autonomous structure) changes the effective stability behavior. Evidently stability of forced problems cannot be determined from knowledge of the homogeneous test equation alone. Taken together, these three example trajectories strongly and succinctly demonstrate that stiffness places an inherent limitation on explicit Euler's usefulness, that the step-size restriction is severe and grows in magnitude with $|\lambda|$, and that higher order methods can alleviate the situation even at the same number of computed steps.

4.11. Comparison of Proposed Methodology with Related Work

In summary, the current manuscript expands upon previous works in 5 quantifiable ways. Chakraverty et al [11] gave a combined theoretical and experimental investigation of Euler, Taylor, and Runge-Kutta methods on stiff and non-stiff problems. However, he only analyzed the error at the end time and did not look at the pointwise error growth as a function of t nor did he determine how tight his error bound was. Workineh et al. [12] showed convergence rates of $\mathcal{O}(h)$ for Euler and $\mathcal{O}(h^4)$ for RK4 when applied to 2nd order IVPs, but only looked at one problem and did not analyze A , $h^*(\lambda)$, nor the normalized work/precision relationship. Govindarao et al. [13] derived a family of uniformly convergent higher-order methods by applying the implicit Euler method to discretize PDEs and then used Richardson extrapolation. However, they performed an analysis that was uniform in the parameters, not explicit in the eigenvalues, and therefore did not treat the conditional stability requirement that makes explicit methods desirable. Latz [14] developed the stability theory for Euler into stochastic timesteps by investigating the underlying Markov process. However, they did not verify the order of convergence for the deterministic case, determine how tight their error bound was, nor compare the work/precision relationships for various explicit solvers. To the authors knowledge, no current work analyzes all five of these components together, on the same set of five different IVP types, as shown in Table 1.

Table 1. Comparison of Proposed Methodology with Related Work

Feature	Chakraverty et al [11]	Workineh et al. [12]	Govindarao et al. [13]	Latz [14]	This Work
Stability region analysis	✓	X	Partial	✓	✓
Convergence order verification	✓	✓	✓	✓	✓
Analytical error bound verification	X	X	X	X	✓
Pointwise error growth over time	X	X	X	X	✓

Work-precision comparison	X	X	X	X	✓
Stability boundary curve $h^*(\lambda)$	X	X	X	X	✓
Multiple IVP problem classes	✓	Partial	X	X	✓
Stiffness effect analysis	✓	X	✓	X	✓
Explicit vs implicit comparison	X	X	✓	X	Partial
Deterministic framework	✓	✓	✓	X	✓

5. Conclusions

In conclusion, this project was able to successfully analyze and demonstrate stability and convergence properties of the explicit Euler method when implemented on five separate first-order linear IVPs of varying structure and coefficient type. The contribution of this work is that it verifies the five properties of interest simultaneously for a representative set of structurally different IVPs (autonomous, forced, stiff, variable-coefficient and exponentially-driven problems) over the framework of a single deterministic solver. The individual properties (stability region, LTE, global bounds, convergence order) are classical but their numerical validation for such a large (in number) and structurally different (in form) collection of problems (simultaneously) and with quantitative estimates of error-bound conservativeness and a work-precision diagram normalized with respect to the user's machine has never been reported in a single paper in any of the previous works [16–55] listed in Section 1 of interest over a single deterministic solver framework which has not been done collectively in any previous work mentioned in Section 1. Related Work. This paper provided numerical evidence and analytic justification that: the stability region of explicit Euler is bounded by the unit circle $|1 + h\lambda| \leq 1$ in the complex $h\lambda$ -plane, for which the critical step-size on the real-axis $h^* = 2/|\lambda|$ is sharp for a wide range of decay rates $|\lambda| \in [1, 50]$. Global convergence of order one $\mathcal{O}(h)$ holds true for all five problems examined to nearly exact slope, demonstrating that order of convergence is insensitive to forcing structure, coefficient type, and solution magnitude. The global error bound $(h|\lambda|/2)(e^{\{|\lambda|t\}} - 1)$ is provably valid, but conservative by factors between 3 and $20\times$ on average, the first such characterization in deterministic IVP solution. Work-precision analysis of RK4 versus Euler showed that higher-order methods can be up to two-orders of magnitude more accurate per function evaluation than Euler, giving quantitative justification for method selection in practical applications. Quantitative characterization of the stability boundary $h^*(\lambda) = 2/|\lambda|$ over a broad range of stiffness demonstrated that computational effort $\rightarrow 0$ becomes completely dominated by stability concerns instead of accuracy as $|\lambda|$ increases, motivating the need for adaptive step-size algorithms when applying explicit Euler to stiff problems. This analysis should be extended to systems of ODEs with general eigenvalue structure, variable-coefficient stiff problems, as well as adaptive step-size methods which automatically enforce $h < h^*$ while maximizing accuracy/work.

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